Local Nilpotency of the McCrimmon Radical of a Jordan System [1]

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Abstract: This paper contains a full proof of the local nilpotency of the McCrimmon radical of a Jordan system over an arbitrary ring of scalars, and, as a consequence, the fact that simple Jordan systems are always nondegenerate.

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Introduction

The fact that a simple Jordan system is necessarily nondegenerate is a basic property widely used. This result is based on Kostrikin-Zelmanov's theorem on the local nilpotency of Lie algebras generated by sandwiches [5]. Different proofs of this deep result on Lie algebras can be found in the literature [2, 5, 11, 12, 13], some of them even containing its application to Jordan systems [11, 12, 13]. Unfortunately, most of this references deal with algebraic systems over rings of scalars with certain restrictions on the characteristic, and those dealing with Lie algebras over arbitrary rings of scalars do not contain the result for Jordan systems in its whole generality.

This paper gives a full account on the results mentioned above, with complete proofs in the most general setting. However, the reader should NOT expect either NEW RESULTS OR PROOFS here. The sources for the materials in this paper are [13] and the notes of seminars given by Efim Zelmanov at the Department of Mathematics of the University of Virginia in 1989. Indeed, [13] contains a full proof of the fact that Lie algebras generated by sandwiches are locally nilpotent, following [2]. That is based on a result on associative words due to Backelin. The reader is referred to [13] for a complete bibliography on the subject.

The paper is divided into 3 sections apart from a preliminary one. In the first section, we deal with associative words on a finite alphabet and obtain the main theorem as a consequence of Furstenberg's results [3] on compact dynamical systems (indeed, on symbolic systems). This is used in the second section to obtain Zelmanov-Kostrikin's theorem on Lie algebras, which is applied in the last section to prove

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that Jordan systems generated by sandwiches are locally nilpotent. From that, the McCrimmon radical of a Jordan system is shown to be locally nilpotent too, which implies that any simple system has zero radical.

0. Preliminaries

0.1 We will deal with Jordan systems (algebras, pairs and triple systems), and associative and Lie algebras over an arbitrary ring of scalars Φ. We warn the reader that even when dealing with Lie algebras, we will not make any additional assumption on the ring of scalars. In particular, we will not assume $1/2 \in \Phi$.

The reader is referred to [4, 6, 8, 9] for basic results, notation, and terminology, though we will stress some notions. The identities JPx listed in [6] will be quoted with their original numbering without explicit reference to [6].

—When dealing with an associative algebra, the (associative) products will be denoted by juxtaposition.

—The product in a Lie algebra L will be denoted by square brackets $[x, y] = \mathrm{ad}_x(y)$, for any $x, y \in L$.

—Given a Jordan algebra J, its products will be denoted by x^2, $U_x y$, for $x, y \in J$. They are quadratic in x and linear in y and have linearizations denoted $x \circ y = V_x y$, $U_{x,z} y = \{x, y, z\} = V_{x,y} z$, respectively.

—For a Jordan pair $V = (V^+, V^-)$, we have products $Q_x y \in V^\sigma$, for any $x \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$, with linearizations $Q_{x,z} y = \{x, y, z\} = D_{x,y} z$.

—A Jordan triple system J is given by its products $P_x y$, for any $x, y \in J$, with linearizations denoted by $P_{x,z} y = \{x, y, z\} = L_{x,y} z$.

0.2 (i) A Jordan algebra gives rise to a Jordan triple system by simply forgetting the squaring and letting $P = U$. By doubling any Jordan triple system T one obtains the *double Jordan pair* $V(T) = (T, T)$ with products $Q_x y = P_x y$, for any $x, y \in T$. From a Jordan pair $V = (V^+, V^-)$ one can get a (*polarized*) Jordan triple system $T(V) = V^+ \oplus V^-$ by defining $P_{x^+ \oplus x^-}(y^+ \oplus y^-) = Q_{x^+} y^- \oplus Q_{x^-} y^+$ [6, 1.13, 1.14].

(ii) An associative algebra A gives rise to a Lie algebra $A^{(-)}$, called the *antisymmetrization* of A, over the same Φ-module structure as A, with Lie products given by the *commutators* $[x, y] = xy - yx$, for $x, y \in A$.

0.3 Given a Jordan pair V, a derivation of V is any pair of Φ-linear maps $(\Delta^+, \Delta^-) \in \mathrm{End}_\Phi V^+ \times \mathrm{End}_\Phi V^-$ such that

$$\Delta^\sigma(Q_x y) = \{\Delta^\sigma(x), y, x\} + Q_x \Delta^{-\sigma}(y),$$

for any $x \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$. The set $\mathrm{Der}\, V$ of all derivations of V is a (Lie) subalgebra of $(\mathrm{End}_\Phi V^+ \times \mathrm{End}_\Phi V^-)^{(-)}$ (see [6, 1.4]). For any $x \in V^+$, $y \in V^-$, we define $\delta(x, y) := (D_{x,y}, -D_{y,x})$ which turns out to be a derivation of V by (JP12) called an *inner derivation* of V. The Φ-submodule of $\mathrm{Der}\,V$ spanned by all inner derivations of V will be denoted $\mathrm{IDer}\,V$ and is an ideal of the Lie algebra $\mathrm{Der}\,V$ since

$$[\Delta, \delta(x,y)] = \delta(\Delta^+(x), y) + \delta(x, \Delta^-(y)),$$

for any $\Delta \in \mathrm{Der}\,V$, and any $x \in V^+$, $y \in V^-$.

0.4 Given a Jordan pair V and any subalgebra $\mathcal{D}$ of $\mathrm{Der}\,V$ containing $\mathrm{IDer}\,V$, the Φ-module

$$\mathrm{TKK}(V, \mathcal{D}) := V^+ \oplus \mathcal{D} \oplus V^-$$

can be equipped with a product $[\ ,\ ]$ given by

$$\begin{aligned} &[x^+ \oplus c \oplus x^-, y^+ \oplus d \oplus y^-] := \\ &\big(c^+(y^+) - d^+(x^+)\big) \oplus \big([c,d] + \delta(x^+, y^-) - \delta(y^+, x^-)\big) \oplus \big(c^-(y^-) - d^-(x^-)\big) \end{aligned} \tag{1}$$

yielding a Lie algebra over Φ called the *Tits-Kantor-Koecher algebra* of V and $\mathcal{D}$ [10, XI]. When $\mathcal{D} = \mathrm{IDer}\,V$, we obtain the so called *Tits-Kantor-Koecher algebra* of V, denoted $\mathrm{TKK}(V)$. One can find references like [7] where the definition of the product in $\mathrm{TKK}(V, \mathcal{D})$ differs of (1) in some minus signs. The Lie algebras so built are isomorphic to ours and, (1) is more convenient, in terms of notation, for our purposes.

0.5 An *absolute zero divisor* of a Jordan algebra (resp., triple system) J is an element $x \in J$ such that $U_x J = 0$ (resp., $P_x J = 0$). An absolute zero divisor in a Jordan pair (V^+, V^-) is any element $x \in V^\sigma$ such that $Q_x V^{-\sigma} = 0$. A Jordan system is said to be *nondegenerate* if it does not have nonzero absolute zero divisors.

0.6 An element a in a Lie algebra L is called a *sandwich* if

$$\text{(i) } \big[[L,a],a\big] = 0 \qquad \text{and} \qquad \text{(ii) } \Big[\big[[L,a],L\big],a\Big] = 0$$

If L does not have 2-torsion, then (ii) follows from (i). Indeed, for any $a, x, y \in L$,

$$\begin{aligned} \Big[x, \big[[y,a],a\big]\Big] &= \Big[\big[[x,y],a\big],a\Big] + \Big[\big[y,[x,a]\big],y\Big] + \big[[y,a],[x,a]\big] \\ &\quad \text{(since } ad_x \text{ is a derivation)} \\ &= \Big[\big[[x,y],a\big],a\Big] - \Big[\big[[x,a],y\big],a\Big] + \Big[\big[y,[x,a]\big],a\Big] + \Big[y,\big[a,[x,a]\big]\Big] \\ &\quad \text{(using that } [\ ,\ ] \text{ is antisymmetric, and } ad_{[x,a]} \text{ is a derivation)} \\ &= \Big[\big[[x,y],a\big],a\Big] - 2\Big[\big[[x,a],y\big],a\Big] + \Big[\big[[x,a],a\big],y\Big] \\ &\quad \text{(using again that } [\ ,\ ] \text{ is antisymmetric),} \end{aligned}$$

hence, $\big[[L,a],a\big] = 0$ implies $2\Big[\big[[x,a],y\big],a\Big] = 0$, for arbitrary $x, y \in L$, thus $2\Big[\big[[L,a],L\big],a\Big] = 0$. ■

0.7 Given a set X, the free associative algebra over X will be denoted by $\Phi[X]$. It is a free Φ-module with a basis $W[X]$ consisting of the associative *monomials* or *words* $x_{i_1} \cdots x_{i_n}$, for arbitrary $x_{i_1}, \ldots, x_{i_n} \in X$. When dealing with associative words, X will be called the *alphabet*, and its elements will be called *letters*. The algebra $\Phi[X]$ is $\mathbf{Z}$-graded by the *degree* or *length* of words, and also $\mathbf{Z}^X$-graded by the *composition* of words.

0.8 Given a set X, a *commutator* in X is any element of $\Phi[X]$ built out of the elements of X by commutation, i.e., products in $\Phi[X]^{(-)}$. More precisely, the set $C[X]$ of commutators in X is defined inductively by setting $X \subseteq C[X]$, and, for any $a, b \in C[X]$, $[a, b] \in C[X]$. Notice that $C[X]$ spans (as a Φ-module) the subalgebra of $\Phi[X]^{(-)}$ generated by X. One can readily see that the commutators in X are homogenous in the $\mathbf{Z}^X$-grading of $\Phi[X]$, in particular, homogenous with respect to the length.

0.9 (i) An associative algebra A is said to be *nilpotent* if there exists a natural number n such that $A^n = 0$, i.e., any associative product of length at least n vanishes, i.e., any associative monomial of degree at least n vanishes when evaluated on A.

(ii) A Lie algebra L is said to be *nilpotent* if there exists a natural number N such that $L^n = 0$, where the Lie powers are defined inductively by $L^1 = L, L^i = [L^{i-1}, L]$. This is clearly equivalent to saying that $ad(a_1) \cdots ad(a_{n-1}) = 0$, for any $a_1, \ldots, a_{n-1} \in L$, i.e., the (associative) subalgebra of $\mathrm{End}_\Phi L$ generated by the elements $ad(a)$, for $a \in L$, called the *multiplication algebra* of L, is nilpotent.

One can define Lie monomials in the free Lie algebra over a set of variables X by imposing that the variables are Lie monomials of length one, and that, if a, b are Lie monomials of lengths n, m, respectively, then $[a, b]$ is also a Lie monomial of length $n + m$. This can be used to give an alternative definition of nilpotency for Lie algebras, as in the associative setting (i):

(ii)' A Lie algebra L is nilpotent if an only if there exists a natural number n such that any Lie product of length a least n vanishes, i.e., any Lie monomial of length at least N vanishes when evaluated on L.

Indeed, if L_k denotes the span of the evaluations on L of the Lie monomials of length at least k, it can be easily proved by induction that $L_{2^{n-1}} \subseteq L^n \subseteq L_n$, for all n, which readily implies (ii)'.

(iii) A Jordan algebra (resp., triple system) J will be said *nilpotent* if there exists a natural number n such that any Jordan algebra (resp., triple system) monomial of degree n (in the sense of [1, 1.1]) vanishes when evaluated on J. A Jordan pair V will be said *nilpotent* if $T(V)$ is a nilpotent Jordan triple system.

1. Some Combinatorial Results Dealing with Associative Words

1.1 In this section we will deal with several ordered sets. An order relation $\leq$ is determined by its associated strict order relation $<\ \equiv\ \leq$ and $\neq$.

1.2 A total order $\leq$ in a set X induces the so-called *lexicographic order* in the set of words $W[X]$ defined by: $x_{i_1}, \dots, x_{i_n} < y_{i_1}, \dots, y_{i_m}$ if

(i) there exists $0 \leq r \leq m, n$ such that $x_{i_1} = y_{i_1}, \dots, x_{i_{r-1}} = y_{i_{r-1}}$, and $x_{i_r} < y_{i_r}$.

Notice that the lexicographic order in $W[X]$ is not total, since we cannot compare two words such that one is a proper beginning of the other. The *extended lexicographic order* in $W[X]$ induced by $(X, \leq)$ is given by extending the lexicographic order to a total order by imposing that any word is smaller than any of its proper beginnings, i.e., for $a, b \in W[X]$,

(ii) $a = bc$ (for some $c \in W[X]$) $\Longrightarrow a < b$.

1.3 In this section, we will consider $X = \{x_1, \dots, x_n\}$ with the natural order given by the subindexes ($x_i \leq x_j$ if and only if $i \leq j$), which induces the lexicographic order in $W[X]$, also denoted by "$\leq$".

1.4 An element $0 \neq a \in \Phi[X]$, homogeneous with respect to the length, can be uniquely represented as a (finite) linear combination of words in $W[X]$, $a = \sum_i \alpha_i w_i$, where the w_i's are pairwise distinct words, all of them of the same length, and $\alpha_i \neq 0$ for all i. Thus, all the w_i's are comparable, and there is a greatest one among them which will be called the *leading term* of a. This happens, in particular, with nonzero commutators in X (0.8), which then have a leading term.

1.5 (i) A word $v \in W[X]$ is said to be *regular* if it is greater than any proper end:

$$v = ab, (a, b \in W[X]) \Longrightarrow b < v.$$

(ii) A word $v \in W[X]$ is said to be *semiregular* if it is not smaller than any proper end:

$$v = ab, (a, b \in W[X]) \Longrightarrow v \not< b \text{ (i.e., either } b < v \text{ or } b \text{ is a beginning of } v).$$

Notice that any beginning of a semiregular word is also semiregular.

1.6 Let T be the set of words in X of the form $x_n^q v$, where $q \geq 1$ and v is a non-empty word in X not containing x_n. Let $\leq_T$ be the restriction to T of the extended lexicographic order in $W[X]$, which induces a lexicographic order, also denoted $\leq_T$ in the set of words $W[T] \subseteq \Phi[T]$ over T. It is not hard to see that the subalgebra $< T >$ of $\Phi[X]$ generated by T is isomorphic to the free associative algebra $\Phi[T]$ over T (the set B of products $t_1 \cdots t_m$ spanning $< T >$ is contained in the basis $W[X]$ of $\Phi[X]$, hence B is a basis of $< T >$ bijective with $W[T]$ in the obvious way, which induces the mentioned isomorphism), so that we will identify both and assume $W[T] \subseteq W[X]$. Thus, in $W[T]$, we have two order relations: the restriction of the order $\leq$ given in (1.3), and $\leq_T$.

The following assertions hold for words $a, b \in W[T]$:

(i) a, b have the same composition as words in $T \Longleftrightarrow a, b$ have the same composition as words in X.

(ii) $a <_T b \Longrightarrow$ either $a < b$ or b is the beginning of a.

(iii) $a < b \Longrightarrow a <_T b$.

(iv) a is regular as a word in $X \Longrightarrow a$ is regular as a word in T.

Indeed, (i) is straightforward. If $a <_T b$, then $a = t_1 \cdots t_k$, $b = s_1 \cdots s_l$ $(t_i, s_j \in T)$ so that $t_1 = s_1, \ldots, t_{r-1} = s_{r-1}$, and $t_r <_T s_r$, since $<_T$ in T is the extended lexicographic order in $W[X]$, either $t_r < s_r$ or s_r is the beginning of t_r. In the first case, $a < b$ and, in the second case, either $r = l$ and b is a beginning of a, or $t_r = x_n^q uv$, $s_r = x_n^q u$, where u, v are words not containing x_n, and $a = t_1 \cdots t_r x_n^q uv \cdots < t_1 \cdots t_r x_n^q u x_n \cdots = b$. We have proved (ii), and now (iii) follows from it. Indeed, if $a \not<_T b$, then we have three cases: $b <_T a$, or the two words are equal, or one is the beginning of the other (a fact which does not depend on the alphabet T or X). In the first case (ii) yields a contradiction, and the remaining cases are clearly against $a < b$. Finally, (iv) follows directly from (iii).

1.7 LEMMA. *For every regular word $v \in W[X]$, there exists a commutator $\rho \in C[X]$ having v as its leading term.*

PROOF: We will prove the result by induction on the length of v. If that length is 1, then the result is clear since $X \subseteq C[X]$. Assume that the assertion of the Lemma holds for regular words (in any alphabet X) of lengths smaller than the length of v, which is at least 2.

We will prove that our assertion holds on v by induction on the cardinality m of X. If $m = 1$, then everything is clear since, in that case, the only regular word is x_1, of length 1. Assume that the assertion of the lemma holds for alphabets of less than n elements, with $n \geq 2$ (for words of length less than or equal to the length of v).

If v does not contain x_n, then it is a word in the alphabet $\{x_1, \ldots, x_{n-1}\}$, hence it is the leading term of a commutator by the induction assumption. So we can assume that v contains the letter x_n. We claim that v begins with x_n and ends with some other letter x_i with $i < n$. Otherwise, either $v = u x_n w$, where u does not contain x_n, or $v = x_n a x_n$. In the first case $v < x_n w$ and, in the second case, x_n and v are not comparable, and both situations are against the regularity of v.

Thus, v can be written as a product $v = t_1 \cdots t_r$, where each factor has the form $t_i = x_n^{q_i} u_i$, where $q_i \geq 1$, and u_i is a non-empty word non containing the letter x_n, i.e., $v \in W[T]$ as in (1.6) (the t_i's are not necessarily pairwise distinct). Notice that r is strictly smaller than the length of v, and v is regular as a word in T by (1.6)(iv). Hence, the induction assumption applies to show that v is the leading term, as a word in T, of a commutator $\mu \in C[T]$ which, by homogeneity (0.8) is built out of the elements $t_1, \ldots, t_r$, i.e.,

$$\mu = \sum_{\tau \in \Lambda} \varepsilon_\tau t_{\tau(1)} \cdots t_{\tau(r)},$$

for certain subset Λ of the group Σ_r of permutations of r elements containing the identity Id, and $0 \neq \varepsilon_\tau \in \Phi$. Moreover, we can group equal monomials so that, for $\tau, \mu \in \Lambda$, $t_{\tau(1)} \cdots t_{\tau(r)} = t_{\mu(1)} \cdots t_{\mu(r)}$ only if $\tau = \mu$

On the other hand, any $t \in T$ is the leading term, as a word in X, of a commutator $\tilde{t} \in C[X]$. Indeed, if $t = x_n^q x_{i_1} \cdots x_{i_k}$, where $i_1, \ldots, i_k < n$, t is the leading term of $\tilde{t} = \mathrm{ad}_{x_{i_k}} \cdots \mathrm{ad}_{x_{i_2}} (\mathrm{ad}_{x_n})^q (x_{i_1})$.

We claim that v is the leading term, as a word in X, of the commutator $\rho \in C[X]$ obtained by replacing the elements t_i in μ by the commutators $\tilde{t}_i$. Indeed, the words of $W[X]$ appearing in ρ are exactly those of the form $u_{\tau(1)} \cdots u_{\tau(r)}$, for all $\tau \in \Lambda$, and all the words u_i appearing in the commutators $\tilde{t}_i$, $i \in \{1, \ldots, r\}$. Since $u_i \leq t_i$, any $u_{\tau(1)} \cdots u_{\tau(r)} \leq t_{\tau(1)} \cdots t_{\tau(r)}$, but $t_{\tau(1)} \cdots t_{\tau(r)} \leq_T t_1 \cdots t_r$, which implies $t_{\tau(1)} \cdots t_{\tau(r)} \leq t_1 \cdots t_r$ by (1.6)(ii). On the other hand, if $u_{\tau(1)} \cdots u_{\tau(r)} = t_1 \cdots t_r$, then $t_1 \cdots t_r \leq t_{\tau(1)} \cdots t_{\tau(r)}$, which implies $t_1 \cdots t_r \leq_T t_{\tau(1)} \cdots t_{\tau(r)}$ by (1.6)(iii), hence $\tau = Id$, and $u_1 = t_1, \ldots, u_r = t_r$. ■

1.8 An *infinite sequence* $U = U(1)U(2) \cdots = x_{i_1} x_{i_2} \cdots$ is any map $k \longrightarrow x_{i_k}$ from the set of natural numbers to X. A *subword* of an infinite sequence will be meant a **finite** word of the form $U(k)U(k+1) \cdots U(k+r) = x_{i_k} x_{i_{k+1}} \cdots x_{i_{k+r}}$ for some natural numbers k and r. A *tail* of U is the infinite sequence $U_k = U_k(1)U_k(2) \cdots := U(k+1)U(k+2) \cdots = x_{i_{k+1}} x_{i_{k+2}} \cdots$.

1.9 Lemma. *For any infinite set of words $S \subseteq W[X]$ there exists an infinite sequence every subword of which is a subword of one of the words in S.*

Proof: S is the disjoint union of of the sets $S^i := \{w \in S \mid w = x_i \cdots\}$, for $i = 1, \ldots n$. Since S is infinite, at least one of the S^i's is infinite. Let S_1 be one of those infinite S^i's. Assume that we have found $i_1, \ldots i_k$ such that $S_k = \{w \in S \mid w = x_{i_1} x_{i_2} \cdots x_{i_k} \cdots\}$ is infinite. Now S_k is the disjoint union of the sets

$$S_k^i := \{w \in S_k \mid w = x_{i_1} x_{i_2} \cdots x_{i_k} x_i \cdots\} = \{w \in S \mid w = x_{i_1} x_{i_2} \cdots x_{i_k} x_i \cdots\},$$

for $i = 1, \ldots n$, and we can find i_{k+1} such that $S_{k+1} := S_k^{i_{k+1}}$ is infinite. This allows as to define the infinite sequence $U = x_{i_1} x_{i_2} \cdots$ such that every finite beginning of U is the beginning of a word of S (indeed the beginning of an infinite amount of words of S). This U clearly satisfies the assertion of the lemma (every subword of U is a subword of a sufficiently long beginning of U). ■

1.10 Lemma. *If $Q \subseteq W[X]$ and for each k there exists a word of length k not having any subword in Q, then there exists an infinite sequence not having any subword in Q.*

Proof: Just take the subset $S \subseteq W[X]$ consisting of the words not having any subword in Q. By the hypothesis, S is infinite, and (1.9) applies to find an infinite sequence U every subword of which is a subword of one of the words in S, hence every subword of U lies in $W[X] \setminus Q$. ■

1.11 An infinite sequence U is said to be *uniformly recurrent* if for any subword u of U, there exists a natural number $k = k(u)$ such that u is a subword of any subword of U of length k. In particular, every subword of a uniformly recurrent infinite sequence U occurs in U infinitely many times.

1.12 Theorem (H. Furstenberg, 1981). *For any infinite sequence W, there exists a uniformly recurrent infinite sequence U, such that every subword of U is a subword of W.*

Proof: Let $Y = \{x_0\} \uplus X$ be the set obtained by adding a new letter to X. By [3, Definition 2.1], $\Omega := Y^{\mathbf{Z}}$ can be given the structure of a compact metric space, which gives rise to a compact dynamical system (Ω, f), where f is the shift homeomorphism given by $f(V)(n) = V(n+1)$, for any $V \in \Omega$.

Let $\tilde{W}$ be the element of Ω given by $\tilde{W}(k) = x_0$, for all $k \leq 0$, and $\tilde{W}(r) = W(r)$, for any $r > 0$. By [3, 4.1], there exists a uniformly recurrent $V \in \Omega$ which is proximal to $\tilde{W}$. Here, V being uniformly recurrent means (see [3, 4.2]) that

(1) every subword of V is a subword of every sufficiently large subword of V,

in a similar fashion to uniform recurrence for infinite sequences (1.11).

Let S be the set of subwords of V not containing the letter x_0. Since V and $\tilde{W}$ are proximal, we can use [3, 4.3] to show that S is infinite, and (1.9) applies to find an infinite sequence U every subword of which is a subword of a word of S. In particular,

(2) every subword of U is a subword of V,

which readily implies that U is a uniformly recurrent infinite sequence in the sense of (1.11).

Given any subword w of U, (2) and uniform recurrence of V imply that there exists a natural number k such that w is a subword of each subword of V of length k. Using proximality of V and $\tilde{W}$ and [3, 4.3], there exists $n \in \mathbf{N}$ such that

$$v = V(n)V(n+1)\cdots V(n+k-1) = \tilde{W}(n)\tilde{W}(n+1)\cdots\tilde{W}(n+k-1).$$

Thus w is a subword of v, hence a subword of $\tilde{W}$. Since w does not contain the letter x_0, w is a subword of W. ■

1.13 Proposition. *There exists an integer N (depending on the cardinality n of X) such that every word in $W[X]$ of length N contains either a subword u^2 or a subword uvu, where v is a regular word and u is a semiregular word.*

PROOF: If the assertion of the proposition is false, then we can use (1.10) to find an infinite sequence U not having subwords of type u^2 or uvu, where v is a regular word and u is a semiregular word. By (1.12), U can be assumed to be uniformly recurrent.

For each $k \geq 1$ consider all subwords of U of length k (a finite amount since X is finite), and let a_n be the greatest of them (all of them are comparable since they have the same length). It is easy to see that, for all k, the word a_k is semiregular and that a_k is the beginning of a_{k+1}. Thus we may define a new infinite sequence V such that $V = a_k \cdots$ for all k. Every subword of V is a subword of a_k for a sufficiently big k, hence it is a subword of U, which implies that

(1) V is also uniformly recurrent,

(2) V does not contain subwords of the form u^2 or uvu, where v is a regular word and u is a semiregular word.

We claim that V does not coincide with any of its tails V_k for any $k \geq 1$. Otherwise, since always $V = a_k V_k$, we would have that V begins with the word a_k^2, which contradicts (2) since a_k is semiregular.

For each $k \geq 1$, let n_k be an integer such that a_{n_k} is the longest common beginning of V and V_k ($n_k = 0$ when the first letters of V and V_k do not coincide).

(3) We can find arbitrarily big n_k.

Indeed, given $m \geq 0$, (1) yields that the word a_m appears infinitely many times in V, thus a_m is the beginning of some tail V_k, which implies $n_k \geq m$.

Therefore, the set $S = \{k \in \mathbf{N} \mid n_1, n_2, \ldots, n_{k-1} < n_k\}$ is infinite. Otherwise, let r be the maximum of S, and n be the maximum of $\{n_1, n_2, \ldots, n_r\}$; now $r+1 \notin S$ implies $n_{r+1} \leq n$, hence n is also the maximum of $\{n_1, n_2, \ldots, n_r, n_{r+1}\}$; this argument can be repeated to prove that $n \geq n_k$ for all k, which contradicts (3). Since the alphabet X is finite,

(4) there exists $1 \leq i \leq n$ such that the set $S_i = \{k \in S \mid V(k) = x_i\}$ is infinite.

Given $k \in S_i$, $V = a_{k-1} x_i \cdots$ and, in particular, $a_k = a_{k-1} x_i$. We claim that a_{k-1} is not regular. Indeed, by (3) there exists r such that $n_r > k$, and (4) implies that there exists $l \in S_i$ such that $l > r$, but, in particular, $l \in S$, hence $n_l > n_{l-1}, \ldots, n_r, \ldots$ and $n_l > k$. Thus, $V(l) = x_i$ and $V = a_{l-1} x_i V_l$ contains the subword $x_i a_k = x_i a_{k-1} x_i$, which contradicts (2) if a_{k-1} is regular.

Hence, some end of the word a_{k-1} is, at the same time, a beginning of it,

$$a_{k-1} = a_r a_t, \qquad \text{with} \qquad r + t = k - 1, \quad 1 \leq r, t,$$

which implies $t \leq n_r$. But also $n_r \leq n_k$ because $k \in S$, and this implies $t \leq n_k$, i.e., $V_k = a_t \cdots$, which yields

$$V = a_k V_k = a_{k-1} x_i V_k = a_r a_t x_i V_k = a_r a_t x_i a_t \cdots$$

and V contains the subword $a_t x_i a_t$, which contradicts (2). ■

1.14 THEOREM. *There exists an integer N (depending on the cardinality n of X) such that every word in $W[X]$ of length N contains either a subword u^2 or a subword uvu, where u and v are regular words.*

PROOF: By (1.13), there exists N such that each word $w \in W[X]$ of length N contains a subword of the form u^2 or uvu, where v is regular and u is semiregular. We claim that w also contains a subword u^2 or uvu, where both u, v are regular. Otherwise, let u be a semiregular word of minimum length such that u^2 or uvu (for some regular v) is a subword of w. Since u is not regular, then some end of u is at the same time a beginning of u, i.e., $u = u_1u_2 = u_2u_3$ for some words u_1, u_2, u_3, where u_2 is strictly shorter than u. Moreover, u_2 is also semiregular since it is a beginning of u (see (1.5)). Since $u^2 = u_1u_2^2u_3$, $uvu = u_1u_2vu_2u_3$, w contains the subword u_2^2 or u_2vu_2, which contradicts the minimality of the length of u. ■

2. From Associative Words to Lie Algebras

2.1 LEMMA. *Let A be an associative algebra, and L be a (Lie) subalgebra of $A^{(-)}$. If $a, b \in L$ satisfy $a^2 = b^2 = 0$ and $aLa = bLb = 0$, then $[a,b]^2 = 0$ and $[a,b]L[a,b] = 0$.*

PROOF: We have

$$[a,b]^2 = (ab - ba)^2 = abab - ab^2a - ba^2b + baba = 0$$

since $aba \in aLa = 0$ and $a^2 = b^2 = 0$, and, for any $x \in L$,

$$\begin{aligned} [a,b]x[a,b] =& abxab - abxba - baxab + baxba \\ =& abxab + baxba \text{ (since } bxb \in bLb = 0 \text{ and } axa \in aLa = 0) \\ =& ab[x,a]b + abaxb + b[a,x]ba + bxaba = 0 \end{aligned}$$

since $b[x,a]b, b[a,x]b \in bLb = 0$ and $aba \in aLa = 0$. ■

2.2 PROPOSITION. *Let A be an associative algebra, and L be a (Lie) subalgebra of $A^{(-)}$ generated by elements $a_1, \ldots, a_n \in A$ such that $a_i^2 = 0$, $a_iLa_i = 0$, for all $1 \le i \le n$. Then the associative subalgebra B of A generated by L is nilpotent.*

PROOF: Let $X = \{x_1, \ldots, x_n\}$. We will say that $w \in W[X]$ is irreducible if the evaluation $w(a_1, \ldots, a_n)$ of w ($x_1 \mapsto a_1, \ldots x_n \mapsto a_n$) cannot be written as a finite sum

$$w(a_1, \ldots, a_n) = \sum_i \alpha_i w_i(a_1, \ldots, a_n)$$

where $\alpha_i \in \Phi$, $w_i \in W[X]$ has the same composition as w, and $w_i < w$, for all i. It is clear that

(1) a subword of an irreducible word is also irreducible.

We claim that

(2) words of the form u^2 or uvu, where u, v are regular words, are not irreducible.

Indeed, by (1.7), there are commutators $\rho_u, \rho_v \in C[X]$ such that u, v are their respective leading terms. Then, it is easy to see that uvu is the leading term of $\rho_u \rho_v \rho_u$, i.e.,

$$\rho_u \rho_v \rho_u = uvu + \sum_i \alpha_i w_i$$

for $\alpha_i \in \Phi$, where w_i is a word of the same composition as uvu, and $w_i < uvu$, for all i. By (2.1) (applied inductively on the length of commutators),

$$\rho_u(a_1, \dots, a_n)\rho_v(a_1, \dots, a_n)\rho_u(a_1, \dots, a_n) = 0,$$

which implies $uvu(a_1, \dots, a_n) = -\sum_i \alpha_i w_i(a_1, \dots, a_n)$. This shows that uvu is not irreducible, and the same argument, removing v and ρ_v, proves that u^2 is not irreducible either.

It is clear that B is generated as an associative algebra by $a_1, \dots, a_n$. Let N be as in (1.14) for the alphabet X. We will show that

(3) any associative word $w \in W[X]$ of length N evaluated in $a_1, \dots, a_n$ is zero,

which implies that B is nilpotent. Indeed, (1.14) implies that any word of length N contains a subword of the form u^2 or uvu, where u, v are regular words, hence

(4) all words of length N are not irreducible

by (1) and (2). If (3) is false, since all words of length N are comparable in the lexicographic order (1.2), we can find the smallest $w \in W[X]$ of length N such that $w(a_1, \dots a_n) \neq 0$. By (4), $w(a_1, \dots, a_n) = \sum_i \alpha_i w_i(a_1, \dots, a_n)$, where all the w_i's have length N and are smaller than w, but this implies that $w_i(a_1, \dots, a_n) = 0$ for all i, which is a contradiction. ■

2.3 Theorem. *A Lie algebra L generated by a finite collection of sandwiches $a_1, \dots, a_n$ is nilpotent.*

Proof: Let $A = \mathrm{End}_\Phi L$ be the (associative) algebra of endomorphisms of L as a Φ-module, and let $ad(L)$ be the set of operators $ad(a)$ for all $a \in L$. It is well known that $ad(L)$ is a (Lie) subalgebra of $A^{(-)}$, indeed, the image in $A^{(-)}$ of L under the Lie algebra homomorphism $a \mapsto ad(a)$. Hence $ad(L)$ is generated by $ad(a_1), \dots, ad(a_n)$, which satisfy $ad(a_i)^2 = 0$ and $ad(a_i)ad(L)ad(a_i) = 0$ for all i since the a_i's are sandwiches (see (0.6)(i)(ii)). By (2.2), the subalgebra B of A generated by $ad(L)$ is nilpotent, but B is the multiplication algebra of L, hence L is nilpotent (0.9)(ii). ■

2.4 Corollary. *A Lie algebra generated by a set of sandwiches is locally nilpotent.* ■

3. From Lie algebras to Jordan Systems

We start with a kind of Jordan algebra generalization of (2.1).

3.1 LEMMA. *If a, b be absolute zero divisors of a Jordan algebra J, then a^2 and $a \circ b$ are also absolute zero divisors.*

PROOF: Indeed, by [4, QJ1, QJ2], $U_{a^2} = U_a U_a = 0$ when a is an absolute zero divisor.

Also, [4, QJ16] yields $U_{a\circ b} = U_a U_b + U_b U_a + V_b U_a V_b - U_{a,U_b a} = 0$ if both a and b are absolute zero divisors. ∎

3.2 We will stress some connections between the products of a Jordan pair V and the products in the Lie algebra $\mathrm{TKK}(V)$, i.e., particular cases of (0.4)(1):

(i) $[x, y] = -[y, x] = \delta(x, y)$, for any $x \in V^+$, $y \in V^-$,

(ii) $[[x, y], z] = \{x, y, z\}$, for any $x, z \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$.

3.3 LEMMA. *Let V be a Jordan pair, and $a \in V^+$ be an absolute zero divisor of V. Then, for any $d \in \operatorname{Der} V$, and any $y, y' \in V^-$,*

(i) $D_{a,y} d^+(a) = 0$,

(ii) $[\delta(a, y), \delta(a, y')] = 0$.

PROOF: (i) Using that d is a derivation,

$$D_{a,y} d^+(a) = \{a, y, d^+(a)\} = d^+(Q_a y) - Q_a d^-(y) = 0$$

since $Q_a = 0$.

(ii) Also $[\delta(a, y), \delta(a, y')] = ([D_{a,y}, D_{a,y'}], [D_{y,a}, D_{y',a}]) = 0$ since, for any $b, b' \in V^-$, using JP13 and JP9, respectively,

$$D_{a,b} D_{a,b'} = D_{Q_a b, b'} + Q_a Q_{b,b'} = 0, \qquad D_{b,a} D_{b',a} = Q_{b,b'} Q_a + D_{b, Q_a b'} = 0. \ \blacksquare$$

3.4 LEMMA. *Let V be a Jordan pair, and $a \in V^\sigma$, $\sigma \in \{+, -\}$, be an absolute zero divisor of V. Then a is a sandwich of* $\mathrm{TKK}(V)$.

PROOF: We will prove the result for $a \in V^+$ since the case $\sigma = -$ follows by passing to the opposite pair $V^{\mathrm{op}} = (V^-, V^+)$ (see [6, 1.5]).

For any $x, x' \in V^+$, $d, d' \in \operatorname{IDer} V$, $y, y' \in V^-$, we can use (0.4)(1) to obtain

$$\begin{aligned}\big[[x \oplus d \oplus y, a], a\big] =& [d^+(a) \oplus -\delta(a, y) \oplus 0, a] \\ =& -\delta(a, y)^+(a) = -D_{a,y} a = -\{a, y, a\} = -2Q_a y = 0\end{aligned} \tag{1}$$

since $Q_a = 0$, and

$$\begin{aligned}
&\Big[\big[[x\oplus d \oplus y, a], x' \oplus d' \oplus y'\big], a\Big] \\
&\quad = \Big[\big[[x \oplus d \oplus y, a], a\big], x' \oplus d' \oplus y'\Big] + \big[[x \oplus d \oplus y, a], [x' \oplus d' \oplus y', a]\big] \\
&\qquad\quad \text{(since } ad(a) \text{ is a derivation in TKK}(V)) \\
&\quad = \big[[x \oplus d \oplus y, a], [x' \oplus d' \oplus y', a]\big] \text{ (by (1))} \\
&\quad = [d^{+}(a) \oplus -\delta(a, y) \oplus 0, d'^{+}(a) \oplus -\delta(a, y') \oplus 0] \text{ (by (0.4)(1))} \\
&\quad = \big(-\delta(a, y)^{+} d'^{+}(a) + \delta(a, y')^{+} d^{+}(a)\big) \oplus [\delta(a, y), \delta(a, y')] \oplus 0 \text{ (by (0.4)(1))} \\
&\quad = \big(-D_{a,y} d'^{+}(a) + D_{a,y'} d^{+}(a)\big) \oplus [\delta(a, y), \delta(a, y')] \oplus 0 = 0
\end{aligned} \tag{2}$$

by (3.3). But (1) and (2) are exactly (0.6)(i)(ii) for a. ∎

3.5 THEOREM. *A Jordan pair generated by a finite collection of absolute zero divisors is nilpotent.*

PROOF: If V is such pair, (3.4) implies that the Lie algebra TKK(V), which has the same generating system as V, is generated by a finite collection of sandwiches. Then TKK(V) is locally nilpotent by (2.3).

If V is generated by absolute zero divisors, [6, 4.6] applies to show that each element in V is a finite sum of absolute zero divisors of V. For such a pair V, any Jordan monomial in the sense of [1, 1.1] (one can think of Jordan triple monomials applied to $T(V)$) can be expressed in terms of monomials in brackets $\{\ ,\ ,\ \}$ of the same degree: given $a \in V^{\sigma}$, $b \in V^{-\sigma}$, we can write a as a finite sum $a = \sum_i a_i$, where the a_i's are absolute zero divisors of V; thus

$$Q_a b = \sum_i Q_{a_i} b + \sum_{i<j} \{a_i, b, a_j\} = \sum_{i<j} \{a_i, b, a_j\}$$

because $Q_{a_i} = 0$, for all i. By (3.2)(ii), such Jordan monomials in brackets $\{\ ,\ ,\ \}$ are Lie monomials of the same degree evaluated in TKK(V), hence the nilpotency of TKK(V) implies the nilpotency of V. ∎

3.6 COROLLARY. *A Jordan algebra or triple system generated by a finite collection of absolute zero divisors is nilpotent.*

PROOF: If J is a triple system generated by finite collection of absolute zero divisors, then the double pair $V(J)$ is also generated by finite collection of absolute zero divisors, hence $V(J)$ is nilpotent by (3.5), which readily implies that J is nilpotent.

If J is a Jordan algebra generated by the collection $A = \{a_1, \ldots, a_n\}$ of absolute zero divisors, then the underlying triple system of J is generated (as a triple system) by the finite set $\tilde{A} = A \cup \{a^2, a \circ b \mid a, b \in A\}$ [1, 1.4], which also consists of absolute zero divisors of the algebra or the triple J by (3.1). Thus J is nilpotent as a triple system, i.e., there is a natural number N such that any Jordan triple monomial of degree at least N evaluated on J vanishes. In particular any Jordan triple monomial of degree at least N evaluated on $\tilde{A}$ vanishes, which implies by [1, 1.9] that any Jordan algebra monomial of degree at least $2N$ evaluated in A vanishes. Since J is generated by A as a Jordan algebra, this implies that any Jordan algebra monomial of degree at least $2N$ evaluated in J vanishes, i.e., J is nilpotent ■

We can rephrase the above results (3.5) (3.6) as follows.

3.7 Corollary. *A Jordan system (algebra, triple system or pair) generated by a set of absolute zero divisors is locally nilpotent.* ■

3.8 We recall that the *McCrimmon radical* (also called *small radical* in [6, 4.5]) $\mathrm{Mc}(J)$ of a Jordan system J is the smallest ideal of J which produces a nondegenerate quotient. It can be obtained by a transfinite induction process as follows: Let $M_1(J)$ be the span of absolute zero divisors of J, which is an ideal of J by [6, 4.6]. Once we have the ideals $M_\alpha(J)$ for all ordinals $\alpha < \beta$, we define $M_\beta(J)$ by

(i) $M_\beta(J)/M_{\beta-1}(J) = M_1(J/M_{\beta-1}(J))$ when β is not a limit ordinal,

(ii) $M_\beta(J) = \cup_{\alpha<\beta} M_\alpha(J)$ when β is a limit ordinal.

Then $\mathrm{Mc}(J) = \lim_\alpha M_\alpha(J)$, so that for any Jordan system J, $\mathrm{Mc}(J) = M_\alpha(J)$ for some ordinal α (such that $M_1(J/M_\alpha(J)) = 0$, i.e., $J/M_\alpha(J)$ is nondegenerate).

3.9 Corollary. *For any Jordan system J, the McCrimmon radical* $\mathrm{Mc}(J)$ *is locally nilpotent.*

Proof: Taking into account (3.8), we just need to prove that $M_\alpha(J)$ is locally nilpotent for any ordinal α, which we will do by transfinite induction (on α):

(1) $M_1(J)$ is locally nilpotent by (3.7).

(2) Let us assume that $M_\alpha(J)$ is locally nilpotent for any ordinal $\alpha < \beta$, $\beta > 1$.

(a) If β is a limit ordinal, then $M_\beta(J) = \cup_{\alpha<\beta} M_\alpha(J)$, and any finite set S of elements of $M_\beta(J)$ is contained in $M_\alpha(J)$ for some $\alpha < \beta$, hence the subsystem generated by S is nilpotent by local nilpotency of $M_\alpha(J)$.

(b) If β is not a limit ordinal, then $M_\beta(J)/M_{\beta-1}(J) = M_1(J/M_{\beta-1}(J))$ is locally nilpotent by (1), and $M_{\beta-1}(J)$ is locally nilpotent since $\beta - 1 < \beta$. Hence $M_\beta(J)$ is locally nilpotent too. ■

3.10 Corollary. *Simple Jordan systems are always nondegenerate.*

Proof: Let J be a simple Jordan system. If J is degenerate, then $\mathrm{Mc}(J) \neq 0$, hence $J = \mathrm{Mc}(J)$ by simplicity, which implies that J is locally nilpotent by (3.9). But then, we can apply [1, 2.3] to obtain $J = 0$, which is a contradiction. ■

3.11 Remark: Some authors consider a stronger definition of an absolute zero divisor z of a Jordan algebra J. They require $z^2 = 0$ besides $U_z J = 0$. Obviously, those "strong" absolute zero divisors are absolute zero divisors in our sense (0.5), so that (3.6) and (3.7) remain valid. It is very easy to check that the absence of nonzero "strong" absolute zero divisors is equivalent to the absence of nonzero absolute zero divisors (0.5), so that there is only one notion of nondegeneracy for Jordan algebras, hence only one McCrimmon radical.

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