A GENERAL THEORY OF RADICALS.*

III. Applications.

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The present paper contains applications of the theory developed in the first two parts ([1], [2]) to theory of rings, in particular to associative rings. The results of the preceding parts [1], and [2] will be quoted by the numbers I, or II, followed by the number of the result quoted.

In section 1 we outline an extension of the theory of nilpotent ideals in associative rings to non associative (and non distributive) systems, obtaining a generalization of Baer's Lowe Radical ([3]) and proving that this generalized radical is also the intersection of all prime ideals, a result which was proved by Levitzki ([8]). Continuing the study of nilpotent ideals, we restrict ourselves in section 2 to associative rings. In this section, some new results on the Lower Radical are obtained as well as some short cuts in the proofs and new results for the semi-nilpotent radical ([7]). In view of the results obtained in section 2, it seems that Baer's Lower Radical is the best 'minimal' extension of the nilpotent radical of rings satisfying the minimum condition for right or left ideals. In section 3 we approach some of the known radicals from the point of view of the present theory. Thus we obtain alternative proofs for: Jacobson's radical ([6]), the locally finite kernel ([10]) and McCoy's radical ([11]). In the last section we utilize the theory for obtaining new radicals, among which is an FI-radical. A ring R is an FI-ring if every homomorphic image R' of R is an I-ring ([10], i. e. every non nil right ideal in R' contains an idempotent). It is shown that this property FI yields a radical which satisfies all the ordinary requirements a radical is supposed to satisfy.

1. Nilpotency and solvability. In the present section we consider rings (semi groups) R not necessarily associative (or distributive) and we outline an extension of the theory of nilpotent ideals to the non-associative case. The whole development here will hold also, without any changes in the proof, in general abstract algebras of the type considered in the last section of II, which are semi groups with respect to a binary operation $f(a, b)$.

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Let Q_1, Q_2 be two subsets of such a ring R. We denote by Q_1Q_2 the set of all element q_1q_2, $q_1 \varepsilon Q_1$, $q_2 \varepsilon Q_2$. In particular we put $Q' = QQ$ and inductively $Q^{(n)} = (Q^{(n-1)})'$. Note that $(Q^{(n)})^{(m)} = Q^{(nm)}$. An ideal P in R is called *a prime ideal* (in R) if: $P_1P_2 \subseteq P$, where $P_i \subseteq P$ are ideals in R, implies that either $P_1 \subseteq P$ or $P_2 \subseteq P$.

Defining a ring R to be a *ζ-algebra* if $R' = 0$,[1] we obtain an H-property ζ of rings. The upper ζ-radical, which we denote by $\zeta(R)$, will be called in this paper the *Baer's radical.* It is known (Levitzki [8]) that in the case of an associative ring R, Baer's radical is the intersection of all the prime ideals in R (McCoy's radical [11]). It will be shown here that this result is true for non associative ring, semi groups and more generally this is true for the abstract algebras of the type mentioned above. That is:

THEOREM 1. 1. *$\zeta(R)$ is the intersection of all the prime ideals of R.*

Proof. Observing that prime ideals are $\bar{\zeta}$-ideals, we deduce from Theorem I. 1. 2 that the intersection X of all the prime ideals of R contains $\zeta(R)$. The theorem will be proved if we show that for every $a \not\in \zeta(R)$ there is a prime ideal Q in R which does not contain the element a, since then $X \subseteq \zeta(R)$. Indeed, put $a_1 = a$; let A_1 be the minimal ideal in R containing a_1 and $\zeta(R)$. Since $a_1 \not\in \zeta(R)$, $A_1 \supset \zeta(R)$. Hence $A'_1 \not\subseteq \zeta(R)$. Let a_2 be any element of A'_1 which does not belong to $\zeta(R)$. In a similar way we can inductively obtain a sequence of elements $a_1, a_2, \cdots$ such that $a_i \not\in \zeta(R)$ and $a_{i+1} \varepsilon A'_i$, where A_i is the minimal ideal in R containing $\zeta(R)$ and a_i. By Zorn's lemma there exists a maximal ideal P in R containing $\zeta(R)$ and excluding the sequence $\{a_i\}$. The ideal P is the required ideal. For if $P_1P_2 \subseteq P$ and $P_i \supset P$ then the maximality of P implies that $a_j \varepsilon P_1$ and $a_k \varepsilon P_2$, for some j and k; hence $P_1 \cap P_2$ contains an element a_l of the sequence $\{a_i\}$. But this implies that $a_{l+1} \varepsilon A'_l \subseteq P_1P_2 \subseteq P$ which is a contradiction.[2]

The preceding proof provides an alternative characterization of $\zeta(R)$. We call a sequence $\{q_i\}$ of elements of R an *m-sequence* if $q_{i+1} \varepsilon Q'_i$, $i = 1, 2, \cdots$, where Q_i denotes the minimal ideal in R containing q_i. An m-sequence is said to be a *null sequence* if it contains only a finite number of non zero elements.

Define a property μ of ideals in R as follows: an ideal in A in R is a *μ-ideal* in R if every m-sequence of elements of A is a null sequence.

We prove:

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[1] Recall that P is said to be a $\bar{\zeta}$-ideal in R if the quotient R/P is ζ-semi simple. For associative rings, the notion of $\bar{\zeta}$-ideals coincide with the notion of radical-ideals of [3].

[2] This proof resembles that of the lemma of [9].

THEOREM 1.2. *The property μ is an RI-property (in the sense of II-section 2) and $\mu(R) = \zeta(R)$.*

Let ϑ be a homomorphism of R onto R^{ϑ} and Q be an ideal in R. Let P be any ideal in R, such $P = Q^{\vartheta}$. First we show that for any m-sequence $\{b_i\}$ in P, there exists an m-sequence $\{a_i\}$ in Q such that $b_i = a_i^{\vartheta}$, $i = 1, 2, \cdots$. Choose a_1, any element of Q such that $a_1^{\vartheta} = b_1$. Suppose that $a_1, \cdots, a_n$ have been so chosen that $a_i \varepsilon Q$, $a_i^{\vartheta} = b_i$, $i = 1, \cdots, n$ and $a_{i+1} \varepsilon A'_i$ where A_i denotes the minimal ideal in R containing a_i. We determine a_{n+1} as follows: since $a_n^{\vartheta} = b_n$ it is evident that $A_n^{\vartheta} = B_n$, where B_n is the minimal ideal in R which contains b_n. Since $(A'_n)^{\vartheta} = B'_n$ and $b_{n+1} \varepsilon B'_n$ we can find $a_{n+1} \varepsilon A'_n$ such that $a_{n+1}^{\vartheta} = b_{n+1}$ and evidently $a_{n+1} \varepsilon Q$. This proceduce determines inductively the required sequence.

Consequently, if every m-sequence of the ideal Q is a null sequence then the ideal $P = Q^{\vartheta}$ has the same property. This proves that μ is an HI-property. That μ is an RI-property will be shown by proving the validity of II. (C). Let A be not a μ-ideal in R, then A contains a non null sequence $\{a_i\}$. Consider the maximal ideal B contained in A and which does not contain the sequence $\{a_i\}$. Evidently $A \supset B$. If $A \supseteq P \supset B$ then the maximality of B implies that $a_i \varepsilon P$ for some i. Hence $a_j \varepsilon P$ for every $j \geq i$. Thus P/B contains the non null sequence $\bar{a}_i, \bar{a}_{i+1}, \cdots$ which proves that P/B cannot be a μ-ideal, and II. (C) is proved.

To prove $\mu(R) = \zeta(R)$, we note first that a ζ-ideal in R is a μ-ideal. By Corollary I. 1. 5 it remains, therefore, to show that ζ-semi simplicity implies μ-semi simplicity. Indeed, let R be a ζ-semi simple ring and let $A \neq 0$ be an ideal in R. Choose a_1 any non zero element of A; since $A'_1 \neq 0$ take $a_2 \neq 0$ any element of A'_1. Inductively we obtain a non null m-sequence $\{a_i\}$ in A, which shows that R is also μ-semi simple.

Remark 1.1. The preceding proof, with the exception of the last part, is readily seen to hold, with some slight changes, for the following similar properties of rings (semi groups, etc.). A sequence $\{p_i\}$ in R is a m_0-sequence if $p_{i+1} \varepsilon P_i$ where P_i is the minimal sub ring of R containing p_i (alternative definitions: $P_i = p_i R$, $P_i = R p_i$, etc.); and R is a μ_0-algeba if every m_0-sequence is a null sequence.

The second aim of the present section is to outline an extension of the theory of nilpotent ideals of associative rings to non associative rings (semi groups, or general multiplicative algebras). It seems that in the general case the property of solvability should replace the property of nilpotency.

We call a subset Q of R a *solvable* subset if $Q^{(n)} = 0$ for some integer n; and a ring R is said to be a *σ-ring* if R is solvable. We call a ring R a

$s\sigma$-*ring* (semi solvable)[3] if every finite subset of R is solvable. Furthermore, an element $a \varepsilon R$ is said to be a nil element if the set (a) containing the element a is solvable, and a ring R is a *nil-ring* (ν-ring) if all its elements are nil elements.

THEOREM 1.3. 1) *σ is an H-property which satisfies* (C) *and* (D_s)[4] *of* II. 2) *$s\sigma$ and ν are R-properties which satisfy* II(D_s) (*and hence* II(D)); *and* $\nu(R) \supseteq s\sigma(R) \supseteq \sigma(R) \supseteq \zeta(R)$.

Proof. Evidently, each of the properties are H-properties and satisfy II. (F_s). σ satisfies II. (C_1), for if A/B is solvable and B is solvable then $A^{(n)} \subseteq B$ for some n and $B^{(m)} = 0$, and hence $A^{(nm)} = 0$. A similar proof holds for $s\sigma$ and ν. Since these last two properties evidently satisfy II. (C_2), it follows by Theorem II. 1. 4 that they are R-properties. The relation between the various radicals follows by Theorem I. 1. 6 as a consequence of the fact that a ζ-ring is solvable, solvability implies semi solvability and, finitely, a semi solvable ring is a nil ring.

There are many other aspects of extending the theory of nilpotent ideals of associative rings to the non associative case. Thus, for instance, nilpotency may be defined as follows: a subset Q of R is nilpotent if there exists an integer n such that all the possible multiples of n elements of Q are zero. Similarly we can define semi nilpotency and nillity. Each of these properties is readily seen to be an H-property. But the main disadvantage of the nilpotency defined in this manner is that it does not satisfy II. (C_1) and, therefore, its allied properties: semi nilpotency and nillity are not, generally, R-properties and do not yield a radical. The validity of II. (C_1) seems the main reason for preferring solvability in the theory of non associative rings over the other generalisations of nilpotency.

Remark 1.2. Radicals of a similar type can be defined in abstract algebras of the type considered in section II, but which need not be ring or semi groups. We assume that these algebras possess a function $f(x_1, \cdots, x_n)$, $n < \infty$, (e. g. Lie triple systems) about which we assume that $f(x_1, \cdots, x_n) = 0$ if one of the $x_i = 0$. As in section II ideals will be the kernels of homomorphisms. An ideal will be said to be prime if

$$f(P_1, \cdots, P_n) = \{f(p_1, \cdots, p_n), p_i \varepsilon P_i\} \subseteq P,$$

where P_i are ideals, implies that at least one $P_i \subseteq P$. Denoting

$$R' = f(R, \cdots, R) = \{f(r_i, \cdots, r_n), r_i \varepsilon R\},$$

we define a ring to be a ζ-ring if $R' = 0$.

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[3] This is a generalisation of the notion of semi-nilpotency defined in [7].

[4] By Theorem II. 8. 3 it follows that σ satisfies also the additivity condition II. (Add) of section 8.

2. Nilpotency and semi-nilpotency of associative rings and semi-groups. We note that when associativity is assumed the two main properties solvability and nilpotency and, therefore, their issues are equivalent. Another point to notice is the difference between the present definition of semi-solvability and the definition of semi nilpotency given in [7]. Our definition seems to be more restrictive by requiring the solvability of every finite subset, but evidently the two definitions are equivalent if associativity is assumed. In the present section we deal with further properties of the radicals defined in the preceding one. We shall consider only associative rings but we remark that the following proofs and results hold also (with slight modifications) for associative semi-groups.

Since in the present case solvability and semi-solvability are equivalent to nilpotency and semi-nilpotency we shall also refer to the latter by the respective notation σ and $\boldsymbol{s}\sigma$.

In addition to the properties of $\boldsymbol{s}\sigma$ given in Theorem 1. 3 we prove:

THEOREM 2. 1. *$\boldsymbol{s}\sigma$ is an RZ-property*[5] *which satisfies* II. (F_r).

In view of Theorem 1. 3 and the validity of (2), it remains to show the validity of II. (F_r). This will follow by proving Lemma II. 3. 1 by a method due to J. Levitzki (compare with [10] Lemma 7. 3). Let J be a right semi-nilpotent ideal in R. Consider the ideal RJ. Let $t_1, \cdots, t_n$ be any finite set of elements of RJ; then $t_i = \sum_k r_{ik}s_{ik}$, $r_{ik} \in R$, $s_{ik} \in J$. Denote by $[t]$, $[r]$ and $[s]$ the module generated by the elements $\{t_i\}$, $\{r_{ik}\}$, and $\{s_{ik}\}$ respectively, then $[t] \subseteq [r][s]$. Hence $[t]^{m+1} \subseteq [r]([s][r])^m[s]$. The module $[s][r] \subseteq [sr]$ and the latter is generated by a finite number of elements of J. Hence $([s][r])^m = 0$ for some m. This proves that $[t]^{m+1} = 0$. Thus RJ is semi nilpotent.

The known properties of the semi nilpotent radical ([7]) follow now immediately by the results of part II. Additional results follow by Theorem II. 2. 2 and II. 6. 3 namely:

COROLLARY 2. 1. 1) *$\boldsymbol{s}\sigma(A) = \boldsymbol{s}\sigma(R) \cap A$ for every ideal A in the ring R.* 2) $\boldsymbol{s}\sigma(R_n) = \boldsymbol{s}\sigma(R)_n$.

Next we turn to a further study of the Baer's radical $\zeta(R)$ of associative rings.

For associative rings, ζ-semi simplicity implies σ-semi simplicity and hence by Corollary II. 1. 6 and by Theorem 1. 3 it follows that $\sigma(R) = \zeta(R)$. We have already seen that σ is an HZ-property which satisfies II. (C_1) and

[5] For definition see section II. 6.

II. (D_s) (and evidently II. (D_r) and II. (D)). Furthermore σ satisfies II. (E) : this follows from the fact that if P is a nilpotent ideal (a σ-ideal!) in an ideal Q of a σ-semi simple ring R, then since $(RPR)^{2n+1} \subseteq RP^nR$, RPR is a nilpotent ideal in R, which implies $RPR = 0$ and consequently $P = 0$. It follows now by Theorem II. 8. 2 that σ^* is an RZ-property which satisfies II. (D_s) and II. (D_r). A ring R was called an L-ring [6] an L-ring if $R = \zeta(R) = \sigma(R)$.

In our notation this is equivalent to the definition of a σ^*-ring. Using the notation of L-rings we obtain, in view of the definition of σ^* and of Lemma II. 8. 1:

Corollary 2. 2. *A ring R is an L-ring if and only if every non zero homomorphic image of R contain non zero nilpotent ideals.*

The validity of (D_s) implies:

Corollary 2. 3. *Every subring of an L-ring is an L-ring.*

We wish now to show that the property L $(=\sigma^*)$ satisfies II. (F_r). Let R be an L-semi simple ring and let J be a right ideal in R such that $L(J) = J$. The annihilator of J in J is the maximal nilpotent ideal in J. For, if Q is a nilpotent ideal in J, then QJ is a right nilpotent ideal in R. The L-semi simplicity of R readily implies that R does not contain non zero right nilpotent ideals and hence $QJ = 0$. Thus $Q \subseteq N$, where N is the left annihilator of J in J. By Theorem II. 1. 1 and by the definition of the upper radical, it is readily seen that $L(J) = N$. Hence $J^2 = 0$. By the L-semi simplicity it follows now that $J = 0$.

Thus, we obtain by Theorem II. 2. 2 and by Theorem II. 6. 3:

Corollary 2. 4. 1) *$L(A) = L(R) \cap A$ for every ideal A in a ring R.* 2) *$L(R_n) = L(R)_n$.*

We conclude with a remark that the preceding results with the exception of the second parts of the Corollaries 2. 1 and 2. 4 hold also for associative semi groups.

3. Various applications. We bring now some alternative proofs of some of the known radicals, in particular those which do not enter in the class of F-radicals, discussed in [4].

A. *Jacobson's radical.* This radical and its generalizations were dealt with from the point of view of F-radicals by several authors. We shall

[6] This is the notation used by J. Levitzki in: "A theorem on polynomial identities," *Proceedings of the American Mathematical Society*, vol. 1 (1950), pp. 334-341.

deal here only with the associative case. Recall that a ring R was called a quasi-regular ring (QR-ring) if for every $r \varepsilon R$ there exists an $s \varepsilon R$ such that $r + s + rs = 0$.[7] This property is readily seen to be an HZ-property and satisfies II. (D) and II. (D_r). Furthermore, QR satisfies the condition of Theorem II. 6. 6. Indeed, if R is not a QR-ring, let $a \varepsilon R$ be not a quasi regular element. Consider the maximal right ideal J containing the ideal $\{x + ax\}$ and excluding a. By the proof of Theorem 17 of [6] it follows that $R/(J:R)$ does not contain right QR-ideals and this proves Theorem II. 6. 6. Hence QR is an RZ-property and satisfies also II. (F_r). The validity of II. (E) follows by Theorem II. 5. 1. Thus most of the properties of the QR-radical obtained in [6] are deduced from the results obtained here.

B. *The locally finite kernel.* An associative algebra R over a field F is said to be *locally finite* (LF-ring) of every finitely generated subalgebra of R is finite over F. Levitzki has recently shown in [10] the existence of the LF-radical, which was called the locally finite kernel. The present theory provides a short cut in Levitzki's proofs and yields further properties of this radical.

The property LF is readily seen to be an HZ-property which satisfies II. (C_2) and II. (D_s) and evidently II. (D). The validity of II. (C_1) follows by [10] Lemma 7. 4, and II. (F_r) follows by [10] Lemma 7. 3. Since LF is also antiisomorphism invariant, a similar condition holds also for left ideals (Remark 1. 1). The validity of all these conditions yields the results on the locally finite kernel obtained in [10]. Additional results now obtained are: $LF(A) = LF(R) \cap A$ for every ideal in an algebra R; and $LF(R)_n = LF(R_n)$.

C. *McCoy's radical.* This radical, which was defined in [11], was proved to be equal to Bear's radical. It can be also approached from the present point of view of HI-properties. Calling an ideal A in a ring an M-ideal if every m-system of A contains the zero, we obtain an HI-property in the sense of section II. 1. The prime ideals are readily seen to be $\bar{M}$-ideals, and if $A \supset B$ are ideals in R such that A/B is not an M-ideal, one can find a prime ideal Q [8] in R such that $A \supset A \cap Q \supseteq B$. It follows now by Theorem I, 2. 4 that M is an RI-property and that the M-radical is the intersection of all the prime ideals.

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[7] Usually, quasi-regularity is defined in a ring but it is known that an ideal A (right, left or two sided) in a ring R is quasi regular in R if and only if A is a quasi-regular ring.

[8] The ideal Q can be chosen as a maximal ideal $Q \supset B$ such that Q/B excludes a certain m-system of A/B.

4. New radicals. A. *PI-radical.* For the present radical we consider only associative algebras over a field F. An algebra R is said to be a PI-ring if it satisfies a polynomial identity $f(x_1,\cdots,x_n)=0$ where the coefficients of f belong to F. This property is readily seen to be an HZ-property which satisfies II. (D_s) and hence also II. (D). Furthermore, PI satisfies II. (C_1) and II. (E). Indeed, if R possesses an ideal A such that R/A satisfies the identity $g(x_1,\cdots,x_n)=0$ and such that A satisfies the identity $f(x_1,\cdots,x_n)=0$, then R satisfies the identity

$$f[g(x_{11},\cdots,x_{1n}),\cdots,g(x_{m1},\cdots,x_{mn})]=0$$

in the nm indeterminates x_{ik}. This proves II. (C_1). To prove II. (E), let R be a PI-semi simple ring and let A be an ideal in R. If Q is an ideal in A which satisfies an identity, then $AQA\subseteq Q$ is a PI-ideal in R, hence $AQA=0$. But the totality of the elements $x\in A$ such that $AxA=0$ constitute a nilpotent ideal in R, hence, by II. (Z), it must be zero; and thus II. (E) is proved.

It follows now by Theorem II. 8. 3 that PI is an additive-property and, therefore, PI^* is an RZ-property and satisfies II. (E) and (D_s). We recall that a ring R is a PI^*-ring if every non zero homomorphic image of R contains non zero ideals which satisfy a polynomial identity.

The property PI^* is an example of an R-property which does not satisfy II. (F_r). For, the ring R of all finite matrices is a simple ring and it is known that this ring does not satisfy a polynomial identity, yet the right ideal $c_{11}R$, i. e., the ring of all one rowed matrices satisfies the identity $(x_1x_2-x_2x_1)x_3=0$.

B. ***π.s.h.**-radicals.* The methods of section II. 7 can be applied to yield a class of new radicals. With each property π of associative rings which satisfies the conditions of Theorem II. 7. 1 or II. 7. 2 we can associate a π.***s.h.***-radical. We recall that a ring R is a π.***s.h.***-ring if every homomorphic image of R does not contain a non zero π-ideal (i. e. is π-semi simple). In particular, Theorem II. 7. 2 can be applied to the following properties: quasi-regularity, nillity, semi-nilpotency and nilpotency.

In particular, a ring R is nilpotent-***s.h.*** if every homomorphic image does not contain nilpotent ideals. Now a regular ring is evidently a ring of this type so that nilpotent-***s.h.*** radical contains the maximal regular ideal defined in [5].

C. *The FI-radicals.* The notions of I-rings and FI-rings were introduced in [10] and the structure of such rings was determined there. An associative ring R was called an I-ring if every non nil right ideal contains an idempotent, and R was called an FI-ring if every homomorphic image of R is an I-ring. Similarly we shall call a ring R an I_S-ring (an I_L-ring) if every right ideal which is not semi-nilpotent (which is not an L-ring) contains an idempotent. In the same way we define FI_S-rings and FI_L-rings.

We wish to show that the three properties: FI, FI_S and FI_L are SRZ-properties and satisfy the conditions II. ($\mathrm{D_r}$). Furthermore, the last two properties satisfy also II. ($\mathrm{F_r}$). The proof will be carried out only for the property FI_S since the proof for the other two properties is similar.

The property I_S satisfies II. (A) and a similar proof to that of [10] Lemma 5. 4 yields that I_S satisfies II. ($\mathrm{C_1}$) and II. ($\mathrm{C_2}$), hence by Lemma II. 7. 1 it follows that FI_S is an R-property. Evidently this property satisfies II. (Z). To prove the validity of II. ($\mathrm{D_r}$) we consider a right ideal J in an FI_S-ring R. Let J/Q be a homomorphic image of J and let P/Q be a right ideal in J/Q. Now, PJ is a right ideal in R and RQJ is a two sided ideal. Since R is an FI_S-ring, R/RQJ is an I_S-ring, so that $(PJ, RQJ)/RQJ$ either contains an idempotent or it is semi-nilpotent. Put $M = PJ \cap RQJ$, then the last quotient is isomorphic with PJ/M. PJM is a two sided ideal in PJ, hence one readily verifies that PJ/PJM is also either semi-nilpotent or contains an idempotent. Since $JRQJ \subseteq JQJ \subseteq Q$ it follows that

$$PJM \subseteq PJ \cap PJRQJ \subseteq PJ \cap Q.$$

Furthermore,

$$(PJ \cap Q)^4 \subseteq PJ(RQJ \cap PJ) = PJM.$$

Hence $PJ/PJ \cap Q$ will also be semi-nilpotent or contain an idempotent respectively what PJ/PJM does. Consequently, $(PJ, Q)/Q$, and hence also P/Q, will have the same property, which proves ($\mathrm{D_r}$). Evidently this implies II. (D), and by Theorem II. 5. 1 it follows that FI_S satisfies also II. (E). Consequently, FI_S is an SRZ-property.

We now prove the validity of ($\mathrm{F_r}$). Let R be an FI_S-semi simple ring and let J be a right FI_S-ideal in R. Consider the ideal $J^* = (J, RJ)$. Let J^*/Q be a homomorphic image of J^* and let P/Q be a right ideal in J^*/Q. We shall prove that either P/Q contains an idempotent or it is semi-nilpotent. Evidently, the same will hold if we replace Q by J^*QJ^*; or, equivalently, we may assume that Q is an ideal in R. In this case one readily verifies that the semi-nilpotent radical N^*/Q of J^*/Q is an ideal in R/Q (by Corollary 2. 1). For every $j \varepsilon J$, $(jP, Q)/Q$ is a homomorphic image of jP. The latter is a right ideal in J, hence either it is semi-nilpotent or contains an idempotent. If the first case holds for every $j \varepsilon J$, then $(JP, Q)/Q$ must be semi nilpotent [9] and hence $JP \subseteq N^*$. Thus

$$P^2 \subseteq (J, RJ)P \subseteq (JP, RJP) \subseteq (N^*, RN^*) \subseteq N^*.$$

This proves that $(P^2, Q)/Q$ and hence also P/Q are semi-nilpotent. If this is not the case, $(jP, Q)/Q$ contains an idempotent e for some $j \varepsilon J$, i. e. $e \equiv jp(Q)$, $j \varepsilon J$ and $p \varepsilon P$. By replacing p by pe we may assume that $pe \equiv p(Q)$. Since $e^2 \not\equiv 0(Q)$, $pj \not\equiv 0(Q)$ and the latter belongs to P and satisfies $(pj)^2 = p(jp)j = pej = pj(Q)$. This proves that P/Q contains an idempotent. Thus J^* is an FI_S-ideal in R; hence the FI_S-semi simplicity of R yields $J^* = 0$ and therefore $J = 0$.

A radical similar to those discussed above is the following: call a ring R an I_0-*ring* if every non zero right ideal in R contains an idempotent, and call R an FI_0-*ring* if every homomorphic images of R are I_0-rings. As in the preceding case FI_0 can be shown to be an R-property, but this property does not satisfy the conditions II. (D_r) and II. (F_r). Yet we can prove that FI_0 satisfies II. (D) and the two conditions II. (G_1) and II. (G_2). Thus FI_0 and the preceding properties FI, FI_S and FI_L satisfy the main results on the radical, in particular Theorem II. 2. 2. Furthermore, FI_0 satisfies Theorem II. 4. 1 and the last two properties satisfy Theorem II. 6. 3.

To prove that FI_0 satisfies II. (D) we consider an FI_0-ring R and an ideal A in R. If A is not an FI_0-ring then A contains an ideal Q such that A/Q is not an I_0-ring and, evidently, A/AQA cannot be an I_0-ring. Hence we may assume that Q is an ideal in R. If J/Q is a right ideal in A/Q which does not contain an idempotent, then since R/Q is an I_0-ring, $JA \subseteq Q$. The set $\{x \mid x \varepsilon A, xA \subseteq Q\}$ is an ideal T in R. If $T \not\equiv 0 \bmod Q$ then it contains an idempotent $e \bmod Q$ but then $e^2 \varepsilon eA \subseteq Q$ which is impossible. This proves that $J \equiv 0 \bmod Q$; that is, the validity of II. (D). Before proceeding with the proof of the conditions II. (G_1) and II. (G_2) we show that the ring R_n is an I_0-ring if and only if R is such. Indeed if R_n is an I_0-ring, one readily verifies that R is also an I_0-ring. Let R be an I_0-ring and let J be a right ideal in R_n. Since R cannot have a left annihilator we have $JR_n \neq 0$. Hence, $J(c_{ii}R) \neq 0$ for some i. We can also see that the coefficients of c_{ii} of the matrices of $J(c_{ii}R)$ constitute a non zero right ideal in R. This ideal contains an idempotent e and one readily observes that the matrix of $Jc_{ii}R$ which is of the form $ec_{ii} + \cdots$ is an idempotent of the ideal J, q. e. d. We turn now to the proof of II. (G_1). Let R be an FI_0-ring.

[9] This proof holds only for the properties FI_S and FI_L and it is not known whether it is valid for FI.

Let Q be an ideal in R_n. Since $R_nQR_n = T_n$, where T is an ideal in R we have $R_n/R_nQR_n \cong (R/T)_n$. The latter is an I_0-ring since R/T is such, hence R_n/R_nQR_n and therefore, also R_n/Q is an I_0-ring which proves that R_n is an FI_0-ring. To prove II. (G_2) let R be an FI_0-semi simple ring. If R_n is not FI_0-semi simple then it contains an ideal Q which is an FI_0-ring. Now Q/R_nQR_n can be an I_0-ring only if $Q = R_nQR_n$. Let T be an ideal in R such that $R_nQR_n = T_n$; then for every ideal P in T we have $Q/P_n \cong (T/P)_n$. Since Q is an FI_0-ring, T/P must be an I_0-ring. This proves that T is an FI_0-ideal in R hence $T = 0$. Therefore $Q = 0$ which proves II. (G_2).

The properties FI (FI_S, FI_L and FI_0) are symmetric, in the sense that if a ring R is I-ring then also every non nil *left* ideal in R contains idempotency ([10]), and conversely.

We conclude with the remark that the following relation exists between these radicals: $FI(R) \supseteq FI_S(R) \supseteq FI_L(R) \supseteq FI_0(R)$ for every ring R, and all these radicals contain the maximal regular ideal of the ring R which was defined in [5]. The last follows from the fact that a regular ring is also an FI_0-ring.

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BIBLIOGRAPHY.

[1] S. A. Amitsur, "A general theory for radicals, I. Radicals in complete lattices," *American Journal of Mathematics*, vol. 74 (1952), pp. 774-786.

[2] ———, "A general theory for radicals, II. Radicals in rings and bicategories," *ibid.*, vol. 76 (1954), pp. 100-125.

[3] R. Baer, "Radical ideals," *ibid.*, vol. 69 (1947), pp. 46-58.

[4] B. Brown and N. H. McCoy, "Some theorems on groups with applications to ring theory," *Transactions of the American Mathematical Society*, vol. 69 (1950), pp. 302-311.

[5] ———, "The maximal regular ideal of a ring," *Proceedings of the American Mathematical Society*, vol. 1 (1950), pp. 165-171.

[6] N. Jacobson, "The radical and semi simplicity for arbitrary rings," *American Journal of Mathematics*, vol. 67 (1950), pp. 300-320.

[7] J. Levitzki, "On the radical of a general ring," *Bulletin of the American Mathematical Society*, vol. 49 (1943), pp. 462-466.

[8] ———, "Prime ideals and the lower radical," *American Journal of Mathematics*, vol. 73 (1951), pp. 25-29.

[9] ———, "A note on prime ideals," *Riveon lemathematica* (Hebrew), vol. 5 (1951-2), pp. 1-3, see also *Mathematical Reviews*, vol. 13 (1), (1952), p. 8.

[10] ———, "On the structure of algebraic algebras and related rings," *Transactions of the American Mathematical Society*, vol. 74 (1953), p. 404.

[11] N. H. McCoy, "Prime ideals in general rings," *American Journal of Mathematics*, vol. 71 (1948), pp. 823-833.