Radical Ideals

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American Journal of Mathematics, Vol. 65, No. 4. (Oct., 1943), pp. 537-568.

RADICAL IDEALS.*

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The radical has been introduced into the theory of rings with the intent of constructing a two-sided nilideal modulo which there do not exist nilpotent right-ideals different from 0. Thus it seems justified to define as a radical ideal every ideal meeting these requirements. Every ring possesses at least one radical ideal; and both the cross-cut and the sum of all the radical ideals are themselves radical ideals which may be called the lower and the upper radical respectively. It is possible that the upper and lower radicals are different and that neither of them is nilpotent. If all the radical ideals are equal, then we say that the radical exists; and this happens e. g., if every right-ideal, not 0, in the quotient ring modulo the lower radical contains a smallest right-ideal different from 0. If this latter condition is satisfied by every quotient ring of the ring under consideration, then a finite or transfinite power of the radical is 0.

Several applications of the theory of radical ideals are given: we prove a criterion for the existence of the identity element in rings that need not satisfy the minimum condition for right- (or left-) ideals; and we deduce the double chain condition for right-ideals from properties considerably weaker than the minimum condition.

It should be noted that we have restricted our attention throughout to the consideration of right-ideals.

1. Existence of the upper and lower radicals. The element x in the ring R is a *nilelement,* if $x^i = 0$ for some exponent i; and the right-ideal J in R may be termed a *nilideal,* if every element in J is a nilelement. The right-ideal J is *nilpotent,* if $J^i = 0$ for some exponent i. It is clear that nilpotent right-ideals are nilideals, though the converse need not be true.[1] It is well known [2] that the sum of a finite number of nilpotent right-ideals is a nilpotent right-ideal; and it is obvious that xJ is a nilpotent right-ideal, whenever J is a nilpotent right-ideal and x an element in R. From these two facts one readily deduces the following well known statement.

* Received May 15, 1942. Presented to the American Mathematical Society. April 16, 1942.

[1] For an example cf. Köthe (1), p. 165 (the numbers in parentheses refer to the bibliography at the end of the paper).

[2] Cf. e. g. v. d. Waerden (1), p. 154.

LEMMA 1. 1. *The sum $N = N(R)$ of all the nilpotent right-ideals in R is a two-sided nilideal in R.*

Note that the ideal $N(R)$ need not be nilpotent, as follows from an example due to Köthe.[3]

The ideal P in the ring R shall be termed *a radical ideal*,[4] if

(1. a) P is a two-sided ideal;

(1. b) P is a nilideal;

(1. c) the quotient-ring R/P does not contain nilpotent right-ideals different from 0.

There exist ideals in R which meet the requirements (1. a) and (1. b), e. g. the null-ideal. Thus we may form the sum[5] $U = U(R)$ of all the ideals P in R satisfying conditions (1. a) and (1. b). This ideal U is clearly a two-sided ideal in R and shall be called *the upper radical of R*.

There exist ideals in R which meet the requirements (1. a) and (1. c), e. g. the ideal $P = R$. Thus we may form the cross-cut $L = L(R)$ of all the ideals P in R which satisfy conditions (1. a) and (1. c). This ideal L is obviously a two-sided ideal in R and shall be termed *the lower radical of R*.

It is an immediate consequence of our definitions that every radical ideal is situated between the upper and the lower radical. But the main justification for our terminology may be seen in the following fact.

THEOREM 1. 2. *The upper and the lower radical of the ring R are radical ideals in R.*

Proof. It has been shown by Köthe[6] that the sum of all the two-sided nilideals in R is a nilideal. Denote by W the uniquely determined two-sided ideal in R which satisfies: $U \leq W$ and $W/U = N(R/U)$. There exists, by Lemma 1. 1, to every element x in W a positive integer i such that x^i is an element in the nilideal U; and hence every element in W is a nilelement. Now it follows from the definition of the upper radical that $W = U$, $N(R/U) = 0$; and thus we have shown that the upper radical of R is a radical ideal in R.

$L \leq U$, since we have just shown that the upper radical U is a radical ideal, and since the lower radical L is certainly part of every radical ideal in R. Consequently L is a nilideal, since U is a nilideal. Every nilpotent right-

[3] Cf. Köthe (1), p. 165.

[4] The existence of radical ideals will be assured by Theorem 1. 2 below.

[5] This ideal has been considered by Fitting (1), p. 21.

[6] Köthe (1), p. 170.

ideal in R/L has the form S/L for S a suitable right-ideal between L and R; and from the nilpotence of S/L we infer the existence of a positive integer i such that $S^i \leq L$. Suppose now that condition (1. c) is satisfied by the two-sided ideal T in R. Then $L \leq T$ and $((T+S)/T)^i = 0$, since $(T+S)^i \leq T + S^i \leq T + L = T$; and it follows from (1. c) that $(T+S)/T = 0$ or $T + S = T$ or $S \leq T$. Thus we have shown that S is part of every ideal T, satisfying (1. a) and (1. c); and hence $S \leq L$, proving[7] that L is a radical ideal in R.

Remarks. 1. The following construction of the lower radical may be of some interest (in particular in case the transfinite induction involved in it happens to stop after a finite number of steps):

(i) $Q(0) = 0$.

(ii) Suppose that the two-sided ideal $Q(u)$ has been defined for every $u < v$.

Case 1. If $v = w + 1$ is not a limit-ordinal, then $Q(v)$ is the uniquely determined ideal in R which contains $Q(w)$ and which satisfies: $Q(v)/Q(w) = N(R/Q(w))$; that $Q(v)$ is a two-sided ideal in R, is an immediate consequence of Lemma 1. 1.

Case 2. If v is a limit-ordinal, then denote by $Q(v)$ the join of all the ideals $Q(u)$ for $u < v$. It is readily verified that $Q(v)$ is a two-sided ideal in R.

(iii) Since the $Q(v)$ form an ascending chain of ideals in R, there exists a (smallest) ordinal z such that $Q(z) = Q(z+1)$; and we put $Q(z) = Q$.

It is clear from our construction of Q that Q is a two-sided ideal in R and that R/Q does not contain nilpotent right-ideals different from 0. Hence it follows from the definition of the lower radical L of R that $L \leq Q$.

Suppose next that the ideal T in R meets the requirements (1. a) and (1. c). It is clear that $Q(0) \leq T$; and thus we may assume that every $Q(u)$ for $u < v$ is part of T. Then it is readily verified that $Q(v)$ is also part of T; and thus it follows by complete induction that Q is part of T. But this shows that $Q \leq L$; and thus we have shown that the ideal Q just constructed is the lower radical L of R.

2.[8] If T is a two-sided ideal between L and U, then the upper radical of R/T is U/T; and if T is a radical ideal, then the lower radical of R/T is 0.

[7] The author is indebted to the referee for this proof, which is much simpler than the author's original proof, of the fact that L is a radical ideal.

[8] This remark is due to the referee.

3. It has been pointed out before that every radical ideal is situated between the upper and lower radical; and it is a consequence of Theorem 1.2 that every subideal of the upper radical is a nilideal. It may happen that there exist two-sided ideals between the upper and the lower radical which are not radical ideals; an example for this phenomenon will be constructed in section **2**.

4. It is a consequence of Theorem 1.2 that *the upper and lower radicals are equal, if U/L is a nilpotent ideal in R/L.* In general, however, the upper and lower radicals may be different, as will be seen from the example constructed in section **2**.

If the upper and the lower radical are equal, then this ideal may be termed *the radical $K = K(R)$ of the ring R*; and we say then that *the radical of R exists.* It should be noted that the radical need not be nilpotent, as may be seen from an example due to Köthe.[9]

2. Existence of different radical ideals. In this section we construct a ring with the following properties:

(i) Every element in the ring is a nilelement so that the ring is its own upper radical U.

(ii) The ring does not contain nilpotent right-ideals different from 0 so that its lower radical is 0.

(iii) The ring contains a two-sided ideal which is not a radical ideal.

Denote by G an abelian group which is the direct sum of a countably infinite number of infinite cyclic groups; and denote by $b(0), b(1), b(-1), \cdots, b(i), b(-i), \cdots$ a basis of G. Then there exists one and only one endomorphism (= homomorphism) $u(i)$, for $i = 1, 2, \cdots$, of G such that

$$b(j)u(i) = \begin{cases} 0, & \text{if } j \equiv 0 \bmod 2^i \\ b(j-1), & \text{if } j \not\equiv 0 \bmod 2^i. \end{cases}$$

The following statement may be easily verified by complete induction:

$$(*) \quad b(j)u(i_0) \cdots u(i_m) = \begin{cases} b(j-m-1), \text{ if } j \not\equiv n \bmod 2^{i_n} \text{ for } 0 \leq n \leq m \\ 0, \text{ if } j \equiv n \bmod 2^{i_n} \text{ for at least one } n \text{ with } 0 \leq n \leq m. \end{cases}$$

Denote by U the ring of endomorphisms of G which is generated by the endomorphisms $u(1), u(2), \cdots$.

Every element $x \neq 0$ in U has the form: $x = \sum_{i=1}^{k} x_i$ where $x_i = \pm u((i, 0))$

[9] Köthe (1), p. 165.

$\cdots u((i, m_i))$, $0 < (i, j)$, $0 \leq m_i$. If h is the maximum of all the numbers $2^{(i,n)}$, then it is an immediate consequence of (*) that $b(j)x_{i_1} \cdots x_{i_h} = 0$ for every j, since the product of endomorphisms contains at least h factors $u(s)$ with $2^s \leq h$. Consequently this product is 0 and this implies that $x^h = 0$. The ring U has therefore property (i).

For a proof of property (ii) we need a closer analysis of the structure of the endomorphisms in U. If x is an endomorphism different from 0 in U, then $x = \sum_{i=0}^{m} y(i)$, $y(i) = \sum_{j=1}^{k(i)} e(i, j)y(i, j)$, $e(i, j) = \pm 1$, $y(i, j) = u((i, j; 0)) \cdots u((i, j; i))$, $0 < (i, j; n)$, $0 \leq k(i)$, and $y(n) \neq 0$. Denote by r some integer satisfying: $m + 1 < 2^r$ and $(i, j; n) \leq r$ for $i = 0, \cdots, m$; $j = 1, \cdots, k(i)$; $n = 0, \cdots, i$.

If h is some preassigned positive integer, then denote by s an integer satisfying $2^{r+1}h \leq 2^s$. We proceed to prove that $(xu(s)^{2^r-m-1})^h \neq 0$; a fact that implies the impossibility of nilpotence of right-ideals different from 0.

It will be convenient to put $z(i, j) = y(i, j)u(s)^{2^r-m-1}$.

Since $y(m) \neq 0$, there exists an integer t such that $b(t)y(m) \neq 0$. Hence we may assume that the positive integer k has been determined in such a way that

$$b(t)y(m, j) = \begin{cases} b(t - m - 1) & \text{for } j = 1, \cdots, k \\ 0 & \text{for } k < j. \end{cases}$$

Then $b(t)y(m) = \sum_{j=1}^{k} e(m, j)b(t - m - 1) \neq 0$; and this shows that $\sum_{j=1}^{k} e(m, j) \neq 0$, a fact we shall have to use later on.

If $0 \leq i_v \leq m$, $1 \leq j_v \leq k(i_v)$ for $1 \leq v \leq h$, and if e is 0 or 1, then it follows from (*) that $b(t + eh2^r)z(i_1, j_1) \cdots z(i_h, j_h)$ is either 0 or $b(t + eh2^r - \sum_{v=1}^{h} (i_v + 1 + 2^r - m - 1)) = b(t + (e - 1)h2^r + mh - \sum_{v=1}^{h} i_v)$ and this latter element is equal to $b(t + (e - 1)h2^r)$ if, and only if, $i_1 = \cdots = i_h = m$.

If $k < j_v$ for some v, then $b(t)y(m, j_v) = 0$; and thus there follows from (*) the existence of an integer n such that $0 \leq n \leq m$ and $t \equiv n \bmod 2^{(m, j_v; n)}$. But then we find that $t + eh2^r \equiv t \equiv n \equiv n + (v - 1)2^r \bmod 2^{(m, j_v; n)}$ and hence we may deduce from (*) that $b(t + eh2^r)z(m, j_1) \cdots z(m, j_h) = 0$. The results obtained in the last two paragraphs may be stated as follows:

If $b(t + eh2^r)z(i_1, j_1) \cdots z(i_h, j_h) = b(t + (e - 1)h2^r)$, then $i_1 = \cdots = i_h = m$ and $1 \leq j_v \leq k$.

Assume now the existence of integers j_v, j'_v such that $1 \leq j_v, j'_v \leq k$ and

$b(t)z(m, j_1)\cdots z(m, j_h) = 0$, $b(t+h2^r)z(m, j'_1)\cdots z(m, j'_h) = 0$. Then we infer from (*) the existence of integers n, v satisfying: $1 \leq v \leq h$ and

$$t \equiv \begin{cases} n + (v-1)2^r \bmod 2^{(m, j_v; n)} & \text{for } 0 \leq n \leq m \\ n + (v-1)2^r \bmod 2^s & \text{for } m < n < 2^r. \end{cases}$$

If $0 \leq n \leq m$, then $t \equiv n \bmod 2^{(m, j_v; n)}$; and we could deduce from (*) that $b(t)y(m, j_v) = 0$ which is impossible, since $j_v \leq k$. Thus there exist integers n, v such that $m < n < 2^r$, $1 \leq v \leq h$ and $t \equiv n + (v-1)2^r \bmod 2^s$; and likewise there exist integers n', v' such that $m < n' < 2^r$, $1 \leq v' \leq h$ and $t + h2^r \equiv n' + (v'-1)2^r \bmod 2^s$. Since $r < s$, it follows from these two congruences that $n \equiv n' \bmod 2^r$; and since $m < n$, $n' < 2^r$, we deduce that $n = n'$. Sonsequently $(v - 1 + h)2^r \equiv (v'-1)2^r \bmod 2^s$ or $h + v - v' \equiv 0 \bmod 2^{s-r}$. Since $1 \leq v$, $v' \leq h$, we have $0 < h + v - v' < 2h \leq 2^{s-r}$, a contradiction. Thus we have shown that there exists an integer t' such that

$$b(t')z(i_1, j_1)\cdots z(i_h, j_h) = b(t' - h2^r)$$

if, and only if, $i_1 = \cdots = i_h = m$ and $1 \leq j_v \leq k$ for $v = 1, \cdots, h$.

Now it is readily verified that

$$b(t')(xu(s)^{2^r-m-1})^h = \left(\sum_{j=1}^{k} e(m, j)\right)^h b(t' - h2^r) + \sum_{i \neq t' - h2^r} c(i)b(i) \neq 0,$$

since the factor of $b(t' - h2^r)$ is, by a previous remark, different from 0.

This completes the proof of the fact that (ii) is satisfied by the ring U.

To prove (iii) let us consider the two-sided ideal T in U which is generated by $u(2), u(3), \cdots$. Every element in T is a linear combination of products $u(i_0)\cdots u(i_m)$ with the restriction that none of these products is a power of $u(1)$.

We note that $b(j)u(1) = \begin{cases} 0 \text{ for even } j \\ b(j-1) \text{ for odd } j \end{cases}$. Hence $u(1)^2 = 0$; and it is readily verified that $U^2 \leq T$. Thus all we have to show is that $u(1)$ is not an element in T.

Every element $x \neq 0$ in T has the form: $x = \sum_{i=2}^{k} c(i)u(i) + x'$ for x' in U^2 so that $b(t)x' = \sum_{j \neq t-1} d(j)b(j)$. If $x = u(1)$, then it follows from $b(1)u(i) = b(0)$ that $\sum_{i=2}^{k} c(i) = 1$; and it follows from $b(2)u(i) = \begin{cases} 0 \text{ for } i = 1 \\ b(1) \text{ for } i \neq 1 \end{cases}$ that $\sum_{i=2}^{k} c(i) = 0$. This contradiction shows that $u(1)$ is not in T so that $U^2 \leq T < U$, as was required.

If we adjoin to the ring U of endomorphisms of the abelian group G the identity-endomorphism, we obtain a ring R whose upper radical is U, whose lower radical is 0 and which contains the two-sided nilideal T such that R/T contains nilpotent right-ideals different from 0.

3. Characterization of the upper radical. The right-ideal J in the ring R is termed *a minimal right-ideal,* if $0 < J$, and if there does not exist a right-ideal J' such that $0 < J' < J$.

THEOREM 3.1. *If T is a radical ideal in the ring R, and if every right-ideal different from 0 in the quotient ring R/T contains a minimal right-ideal, then T is the upper radical U of R and the upper radical of R contains every nilideal*[10] *in R.*

Proof. If the nilideal J in R is not part of T, then there exists an ideal V between T and $T+J$ such that V/T is a minimal right-ideal in R/T. Since V/T is part of $(T+J)/T$, and since J is a nilideal, V/T is a nilideal. A minimal right-ideal is either nilpotent or idempotent; and since T is a radical ideal in R, it follows that V/T is idempotent. But it is well known[11] that idempotent minimal right-ideals contain idempotent elements not 0. Thus the nilideal V/T contains an idempotent element not 0, a contradiction. Hence it follows that T contains every nilideal in R. Since T is a radical ideal, it is a two-sided nilideal and therefore part of the upper radical. Since the upper radical is the sum of all the two-sided nilideals, it follows from what we have shown just now that U is part of T, i. e. $U = T$ contains every nilideal.

The right-ideal J is termed *a maximal right-ideal in the ring R,* if $J < R$, and if there does not exist a right-ideal J' such that $J < J' < R$. The element e in R is a *left-identity element,* if $ex = x$ for every element x in the ring R.

LEMMA 3.2. *If the ring R contains a left-identity element e, then each nilideal is part of every maximal right-ideal.*

Proof. If the nilideal V were not part of the maximal right-ideal J, then $R = V + J$ and there would exist elements v and j in V and J respectively such that $e = v + j$. By complete induction we may show the existence of elements $j(i)$ in J such that $e = v^i + j(i)$ since

$$e = e^2 = (j+v)v^i + ej(i) = v^{i+1} + jv^i + j(i) = v^{i+1} + j(i+1).$$

[10] This shows that under the hypotheses of Theorem 3.1 the upper radical meets all the requirements, concerning right-ideals, imposed upon the radical by Köthe (1), p. 169.

[11] Cf. e. g. v. d. Waerden (1), p. 157, Hilfssatz 3.

But v is a nilelement; and hence it follows that e is in J and that therefore $J = R$, an impossibility which proves our contention.

THEOREM 3.3.[12] *If the ring R contains a left-identity element, and if every right-ideal different from 0 in the quotient ring R/U contains a minimal right-ideal, then the upper radical U of R is the cross-cut of all the maximal right-ideals in R.*

Remark. The example in section **2** shows the impossibility of omitting the hypothesis concerning R/U; and Theorem 9.6 below shows the need for assuming the existence of a left-identity element in R.

Proof. If the intersection J of all the maximal right-ideals in R/U were different from 0, then J would contain a minimal right-ideal J'; and consequently we could infer[13] the existence of an idempotent $e \neq 0$ in J'. It is well known that R/U is the direct sum of the right-ideals J' and Z where J' consists of all the elements $v = ev$ and where Z consists of the elements z satisfying $ez = 0$. Thus the crosscut of Z and J' is 0. Since J' is a minimal right-ideal in R/U, Z is a maximal right-ideal in R/U. Hence $J' \leq J \leq Z$, a contradiction showing that $J = 0$. Thus we have proved that U is the intersection of all the maximal right-ideals in R which contain U; and it follows from Lemma 3.2 that U is part of every maximal right-ideal in R, since U is, by Theorem 1.2, a nilideal.

4. The anti-radical. We state the following well known fact[14] for future reference.

LEMMA 4.1. *If N is a sum of (a finite or infinite number of) minimal right-ideals in the ring R, then every right-ideal contained in N is a direct summand of N and is itself a direct sum of minimal right-ideals in R.*

The sum[15] $M = M(R)$ of all the minimal right-ideals in the ring R shall be called *the anti-radical of R*; and we put $M(R) = 0$, if there are no minimal right-ideals in R. This definition may be justified by the fact that the upper radical is, under not too narrow assumptions, just the cross-cut of

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[12] The author proved this theorem originally, using a stronger hypothesis. He is indebted to the referee for supplying him with a proof for the theorem in its present form.

[13] Cf. e.g. v. d. Waerden (1), p. 157, Hilfssatz 3.

[14] Cf. e.g. MacLane (1), p. 458, Theorems 3 and 7.

[15] This ideal has been investigated by Hopkins (1) under the hypothesis that the minimum condition be satisfied by the right-ideals in the ring.

all the maximal right-ideals in R (see Theorem 3. 3); and by the fact which we shall prove immediately that radical ideals and the anti-radical annihilate each other.

If J is a minimal right-ideal in the ring R, x an element in R, then xJ is either 0 or a minimal right-ideal in R; and this shows that *the anti-radical is a two-sided ideal.*

THEOREM 4. 2. *If M is the anti-radical of the ring R and J a nilideal in R, then $MJ = 0$.*

Proof. If b is an element in the minimal right-ideal B, j an element in the nilideal J, and if bj were different from 0, then bj would be an element not 0 in the minimal right-ideal B so that B would be the smallest right-ideal containing bj. Consequently there exists an element r in R such that[16] $b = bjr \pm bj \pm \cdots \pm bj = b(jr \pm j \pm \cdots \pm j) = bj'$ where j' is an element in J, since j is an element in the right-ideal J. There exists an integer i such that $j'^i = 0$ and this leads to the contradiction: $0 \neq bj = bj'j = \cdots = bj'^ij = 0$. We have shown, therefore, that $BJ = 0$ for every minimal right-ideal B in R; and this fact clearly implies $MJ = 0$.

The following condition will be imposed frequently upon the rings under consideration.

(4. A) *If x is an element in the ring R, then x is contained in the right-ideal xR.*

This requirement is met e. g. by all the rings which contain a right-identity element; and (4. A) is satisfied by the ring R if, and only if, $J = JR$ for every right-ideal J in R.

THEOREM 4. 3. *If condition (4. A) is satisfied by the ring R, if T is a two-sided ideal in R, and if R/T is a sum of minimal right-ideals, then the anti-radical M contains every element x, satisfying: $xT = 0$.*

Proof. If $xT = 0$ is satisfied by the element x in R, and if the right-ideal J in R contains T and is a minimal right-ideal modulo T, then either $xJ = 0$ or xJ is a minimal right-ideal in R. The right-ideal xR is consequently a sum of minimal right-ideals in R, since R/T is a sum of minimal right-ideals in R/T; and as a sum of minimal right-ideals xR is part of the sum M of all the minimal right-ideals in R. Applying condition (4. A) to $xR \leq M$ we find finally that x is in M whenever $xT = 0$.

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[16] This more complicated form for the most general element in the right-ideal generated by bj is due to the fact that no identity element need exist in R.

Remark. The impossibility of omitting condition (4. A) in Theorem 4. 3 may be seen from the following example: Denote by R the ring of all the multiples of the prime number p considered modulo p^3 and denote by T the ideal of all the multiples of p^2 in R. It is obvious that T is a two-sided ideal in R such that R/T is a minimal ideal and such that $RT = 0$, though R is not a sum of minimal ideals.

The following statement is an immediate consequence of Theorems 1. 2, 4. 2 and 4. 3.

THEOREM 4. 4. (a) $MU = 0$.

(b) *If condition* (4. A) *is satisfied by the ring R, and if R/U is a sum of minimal right-ideals, then the anti-radical is exactly the set of all the the elements x which satisfy:* $xU = 0$.

Remark. The impossibility of omitting the second hypothesis in (b) may be seen from the example of the ring of all the integers whose radical and anti-radical are both 0.

The following lemma will prove useful later on.

LEMMA 4. 5. *If the right-ideal J in the ring R is contained in the anti-radical M of R, then $J^2 = J^3$.*

Proof. It is a consequence of Lemma 4. 1 that J is a sum of minimal right-ideals. If Z is a minimal right-ideal in R, then Z^2 is a subideal of Z and consequently either $Z^2 = Z$ or $Z^2 = 0$. If Z is a minimal right ideal contained in J, then $Z^2 = Z$ implies that $Z \leq J^2$, and $Z^2 = 0$ implies that Z is part of the cross-cut C of J and the upper radical U. From these facts we deduce that $J = J^2 + C$. From Theorem 4. 2 we infer that $JC \leq MU = 0$. Consequently $J^2 = J(J^2 + C) = J^3 + JC = J^3$.

M. Hall has shown [17] that every algebra may be decomposed in one and only one way into the sum of a semi-simple and a "bound" algebra. The following concepts will be needed for an extension of his theorem.

We denote by $A = A(R)$ the set of all the elements x in R which satisfy: $xU = Ux = 0$; and we denote by $B = B(R)$ the set of all the elements b in R which satisfy: $bA^2 = A^2b = 0$. It is obvious that A and B are two-sided ideals in R, and that $U \leq B$.

THEOREM 4. 6. *Suppose that condition* (4. A) *is satisfied by the ring R.*

(a) *If R is the direct sum of the two-sided ideals S and T, if $U \leq T$, and if the cross-cut of A and T is part of U, then $S = A^2$ and $T = B$.*

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[17] Hall (2), Theorem 2. 2.

(b) *If R/U is a sum of minimal right-ideals, then the cross-cut of A^2 and B is 0; and the cross-cut of A and B is part of U.*

(c) *If R/U is a sum of minimal right-ideals, and if the cross-cut of A^2 and the anti-radical M of R is the sum of a finite number of minimal right-ideals,*[18] *then R is the direct sum of A^2 and B.*

Proof. If the two-sided ideals S and T meet the requirements of (a), then $ST = TS = 0$ and consequently $S \leq A$. If D is the cross-cut of A and T, then A is the direct sum of S and D, and D is part of U. Thus $D^2 \leq AU = 0$ and $A^2 = S^2 + D^2 = S$; and this implies $B = T$.

If R/U is the sum of minimal right-ideals, then it follows from Theorem 4. 4, (b) that A is part of the anti-radical M of R; and hence it follows from Lemma 4. 5 that $A^2 = A^3$. If W is the cross-cut of A^2 and B, then we deduce from Lemma 4. 1 the existence of a right-ideal V such that A^2 is the direct sum of W and V. Clearly $A^2W = WA^2 = 0$ and hence we have $A^2 = A^4 = (V + W)^2 = V^2$, proving that $W = 0$. The cross-cut of A and B is part of M and is therefore the sum of minimal right-ideals. If Z is a minimal right-ideal, then either $Z^2 = Z$ or $Z^2 = 0$; and if $Z = Z^2$, then Z is part of A^2 and is therefore not contained in B. If $Z^2 = 0$, then Z is nilpotent and is therefore part of U; and this completes the proof of (b).

If the requirements of (c) are met by the ring R, then A^2 is the sum of a finite number of minimal right-ideals; and since the cross-cut of A^2 and B is 0, it follows from $U \leq B$ that the minimal right-ideals contained in A^2 are idempotent. Now we may deduce by the customary arguments[19] the existence of an idempotent e such that $A^2 = eR$. Denote by E the set of elements x in A^2 such that $xe = 0$. Clearly E is a left-ideal and the two-sided ideal ER satisfies: $(ER)^2 = ERER \leq E^2R \leq EA^2 = EeR = 0$, showing that $ER \leq U$. Applying (4. A) we infer now that $E \leq ER \leq U$; and since the cross-cut of A^2 and U is part of the cross-cut 0 of A^2 and B, it follows that $E = 0$; and from this fact we deduce that $A^2 = eR = eA^2 = Re = A^2e$; and this shows that the element b in R belongs to B if, and only if, $be = eb = 0$. If r is any element in R, then $e(r - ere)$ and $(r - ere)e$ belong both to $eR = Re$ and satisfy therefore: $e(r - ere) = e(r - ere)e = 0$ and $(r - ere)e = e(r - ere)e = 0$. Consequently ere belongs to $A^2 = eRe$ and $r - ere$ belongs to B, i. e. R is the direct sum of A^2 and B.

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[18] It is not known to the author whether or not this hypothesis is needed for the validity of the proposition (c).

[19] E. g. v. d. Waerden (1), pp. 156-158.

5. The anti-radical series. An ascending chain of two-sided ideals $M_v = M_v(R)$ in the ring R may be defined by complete (transfinite) induction as follows:

(i) M_1 is the anti-radical M of R.

(ii) M_{v+1} is the uniquely determined two-sided ideal in R which contains M_v and which satisfies: $M(R/M_v) = M_{v+1}/M_v$.

(iii) If v is a limit-ordinal, then M_v is the join (and therefore the sum) of all the ideals M_u for $u < v$.

(iv) There exists a smallest ordinal $m = m(R)$ such that $M_m = M_{m+1}$.

THEOREM 5.1. *$R = M_m$ for some (finite or infinite) ordinal m if, and only if, the following condition is satisfied by every quotient-ring of R:*

(5.B) *Every right-ideal different from* 0 *contains a minimal right-ideal.*

Proof. If condition (5.B) is satisfied by every quotient ring of R, and if $M_v < R$, then R/M_v contains a minimal right-ideal so that $M_v < M_{v+1}$. Since $M_m = M_{m+1}$ for some ordinal m, this shows the sufficiency of the condition.

If $R = M_m$ for some ordinal m, and if T is a two-sided ideal different from R, then it is easy to construct a well-ordered ascending chain of right-ideals $J(v)$ with the following properties:

(a) $J(0) = T$.

(b) $J(v) < J(v+1)$ and there does not exist a right-ideal J satisfying: $J(v) < J < J(v+1)$.

(c) If v is a limit-ordinal, then $J(v)$ is the join of all the right-ideals $J(u)$ for $u < v$.

(d) $J(w) = R$ for some (finite or infinite) ordinal w.

If J is some right-ideal in R such that $T < J$, then there exists a smallest ordinal z such that the cross-cut of J and $J(z)$ is different from T, since the cross-cut J of J and $J(w) = R$ is different from T. It is an obvious consequence of (c) that z cannot be a limit-ordinal; and we may readily verify that the cross-cut V of J and $J(z)$ is a minimal right-ideal modulo T, as was to be shown.

COROLLARY 5.2. *If $R = M_m$ for some ordinal m, then the radical K of the ring R exists.*

Proof. If $R = M_m$, then it follows from Theorem 5.1 that (5.B) is satisfied by R/L. Hence it follows from Theorems 1.2 and 3.1 that the lower radical L and the upper radical U of the ring R are equal, i.e. $U = L$ is the radical K of R.

We note that under the hypotheses of Corollary 5.2 every nilideal is contained in the radical K of R.

It should be mentioned finally that (5.B) is satisfied by every quotient ring of R, whenever the minimum condition is satisfied by the right-ideals in R.

6. The powers of the radical. If J is a right-ideal in the ring R, then the (finite and transfinite) powers J^v of J are defined by transfinite induction as follows:

(i) $J^1 = J$.

(ii) $J^{v+1} = JJ^v$.

(iii) If v is a limit-ordinal, then J^v is the cross-cut of all the J^u for $u < v$.

(iv) There exists a (smallest) ordinal $s = s(J)$ such that $J^s = J^{s+1}$.

It is clear that the powers of a right-ideal are right-ideals; and that the powers of a two-sided ideal are two-sided ideals.

If J is a right-ideal in R, and if $u \leq v$, then $J^v \leq J^u$.

Proof. We prove this contention by complete induction with regard to v. It is certainly true for $v = 1$; and thus we may assume that $J^w \leq J^u$ for $u \leq w < v$.

Case 1. v is a limit-ordinal.

Then J^v is the cross-cut of all the J^u for $u < v$ so that $u < v$ implies $J^v \leq J^u$.

Case 2. $v = w + 1$ for w a limit-ordinal.

Then $J^v = JJ^w \leq JJ^u = J^{u+1}$ for every $u \leq w$. Thus J^v is part of the cross-cut of all the J^{v+1} for $u < w$; and this cross-cut is just J^w, since w is a limit-ordinal.

Case 3. $v = z + 2$ for some ordinal z.

Then $J^v = JJ^{z+1} \leq JJ^z = J^{z+1} \leq J^w$ for $w \leq z + 1 < v$; and this completes the proof.

THEOREM 6.1. *If J is a nilideal, then $M_v J^v = 0$ for every ordinal v.*

Proof. It is a consequence of Theorem 4. 2 that $M_1 J^1 = MJ = 0$. Thus we may assume that our assertion holds for every $u < v$.

Case 1. $v = w + 1$ is not a limit-ordinal.

Then M_v/M_w is the anti-radical of R/M_w and $(J + M_w)/M_w$ is a nil-ideal in the quotient ring R/M_w. Hence it follows from Theorem 4. 2 that their product is 0; and from this fact we deduce $M_v J \leq M_w$. From the induction-hypothesis we infer $M_w J^w = 0$. Consequently we find that $M_v J^v = M_v J J^w \leq M_w J^w = 0$.

Case 2. v is a limit-ordinal.

If x is any element in M_v, then we deduce from the definition of M_v as the join of the M_u with $u < v$ the existence of an ordinal $u < v$ such that x is an element in M_u. It is a consequence of the definition of J^v that $J^v \leq J^u$; and hence it follows from the induction-hypothesis that $xJ^v \leq M_u J^u = 0$; and thus we have shown that $M_v J^v = 0$.

COROLLARY 6. 2. *If $R = M_m$ for some ordinal m, then $K^{m+1} = 0$.*

Remark. The existence of the radical K of R is assured by Corollary 5. 2.

Proof. It is an immediate consequence of Theorem 6. 1 that

$$K^{m+1} = KK^m \leq RK^m = M_m K^m = 0.$$

Note that we could infer $K^m = 0$ from $R = M_m$, if 0 were the only element x in R satisfying $Rx = 0$.

That $K^k = 0$ need not imply $R = M_m$, may be seen from the example of the ring of all the integers where $M_v = K^v = 0$ for every v.

THEOREM 6. 3. *If $R = M_m$ for some ordinal m, then each of the following properties of the right-ideal J in R implies all the others:*

(i) *J is a nilideal.*

(ii) *J is a part of the radical K of R (whose existence is assured by Corollary* 5. 2).

(iii) *$J^k = 0$ for some ordinal k.*

Proof. It is a consequence of Theorems 5. 1 and 3. 1 that every nilideal is part of the radical. If the right-ideal J is part of the radical K of R, then we deduce from Corollary 6. 2 that $J^{m+1} \leq K^{m+1} = 0$. If finally $J^k = 0$ for some ordinal k, then we denote by J_v the cross-cut of J and of M_v, and we put $J_0 = 0$. We proceed to prove by complete (transfinite) induction that every J_v is a nilideal, a fact that is clearly true for $v = 0$. Hence we may assume that J_u is a nilideal for every $u < v$.

Case 1. $v = w + 1$ is not a limit-ordinal.

The right-ideal $J^* = (M_w + J_v)/M_w$ is part of the anti-radical M_v/M_w of the ring R/M_w; it is, therefore, the sum of minimal right ideals in R/M_w. If Z^* is a minimal right-ideal in R/M_w, then either $Z^{*2} = 0$ or $Z^* = Z^{*2}$. If the idempotent minimal right-ideal Z^* were part of J^*, then there would exist an element e in J_v which is not contained in J_w, though $e - e^2$ is an element in J_w, since every idempotent minimal right-ideal contains an idempotent not 0.[20] It is a consequence of the induction-hypothesis that J_w is a nilideal and that therefore $e - e^2$ is a nilelement. Consequently we are able to deduce from a theorem of G. Köthe [21] the existence of an idempotent $j \neq 0$ in $J_v \leq J$, a fact that is clearly incompatible with $J^k = 0$. Thus we have shown that J^* is the sum of minimal right-ideals whose squares are 0; and from this result it is easily deduced that $J^{*2} = 0$. But $J_v{}^2 \leq M_w$ is an immediate consequence of $J^{*2} = 0$. Since M_w is a nilideal, the square of every element in J_v is a nilelement, i. e. J_v is a nilideal.

Case 2. v is a limit-ordinal.

Then every element in J_v is contained in some J_u for $u < v$; and it follows from the induction-hypothesis that every element in J_v is a nilelement, i. e. that J_v is a nilideal.

This shows that J is a nilideal, since $J = J_m$ is a consequence of $R = M_m$.

Remark. If J is an ideal, neither 0 nor 1, in the ring of natural integers, then $J^\omega = 0$, though J is not a nilideal; and this shows that the hypothesis $R = M_m$ cannot be omitted in Theorem 6. 3.

THEOREM 6. 4. *If condition* (4. A) *is satisfied by the ring R, and if R/U is a sum of minimal right-ideals, then M_i is, for every positive integer i, exactly the set of all the elements x in R which satisfy: $xU^i = 0$.*

Proof. The validity of our contention for $i = 1$ is an immediate consequence of Theorem 4. 4, (b). Thus we may assume that M_{i-1} is exactly the set of all the elements x in R which satisfy: $xU^{i-1} = 0$. The ideal M_i/M_{i-1} is the anti-radical of the ring R/M_{i-1}; and $(U + M_{i-1})/M_{i-1}$ is, by Theorem 1. 2, a two-sided nilideal in R/M_{i-1} modulo which this ring is a sum of

[20] E. g. v. d. Waerden (1), p. 157, Hilfssatz 3.
[21] Köthe (1), p. 168, Hilfssatz 3.

minimal right-ideals. Hence it follows from Theorems 4.2 and 4.3 that M_i is exactly the set of all the elements x in R which satisfy: $xU \leq M_{i-1}$. But it is a consequence of the induction-hypothesis that xU is part of M_{i-1} if, and only if, $0 = xUU^{i-1} = xU^i$; and x is therefore an element in M_i if, and only if, $xU^i = 0$, as was to be shown.

THEOREM 6.5. *If the right-ideal J in R is part of M_i for i a positive integer, then $J^{2^i} = J^{2^i+1}$.*

Proof. It is a consequence of Lemma 4.5 that our theorem is true for $i = 1$. Thus we may assume the validity of the theorem for subideals of M_i and we have to derive it from this induction hypothesis for the subideals of M_{i+1}.

Assume now that the right-ideal J in R is part of M_{i+1}. The right-ideal $(M_i + J)/M_i$ in R/M_i is part of the anti-radical $M(R/M_i) = M_{i+1}/M_i$ of the quotient ring R/M_i; and hence we deduce from Lemma 4.5 that $M_i + J^2 = M_i + J^3$. The cross-cut C of M_i and J^2 is a subideal of M_i and consequently we may infer from the induction hypothesis that $C^{2^i} = C^{2^i+1}$; and the above equation may be restated as $J^2 = C + J^3$, since $J^3 \leq J^2$ and since we may apply the modular (Dedekind's) law. Expanding $(C + J^3)^{2^i-1}$ we obtain

$$J^{2^{i+1}-2} = (C + J^3)^{2^i-1} = C^{2^i-1} + \sum_{j=1}^{2^i-1} V_j$$

where every summand V_j is the sum of products of $2^i - 1$ ideals of which j are equal to J^3 and the remaining $2^i - 1 - j$ factors are equal to C. Since $C \leq J^2$, it follows that $V_j \leq J^{3j+2(2^i-1-j)} = J^{2^{i+1}+j-2} \leq J^{2^{i+1}-1}$ or

$$CV_j \leq J^{2+2^{i+1}-1} = J^{2^{i+1}+1}.$$

Consequently we find that

$$\begin{aligned} J^{2^{i+1}} = J^2 J^{2^{i+1}-2} &= (C + J^3) J^{2^{i+1}-2} = C(C^{2^i-1} + \sum_{j=1}^{2^i-1} V_j) + J^{2^{i+1}+1} \\ &= C^{2^i} + J^{2^{i+1}+1} = C^{2^i+1} + J^{2^{i+1}+1} \\ &= J^{2^{i+1}+1}, \end{aligned}$$

since $C^{2^i+1} \leq J^{2^{i+1}+2} \leq J^{2^{i+1}+1}$; and this completes the proof.

7. Nilpotent ideals. It is an immediate consequence of the definition of radical ideals (cf. condition (1.c)!) that they contain every nilpotent right-ideal. On the other hand it has been pointed out that radical ideals need not be nilpotent. Thus we shall give in this section several criteria for a nilideal, in particular a radical ideal, to be nilpotent.

THEOREM 7.1. *Every nilideal contained in M_i for i a positive integer is nilpotent.*

Proof. It is a consequence of Theorem 6.1 that a nilideal J contained in M_i satisfies: $J^{i+1} = JJ^i \leq M_i J^i = 0$, and J is therefore nilpotent.

THEOREM 7.2. *If the maximum-condition*[22] *is satisfied by the nilpotent right-ideals in the ring R, then the lower radical of R is nilpotent.*

Remark. That the hypothesis of this theorem is not sufficient for proving that the upper radical is nilpotent, may be seen from the example in section **2**. It seems to be an open question whether or not the maximum condition for right-ideals is sufficient for nilpotence of the upper radical.

Proof. There exists a greatest nilpotent right-ideal G in R. If J is a nilpotent right-ideal in R, then $G + J$ is a nilpotent right-ideal so that J is part of G. Consequently G is the sum of all the nilpotent right-ideals in R. It is a consequence of Lemma 1.1 that G is a two-sided ideal in R. There cannot exist nilpotent right-ideals different from 0 in R/G, since $Z^i \leq G$ and the nilpotence of G imply the nilpotence of Z. This shows that the nilpotent ideal G is a radical ideal; and it is readily verified that G is the lower radical of the ring R.

THEOREM 7.3. *If $R = M_m$ for some ordinal m, and if at least one of the two chains M_v and K^v is finite, then the radical K of R is nilpotent.*

Remark. The existence of the radical K is a consequence of Corollary 5.2.

Proof. If $R = M_m$ for some finite ordinal m, then the nilpotence of the radical is a consequence of Theorem 7.1, since the radical is by Theorem 1.2 a nilideal.

If $R = M_m$ for some ordinal m, then it follows from Corollary 6.2 that $K^{m+1} = 0$. If the chain K^v is finite, then there exists a finite ordinal i such that $K^i = K^{i+1} = \cdots = K^{m+1} = 0$.

The next theorem is a partial converse of Theorem 7.3.

THEOREM 7.4. *If condition* (4.A) *is satisfied by the ring R, if the radical K of R exists and is nilpotent, and if R/K is a sum of minimal right-ideals, then $R = M_i$ for some positive integer i.*

Proof. There exists a positive integer i such that $K^i = 0$; and it is a

[22] It states that every not vacuous class of nilpotent right-ideals contains at least one greatest ideal.

consequence of Theorem 6.4 that M_i is exactly the set of all the elements x in R which satisfy: $xK^i = 0$. Since $RK^i = R0 = 0$, it follows that $M_i = R$.

The third condition in Theorem 7.4 cannot be omitted, since in the ring of all the integers radical and anti-radical are both 0.

COROLLARY 7.5. *Suppose that condition* (4. A) *is satisfied by the ring R, that R/U is a sum of minimal right-ideals, and that* 0 *is the only element x in R that satisfies $Rx = 0$. Then*

$$U^i = 0 \text{ if, and only if, } M_i = R \text{ (for } i \text{ a positive integer).}$$

Proof. That $U^i = 0$ implies $M_i = R$, is an immediate consequence of Theorem 6.4; and that $M_i = R$ implies $U^i = 0$, may be inferred from Theorem 6.1, since $0 = M_i U^i = RU^i$, and since $S = 0$ is a consequence of $RS = 0$.

8. Maximum and minimum conditions.[23] In the results of the previous sections there occurred hypotheses that are connected in various ways with maximum and minimum conditions. In this section we investigate the relations between these properties.

THEOREM 8.1. *Suppose that the maximum condition is satisfied by the right-ideals in the ring R. Then the minimum condition is satisfied by the right-ideals in R if (and only if) condition* (5. B) *is satisfied by every quotient-ring of R.*

Proof. If condition (5. B) is satisfied by every quotient ring R, then there exists, by Theorem 5.1, an ordinal m such that $R = M_m$. Since the ideals M_v form an ascending chain, and since the maximum condition is satisfied by the right-ideals in R, there exists a positive integer i such that $M_i = M_{i+1}$. Since $R = M_m$ for some ordinal m, it follows that $R = M_i$ for i a finite ordinal. It is a consequence of the definition of the series M_v that M_{v+1}/M_v is a sum of minimal right-ideals in R/M_v; and we infer from the maximum condition for right-ideals that M_{v+1}/M_v is a sum of a finite number of minimal right-ideals. Consequently there exists a finite composition series [24] of right-ideals in R; and it is well known that the minimum condition for right-ideals is a consequence of this fact.

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[23] The maximum (minimum) condition is satisfied by the right-ideals in the ring R, if in every not vacuous set of right-ideals there exists a right-ideal which is not smaller (greater) than any other right-ideal in the set.

[24] A composition series is a densest finite ascending chain of right-ideals.

If there exists in the ring R an infinite independent set [25] of right-ideals, then there exists in R a countably infinite, independent set of right-ideals $J(i) \neq 0$ for $i = 1, 2, \cdots$. The chain of right-ideals $\sum_{j < i} J(j)$ is an infinite ascending chain of right-ideals; and the chain $\sum_{i < j} J(j)$ is an infinite descending chain of right-ideals. Thus the maximum as well as the minimum condition for right-ideals implies the following property of rings R·

(8. C) *There does not exist an infinite independent set of right-ideals.*

If (8. C) is satisfied by the ring R, then it follows from Lemma 4. 1 that the anti-radical M of R is the (direct) sum of a finite number of minimal right-ideals. The following statement is a partial converse of this fact.

THEOREM 8. 2. *If condition* (5. B) *is satisfied by the ring R, and if the anti-radical M is a sum of a finite number of minimal right-ideals in R, then* (8. C) *is satisfied by R.*

Proof. Suppose that S is an independent set of right-ideals different from 0 in R. If J is a right-ideal in the set S, then J contains a minimal right-ideal J'. The set S' of these minimal right-ideals J' is independent and contains as many elements as S. Since there does not exist an infinite independent set of minimal right-ideals in R, it follows that S is a finite set and that therefore (8. C) is satisfied by R.

LEMMA 8. 3. *If conditions* (5. B) *and* (8. C) *are satisfied by the ring R, and if R does not contain nilpotent right-ideals different from* 0, *then R is the sum of a finite number of minimal right-ideals.*

Remark. This is a generalization of the so-called "Fundamental Theorem on Semi-simple Rings." [26]

Proof. Suppose that we have constructed idempotents $e_1, \cdots, e_n$ meeting the following requirements:

(a) e_iR is a minimal right-ideal.

(b) $e_ie_j = 0$ for $i \neq j$.

This is certainly possible for $n = 0$.

We note that the right-ideals e_iR form an independent set of right-ideals.

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[25] The set S of ideals is said to be *independent,* if the cross-cut of any ideal J in S with the sum of the other ideals in S is 0.

[26] Cf. e. g. v. d. Waerden (1), p. 156.

Let $W(n)$ be the right-ideal of all the elements x satisfying: $\sum_{i=1}^{n} e_i x = 0$ ($W(0) = R$). Then it is readily seen that $R = W(n) + \sum_{i=1}^{n} e_i R$; and we have effected the proof of our lemma, if $W(n) = 0$. Should $W(n)$ be different from 0, then it contains a minimal right-ideal J; and J contains, by a well known theorem,[27] an idempotent $e \neq 0$, since $J = J^2$. Put $e_{n+1} = e - e \sum_{i=1}^{n} e_i$. It is readily verified that e_{n+1} is an idempotent, satisfying $e_i e_{n+1} = e_{n+1} e_i = 0$ for $i = 1, \cdots, n$, and that $J = e_{n+1} R$; and thus the idempotents $e_1, \cdots, e_n, e_{n+1}$ meet the requirements (a) and (b).

If this construction would never stop, i. e. if the right-ideal $W(n)$ were different from 0 for every positive n, then we would be led to the infinite independent set of minimal right-ideals $e_1 R, \cdots, e_i R, \cdots$, contradicting condition (8. C). This completes the proof.

THEOREM 8. 4. *If condition* (4. A) *is satisfied by the ring* R, *then each of the following properties implies all the others:*

(1) *The maximum and the minimum condition are satisfied by the right-ideals in* R.

(2) *The minimum condition is satisfied by the right-ideals in* R.

(3) *Conditions* (5. B) *and* (8. C) *are satisfied by every quotient-ring of the ring* R; *and the descending chain of the powers of the radical*[28] K *of* R *is finite.*

Proof. It is obvious that (1) implies (2); and that (3) is a consequence of (2) is a consequence of facts we mentioned when introducing condition (8. C).

If (3) is satisfied by the ring R, then it follows from Theorem 5. 1 that $R = M_m$ for some ordinal m; and from Lemma 8. 3 that R/K is the sum of a finite number of right-ideals. There exists a positive integer i such that $K^i = K^{i+1}$; and it is a consequence of Theorem 6. 4 that M_n is, for finite n, exactly the set of all the elements x, satisfying $xK^n = 0$. This shows the equality of $M_i = M_{i+1}$. Hence we have $M_i = M_{i+1} = \cdots = M_m = R$. Since there do not exist infinite independent sets of right-ideals in R/M_n, it follows from Lemma 3. 3 that M_{n+1}/M_n is the sum of a finite number of minimal right-ideals. Consequently there exists a finite composition series of right-ideals in R, a fact which is equivalent to our property (1).

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[27] Cf. e. g. v. d. Waerden (1), p. 157.

[28] The existence of the radical K is assured by Theorem 5. 1 and Corollary 5. 2.

Remarks. 1. In proving that (1) is a consequence of (3) we did not use condition (3) in its entirety. That condition (5. B) is satisfied by every quotient ring of R, and that the powers of the radical form a finite chain, are hypotheses indispensable for the above proof. But it is not necessary to assume that (8. C) is satisfied by every quotient ring of R. It would have been sufficient to make sure that the quotient rings R/M_i for finite i and R/K meet the requirement (8. C); and for the latter assumption we could have substituted the weaker hypothesis that R/K is a sum of minimal right-ideals.

2. It has been shown elsewhere [29] that a ring with minimum-condition for right-ideals possesses a right-identity element if, and only if, it satisfies condition (4. A); and Ch. Hopkins [30] has shown that the maximum condition for right-ideals is a consequence of the minimum condition for right-ideals, provided there exists a right-identity element. It should be noted, however, that condition (4. A) is indispensable for the validity of this Theorem 8. 4, as may be seen from the following example: denote by R any infinite abelian group without elements of infinite order which contains only a finite number of elements of order a prime; such a group is the direct sum of a finite abelian group and of a finite number of groups of type [31] p^∞. If we define $xy = 0$ for every pair of elements x and y in R, then R is a commutative ring, satisfying $0 = R^2$. The ideals in R are just the subgroups of the additive group R; and thus it becomes apparent that the minimum condition is satisfied by the ideals in R, but not the maximum condition.

9. Existence of the identity. The following statement is basic for the considerations of this section.

LEMMA 9. 1. *If conditions* (5. B) *and* (8. C) *are satisfied by* [32] R/K, *then there exists an idempotent e in R such that* $x \equiv ex \equiv xe$ modulo K *for every element x in R.*

Proof. R/K is a ring without nilpotent right-ideals different from 0, by Theorem 1. 2; and R/K is, by Lemma 8. 3, the sum of a finite number of minimal right-ideals. Hence there exists, by a well known theorem,[33] an identity element in R/K, i. e. there exists an element f in R such that

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[29] Baer (1), Corollary to Theorem 6.

[30] Hopkins (1), p. 726, Theorem 6. 4.

[31] The groups of type p^∞ have been discovered by H. Prüfer; they are generated by a countable number of elements g_i subject to the relations: g_0 is an element of order p; $g_{i-1} = g_i p$.

[32] The existence of the radical K is assured by Theorem 5. 1 and Corollary 5. 2.

[33] Cf. e. g. v. d. Waerden (1), p. 156.

$x \equiv xf \equiv fx$ modulo K for every x in R. If f is any element meeting this requirement, then $f^2 - f = f'$ is an element in K; and we deduce from Theorem 1.2 the existence of a positive integer $n = n(f)$ such that $f'^n = 0$. Thus there exists among the elements which represent an identity element modulo K one, say e, with minimal $n(e)$. It is readily verified that e is an idempotent (which clearly meets all our requirements), since otherwise[34] $e_1 = e - 2ee' + e'$ [for $e' = e^2 - e$] would be an element which represents the identity element modulo K, though $n(e_1)$ were smaller than $n(e)$.

THEOREM 9.2. *If $R = M_m$ for some (finite or infinite) ordinal m, and if condition (8.C) is satisfied by the ring*[35] *R/K, then the property that every element x in R is contained in Rx is a necessary and sufficient condition for the existence of a left-identity-element in R.*

Proof. If e is a left-identity-element in R, then $x = ex$ belongs to Rx, showing the necessity of the condition. We assume now that the condition: "x belongs to Rx" is satisfied by the ring R. It is a consequence of Theorem 5.1 that condition (5.B) is satisfied by every quotient ring of R; and hence we may deduce from Lemma 9.1 the existence of an idempotent e such that $x \equiv ex \equiv xe$ modulo K for every element x in R. We denote by W the set of all the elements x in R such that $ex = 0$. Since $x \equiv ex$ modulo K, it follows that W is part of K. Finally we have $R = Re + K$.

We proceed to prove by complete (transfinite) induction that $W \leq K^v$ for every v. This is certainly true for $v = 1$; and thus we may assume that $W \leq K^u$ for every $u < v$.

Case 1. $v = w + 1$ is not a limit-ordinal.

Then we deduce from the hypothesis that x is an element in Rx and from the induction hypothesis the inequality:

$$W \leq RW = ReW + KW = KW \leq KK^w = K^v.$$

Case 2. v is a limit-ordinal.

Then K^v is the cross-cut of all the K^u for $u < v$; and $W \leq K^v$ is an immediate inference from the induction-hypothesis.

It is a consequence of $R = M_m$ and Corollary 6.2 that $K^{m+1} = 0$. Thus W, as a part of K^{m+1}, is 0.

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[34] This construction is due to Köthe (1), p. 169 and Dickson (1), p. 123.

[35] The existence of the radical K is assured by Theorem 5.1 and Corollary 5.2.

Since $x - ex$ belongs to W for every x in R, it follows that $x = ex$ for every x in R, i. e. that e is a left-identity-element in R.

Remark. The condition: $R = M_m$ was only needed in the proof to assure that condition (5. B) is satisfied by R/K and that $K^{m+1} = 0$; and these two apparently weaker conditions may be substituted for $R = M_m$.

The *identity* is an element e in R that satisfies $ex = xe = x$.

COROLLARY 9. 3. *If $R = M_m$ for some (finite or infinite) ordinal m, and if condition* (8. C) *is satisfied by the ring*[35] *R/K, then the following two conditions are necessary and sufficient for the existence of the identity* 1 *in the ring R:*

(i) *$xR = 0$ implies $x = 0$; and*

(ii) *Rx contains x.*

Proof. The necessity of the conditions (i) and (ii) is obvious. If these conditions are satisfied by the ring R, then we deduce from Theorem 9. 2 the existence of a left-identity-element e in R; and e is the identity 1 in R, since

$$(x - xe)R = (x - xe)eR = (xe - xe)R = 0 \quad \text{implies} \quad x = xe \quad \text{by} \quad \text{(i)}.$$

THEOREM 9. 4. *If $R = M_m$ for some (finite or infinite) ordinal m, and if condition* (8. C) *is satisfied by the ring*[36] *R/K, then condition* (4. A) *is a necessary and sufficient condition for the existence of a right-identity-element in R.*

Proof. If e is a right-identity-element in R, then $x = xe$ is contained in the right-ideal xR, showing the necessity of condition (4. A). If condition (4. A) is satisfied by the ring R, then we deduce from Theorem 5. 1 the validity of condition (5. B) in every quotient ring of R; and hence we may infer from Lemma 9. 1 the existence of an idempotent e satisfying $x \equiv ex \equiv xe \bmod K$ for every element x in R. Clearly $R = eR + K$.

It will be convenient to put $0 = M_0$. Then we prove by complete (transfinite) induction that $M_v = M_v e$ and that 0 is the only element x in M_v satisfying $xe = 0$. This fact is patently true for $v = 0$ and thus we assume it to be true for every $u < v$.

Case 1. $v = w + 1$ is not a limit-ordinal.

Then $M_w = M_w e$ and 0 is the only element x in M_w such that $xe = 0$.

[36] The existence of the radical K is assured by Theorem 5. 1 and Corollary 5. 2.

If y is an element in M_v such that $ye = 0$, then $yR = yeR + yK = yK$; and since M_v/M_w is the anti-radical of the ring R/M_w, it follows from Theorem 4.4, (a) that $yR = yK \leq M_w$. But it is a consequence of (4. A) that y is an element in yR and therefore in M_w; and now we deduce from the induction-hypothesis that $y = 0$. Consequently 0 is the only element x in M_v such that $xe = 0$; and $M_x = M_x e$ is an obvious consequence of this fact.

Case 2. v is a limit-ordinal.

If x is an element in M_v, then there exists an ordinal $u < v$ such that x is an element in M_u. Hence it follows from the induction-hypothesis that $x = xe$ showing that $M_v = M_v e$ and that therefore 0 is the only element z in M_v such that $ze = 0$.

Now it is evident that $R = M_m = M_m e = Re$ and that consequently e is a right-identity-element for R.

By essentially the same arguments as the ones used in the proof of Corollary 9.3 we deduce the following statement from Theorem 9.4.

COROLLARY 9.5. *If $R = M_m$ for some (finite or infinite) ordinal m, and if condition* (8. C) *is satisfied by the ring*[36] *R/K, then the following two conditions are necessary and sufficient for the existence of the identity* 1 *in the ring R:*

(i) *$Rx = 0$ implies $x = 0$; and*

(ii) *xR contains x.*

It has been assumed in Theorems 9.2 and 9.4 and in Corollaries 9.3 and 9.5 that condition (8. C) is satisfied by the ring R/K. The impossibility of omitting this hypothesis may be seen from the following example: Denote by F a field, by G an abelian group which admits the elements in F as operators and whose rank over F is infinite, and by R the ring of all the (proper and improper) automorphisms of the group G over F which map G upon a subgroup of finite rank. It is readily seen that R contains neither a left-identity-element nor a right-identity-element, that $R = M$ and that $K = 0$. If r is any element in R, then denote by G_r the set of all the elements in G which are mapped upon 0 by r. Clearly G^r and G/G_r are isomorphic groups of finite rank over R; and hence there exist idempotents e, f in R such that every element in G^r is left invariant by e, G_r is mapped upon 0 by f and every coset of G/G_r is mapped by f upon an element in itself, since both G^r and G_r are direct summands of G. Clearly $r = re = fr$ showing that every subset S of R is contained in both SR and RS.

THEOREM 9. 6. *If conditions* (5. B) *and* (8. C) *are satisfied by*[36] R/K, *then the following pair of properties is a necessary and sufficient condition for the existence of a left-identity-element in* R:

(a) *Every right-ideal different from* R *is contained in a maximal right-ideal in* R.

(b)[37] *The radical* K *of* R *is the cross-cut of all the maximal right-ideals in* R.

Proof. Suppose that there exists a left-identity-element e in R. If the right-ideal J in R is different from R, then there exists a greatest right-ideal G in R which contains J, but which does not contain e. Clearly G is a maximal right-ideal in R, since $R = eR$. It is a consequence of Theorem 1. 2 and of Lemma 8. 3 that R/K is a sum of minimal right-ideals; and hence it may be inferred from Theorem 3. 4 that K is the cross-cut of all the maximal right-ideals in R.

Suppose now, conversely, that the conditions (a) and (b) are satisfied by the ring R. There exists by Lemma 9. 1 an idempotent e in R such that $x \equiv ex \equiv xe$ modulo K for every element x in R. Since the elements v satisfying $ev = 0$ are certainly contained in K, it follows that $R = eR + K$. If the right-ideal eR in R were different from R, then there would exist a maximal right-ideal G in R which contains eR. It is a consequence of (b) that $K \leq G$; and thus we are led to the contradiction: $R = eR + K \leq G < R$. Hence $R = eR$ and e is a left-identity-element in R.

Remarks. 1. If R is the ring of all the even rational integers, then the radical of R is 0; and it is readily seen that conditions (a) and (b) are satisfied by R. Condition (8. C) is satisfied too; but R does not contain an identity. This shows the impossibility of omitting the hypothesis that condition (5. B) be satisfied by R/K.

2. Suppose that the abelian group R is the direct sum of a group K of type 2^∞ and of a cyclic group of order 2 which is generated by an element e. In R we define a commutative multiplication by the rules: $uv = 0$, if at least one of the factors u and v is in K; and $e = e^2$. It is readily seen that K is the only maximal ideal in R, and that K is the radical in R. Thus conditions (5. B) and (8. C) are satisfied by R/K and condition (b) is satisfied by R. But there does not exist an identity element in R; and this shows the impossibility of omitting condition (a) in Theorem 9. 6.

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[37] Considering this condition (b) and Theorem 4. 4 it may be shown that Theorem 9. 6 is a generalization of a theorem due to M. Hall; cf. Hall (1), p. 362, Theorem 3. 2.

10. The quasi-regular elements. S. Perlis [38] has recently discovered a characterization of the radical of an algebra which is rather different from the usual criteria. We devote this section to a generalization of his theory.

An element x in the ring R is termed *quasi-regular*,[39] if there exists an element y in R satisfying: $x + y + xy = 0$. If the ring R possesses an identity element 1, then a necessary and sufficient condition for quasi-regularity of the element x in R is the existence of a solution z of the equation: $(1 + x)z = 1$.

The *right-ideal* J in the ring R shall be termed *quasi-regular*, if every element in J is quasi-regular.

THEOREM 10. 1. *If z is an element in the quasi-regular right-ideal J, then there exists one and only one solution $x = z'$ of the equation $z + x + zx = 0$; and such a solution satisfies $zz' = z'z$.*

Proof. If z is an element in the quasi-regular right-ideal J, then there exists an element y in R such that $z + y + zy = 0$. If y is some solution of this equation, then $y = -(z + zy)$ is an element in J, since z is an element in the right-ideal J. Since J is quasi-regular, so is y; and hence there exists an element w in R such that $y + w + yw = 0$. Consequently we find that

$$\begin{aligned} z &= z + y + w + yw + z(y + w + yw) \\ &= z + y + zy + (y + z + zy)w + w = w; \end{aligned}$$

and this shows that $0 = y + w + yw = y + z + yz$; and we deduce $zy = yz$ from $y + z + zy = 0$.

Suppose now that $x = y$ and $x = y'$ are solutions of the equation: $z + x + zx = 0$. From what we have shown in the previous paragraph of the proof, it follows that $0 = z + y + yz$; and thus we find that

$$\begin{aligned} y' &= y' + z + y + yz + (z + y + yz)y' \\ &= z + y' + zy' + y(z + y' + zy') + y = y; \end{aligned}$$

and this completes the proof.

We denote by $S = S(R)$ *the sum of all the quasi-regular right-ideals in* R.

THEOREM 10. 2. *$S(R)$ is a quasi-regular right-ideal in the ring R.*

Proof. It is readily seen that it suffices to prove the following statement: If u is contained in a quasi-regular right-ideal, and if v is contained in a quasi-regular right-ideal, then $u + v$ is an element of some quasi-regular right-ideal.

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[38] Perlis (1).

[39] Perlis (1), p. 129.

We note that (in the absence of identity elements) the right-ideal generated by the element w consists of all the elements of the form: $wi + wr$ for r in R and i an ordinary rational integer (i need not be an element in R). Thus we have to show that $(u+v)i + (u+v)r$ is quasi-regular for every r in R and for every rational integer i, whenever both u and v are contained in quasi-regular right-ideals.

Since $ur + ui$ belongs to the right-ideal generated by u, it is quasi-regular; and hence there exists an element s such that

$$(ur + ui) + s + (ur + ui)s = 0;$$

and since $v(r + rs + ri) + vi$ belongs to the right-ideal generated by v, it is quasi-regular too; and this assures the existence of an element t satisfying:

$$v(r + rs + si) + vi + t + (v(r + rs + si) + vi)t = 0.$$

Hence

$$\begin{aligned}((u+v)r + (u+v)i) + (s + t + st) + ((u+v)r + (u+v)i)(s + t + st) \\ = ur + ui + s + (ur + ui)s + (ur + ui + s + (ur + ui)s)t \\ + v(r + rs + si) + vi + t + (v(r + rs + si) + vi)t = 0;\end{aligned}$$

and this proves that $(u+v)r + (u+v)i$ is quasi-regular for every r in R and for every natural integer i; i. e. $u + v$ is contained in a quasi-regular right-ideal in R, as was to be shown.

THEOREM 10. 3. *If the right-ideal J in R consists of elements that are quasi-regular modulo*[40] *$S(R)$, then $J \leq S(R)$.*

Proof. If the element x in R is quasi-regular modulo $S(R)$, then there exists an element y such that $x + y + xy$ is in $S(R)$; and it follows from Theorem 10. 2 that $x + y + xy$ is quasi-regular. Consequently there exists an element z in R such that

$$0 = x + y + xy + z + (x + y + xy)z = x + (y + z + yz) + x(y + z + yz);$$

and x is therefore a quasi-regular element. The right-ideal J is thus quasi-regular, if each of its elements is quasi-regular modulo $S(R)$.

COROLLARY 10. 4. *If the right-ideal J in R is a nilideal modulo*[41] *$S(R)$, then $J \leq S(R)$.*

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[40] I. e. to every element j in J there exists an element h in R such that $j + h + jh$ belongs to $S(R)$.

[41] I. e. to every element j in J there exists a positive integer n such that j^n belongs to $S(R)$.

Proof. If the element w is a nilelement modulo $S(R)$, then there exists a positive integer n such that w^{2n+1} is an element in $S(R)$. Since

$$w+\sum_{i=1}^{2n}(-1)^i w^i + w\sum_{i=1}^{2n}(-1)^i w^i = w^{2n+1},$$

it follows that w is quasi-regular modulo $S(R)$; and this shows that our contention is an immediate consequence of Theorem 10. 3.

LEMMA 10. 5. *If the element e in $S(R)$ is an idempotent modulo the two-sided ideal T in R, then e belongs to T.*

Proof. If e is an element in $S(R)$, then $-e$ is quasi-regular. Hence there exists an element f in R such that $-e+f-ef=0$. If e is furthermore an idempotent modulo the two-sided ideal T, then

$$e\equiv e^2\equiv e(f-ef)\equiv ef-e^2f\equiv ef-ef\equiv 0 \bmod T,$$

i. e. e belongs to T.

THEOREM 10. 6. *If the two-sided ideal T in R is part of $S(R)$, if every right-ideal, not 0, in R/T contains a minimal right-ideal, and if 0 is the only nilpotent right-ideal in R/T, then $T=S(R)$.*

Proof. If T were different from $S(R)$, then there would exist a right-ideal J between T and $S(R)$ such that J/T is a minimal right-ideal in R/T. Since 0 is the only nilpotent right-ideal in R/T, it follows that $J/T=(J/T)^2$; and hence we may deduce from known theorems [42] that J/T contains an idempotent different from 0. But it follows from Lemma 10. 5 that this is impossible; and this shows that $T=S(R)$.

COROLLARY 10. 7. *If every right-ideal, not 0, in R/U contains a minimal right-ideal, then $S(R)$ is the upper radical U of R.*[43]

Proof. It is a consequence of Theorem 1. 2 that U is a nilideal and that R/U does not contain nilpotent right-ideals different from 0; it is a consequence of Corollary 10. 4 that U is part of $S(R)$; and hence we may deduce from Theorem 10. 6 that $U=S(R)$.

We denote by $S^*=S^*(R)$ *the sum of all the quasi-regular two-sided ideals in R.* Clearly S^* is a two-sided ideal which is part of $S(R)$; and hence it follows from Theorem 10. 2 that $S^*(R)$ is quasi-regular. It is a consequence of Theorem 10. 3 that $S^*(R/S^*(R))=0$.

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[42] Cf. e. g. v. d. Waerden (1), p. 157, Hilfssatz 3.

[43] This is readily seen to be an extension of Perlis' (1) Theorems 1 and 2.

COROLLARY 10. 8. *If every right-ideal, not 0, in $R/S^*(R)$ contains a minimal right-ideal, then $S^*(R) = S(R)$.*

Proof. Denote by H the two-sided ideal which contains $S^*(R)$ and which satisfies $H/S^* = N(R/S^*)$ (= sum of all the nilpotent right-ideals in R/S^*). That H is a two-sided ideal, is a consequence of Lemma 1. 1; and it follows from Lemma 1. 1 that H is a nilideal modulo S^*. Hence we deduce from Corollary 10. 4 that H is part of $S(R)$; and we infer from Theorem 10. 2 that the two-sided ideal H is quasi-regular. Hence $H = S^*$ and 0 is the only nilpotent right-ideal in R/S^*. Applying Theorem 10. 6 we see that $S^* = S$.

THEOREM 10. 9. *If the ring R possesses an identity element 1, then $S(R)$ is part of every maximal right-ideal in R.*

Proof. If $S(R)$ were not contained in the maximal right-ideal G in R, then $R = G + S$ and there exist elements g and s in G and S respectively such that $1 = g + s$. It is a consequence of Theorem 10. 2 that $-s$ is quasi-regular and that therefore $-s + t - st = 0$ for some t in R. Hence $1 + t = (g + s)(1 + t) = g + s + gt + st$ or $1 = g + gt$ so that 1 is an element in the right-ideal $G < R$, a contradiction.

COROLLARY 10. 10. *If the ring R possesses an identity element 1, and if R/S^* is a sum of minimal right-ideals, then $S(R) = S^*(R)$ is the cross-cut of all the maximal right-ideals in R.*

This is an immediate consequence of Lemma 3. 3, Corollary 10. 8 and Theorem 10.9.

Remark. The upper radical U of the ring R is by Theorem 1. 2 a two-sided nilideal; and hence it is a consequence of Theorem 10. 2 and of Corollary 10. 4 that $U \leq S^*(R)$. That the ideals U and S^* need not be equal may be seen from the following *example*: R is the ring of all the (formal) power series $\sum_{i=0}^{\infty} c_i t^i$ in one indeterminate t with coefficients c_i from some commutative field. An element in R possesses an inverse in R if, and only if, "its absolute term" $c_0 \neq 0$; and an element in R is quasi-regular if, and only if, $1 + c_0 \neq 0$. This shows that $S^*(R) = S(R) = tR$ whereas $U(R) = 0$.

THEOREM 10. 11. *If J is a quasi-regular right-ideal in the ring R, then $M_v J^v = 0$ for every ordinal v.*

Remark. It is a consequence of Corollary 10. 4 that Theorem 6. 1 is a special case of this Theorem 10. 11.

Proof. A. If Z is a minimal right-ideal in R, and if z is any element in Z, then zJ is a right-ideal in R which is part of Z. Thus zJ is either 0 or Z. If zJ were equal to Z, then there would exist an element w in J such that $zw = z$. Since w is an element in the quasi-regular right-ideal J, it follows that $-w$ is quasi-regular; and hence there exists an element v in R such that $w = v - wv$. Consequently

$$z = zw = z(v - wv) = zv - zv = 0,$$

an impossibility. Thus we have shown that $zJ = 0$ for every z in Z; and this shows that $ZJ = 0$ for every minimal right-ideal Z in R. Since the anti-radical $M = M(R)$ is the sum of all the minimal right-ideals in R, it follows that

$MJ = 0$ *for every quasi-regular right-ideal* J *in* R.

B. We proceed to prove the theorem by complete induction with regard to v. That the theorem holds true for $v = 1$, has been shown under A.; and thus we may assume that our assertion is valid for every $u < v$.

Case 1. $v = w + 1$ is not a limit-ordinal.

Then M_v/M_w is the anti-radical of R/M_w and $(J + M_w)/M_w$ is a quasi-regular right-ideal in R/M_w. Hence it follows from A. that their product is 0; and from this fact we deduce that $M_vJ \leq M_w$. From the induction-hypothesis we infer $M_wJ^w = 0$; and thus we find that

$$M_vJ^v = M_vJJ^w \leq M_wJ^w = 0.$$

Case 2. v is a limit-ordinal.

If x is any element in M_v, then we deduce from the definition of M_v as the join of the M_u with $u < v$ the existence of an ordinal $d < v$ such that x is an element in M_d. It is a consequence of the definition of J^v that it is part of J^d; and hence it follows from the induction-hypothesis that $xJ^v \leq M_dJ^d = 0$; and thus we have shown that $M_vJ^v = 0$.

11. Rings admitting operators. The ring R is said to admit the elements in the system $\boldsymbol{V}$ as operators, if to every element r in R and to every element $\boldsymbol{v}$ in $\boldsymbol{V}$ there exists a uniquely determined element $r\boldsymbol{v}$ in R meeting the following requirements:

(i) $(r \pm s)\boldsymbol{v} = r\boldsymbol{v} \pm s\boldsymbol{v}$;
(ii) $(rs)\boldsymbol{v} = r(s\boldsymbol{v}) = (r\boldsymbol{v})s$.

It is readily verified that $J\boldsymbol{v}$ is a right-(left-) ideal in R whenever J is a right-(left-) ideal in R; that $r^i = 0$ implies $(r\boldsymbol{v})^i = 0$; and that therefore $J\boldsymbol{v}$ is a nilideal (a nilpotent right-ideal) whenever J is a nilideal (a nilpotent right-ideal). Saying that an ideal J in R is *$\boldsymbol{V}$-admissible,* if $J\boldsymbol{V} \leq J$, it is now easy to prove the following statement:

If the ring R admits the elements in the system $\boldsymbol{V}$ as operators, then the upper and the lower radical and all the ideals M_v in the anti-radical series are $\boldsymbol{V}$-admissible.[44]

If the ring R satisfies condition (4. A), then there exists to every element x in R an element x^* in R such that $x = xx^*$; and this shows that $x\boldsymbol{v} = x(x^*\boldsymbol{v})$ for every x in R. Hence we have proved the following theorem:

If condition (4. A) *is satisfied by the ring R, then every right-ideal in R is $\boldsymbol{V}$-admissible.*

On the basis of this theorem it becomes evident that most of the theorems derived in this paper may be applied to rings admitting operators.

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Added in proof: (October, 1943): Since this paper has been submitted for publication, the following investigations which are somewhat related to the present article have appeared in print.

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[Footnotes]

[15] **Rings With Minimal Condition for Left Ideals**
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