



NORTH-HOLLAND

## Jordan Canonical Form for Finite Rank Elements in Jordan Algebras

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### ABSTRACT

The usual Jordan canonical form for matrices is extended first to nilpotent elements of the socle of a nondegenerate Jordan algebra and then to elements of a nondegenerate Jordan algebra which is reduced over an algebraically closed field.  
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### 1. INTRODUCTION

As presented in [8], there are two classical approaches to canonical forms of matrices. We can get the Jordan form of a matrix having all its characteristic roots in the base field, either by using elementary methods of linear algebra, or as a consequence of the structure of finitely generated modules over a euclidean domain. In this paper we present a new approach to canonical forms that could be considered more intrinsic. We begin by stating the key result for nilpotent matrices in a way that it is suitable for our purposes.

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Let  $a \in \mathcal{M}_n(F)$  be a nonzero nilpotent matrix. Then there exist  $r$  orthogonal idempotents  $e_1, \dots, e_r$  in  $\mathcal{M}_n(F)$  such that  $a = a_1 + \dots + a_r$ , where each  $a_i \in e_i \mathcal{M}_n(F) e_i$  is indecomposable and  $n_1 \geq n_2 \geq \dots \geq n_r > 1$  with  $n_i$  being the index of nilpotence of  $a_i$ . Moreover, any other such decomposition of  $a$  has the same invariants  $n_1 \geq n_2 \geq \dots \geq n_r$ .

It is then natural to ask oneself whether this result can be extended to nilpotent elements of more general algebras than that of the full algebra of matrices over a field.

As it will be clear later, a nilpotent will have a Jordan canonical form if its “rank” is finite, but elements having finite rank are precisely those elements in the socle. On the other hand, since all the notions involved in the result are symmetric, we can deal with Jordan algebras.

Finally, a Jordan canonical form is provided for any element of a nondegenerate Jordan algebra which is reduced over an algebraically closed field—in particular, for elements of the socle of a Jordan normed algebra over the complex field.

## 2. PRELIMINARIES AND NOTATION

All the algebras we consider here are over a ring of scalars  $\Phi$  containing  $\frac{1}{2}$ . A (nonassociative) algebra  $J$  with product  $x \cdot y$  satisfying

- (1)  $x \cdot y = y \cdot x$ ,
- (2)  $x^2 \cdot (y \cdot x) = (x^2 \cdot y) \cdot x$  (Jordan identity)

is called a (linear) *Jordan algebra*. The reader is referred to the books [10, 11, 21] for basic results and notation on Jordan algebras. For an expository survey of the theory of Jordan algebras the reader is referred to [18]. Every associative algebra  $A$  gives rise to a Jordan algebra  $A^+$  under the new multiplication defined by

$$x \cdot y = \frac{1}{2}(xy + yx).$$

Jordan algebras which are subalgebras of a Jordan algebra  $A^+$  are called *special Jordan algebras*. For every associative algebra  $A$  with involution  $*$ :  $A \rightarrow A$  the set of all hermitian elements  $H(A, *) = \{a \in A : a = a^*\}$  is a subalgebra of  $A^+$ , and therefore special.

Every Jordan algebra which is not special is called an *exceptional Jordan algebra*. Let  $C$  be a Cayley-Dickson algebra over a field  $K$  ( $C$  is an 8-dimensional alternative algebra obtained by doubling a quaternion algebra by the Cayley-Dickson process). Then the set  $H_3(C)$  of all  $3 \times 3$  matrices in  $C$  which are hermitian under the involution  $X^* = \bar{X}^t$  is a simple 27-dimensional exceptional Jordan algebra.

In order to obtain identities in Jordan algebras is very useful the following theorem due to Shirshov and Macdonald.

*Any polynomial identity in three variables with degree at most 1 in one variable, and which holds in all special Jordan algebras, holds in all Jordan algebras.*

Throughout the paper we will use this theorem without making mention of it. Another important result is Shirshov and Cohn's theorem.

*Any Jordan algebra generated by at most two elements is special, indeed it has the form  $H(A, *)$  for an associative algebra  $A$ .*

We write  $U_x$  to denote the quadratic  $U$ -operator  $U_x y = 2x \cdot (x \cdot y) - x^2 \cdot y$ , and

$$\{x, y, z\} := \frac{1}{2}(U_{x+z} y - U_x y - U_z y).$$

Recall that  $J$  is said to be *nondegenerate* if  $U_a = 0$  implies  $a = 0$  for  $a \in J$ . Since  $U_a b = aba$  for a special Jordan algebra  $A^+$ , we have that  $A^+$  is nondegenerate if and only if  $A$  is semiprime.

Any Jordan algebra  $J$  gives rise to a *Jordan pair*  $(J, J)$ , so we can borrow results from the theory of Jordan pairs (the standard reference for Jordan pairs is [13]).

The *socle* of a nondegenerate Jordan algebra  $J$ , defined as the sum of all minimal inner ideals of  $J$ , will be denoted by  $\text{Soc}(J)$ . For semiprime  $A$ ,  $\text{Soc}(A^+) = \text{Soc}(A)^+$ , and if  $A$  has an involution  $*$  then  $\text{Soc}(H(A, *)) = H(\text{Soc}(A), *)$  [2, Proposition 2.6]. The notion of socle was introduced in [20] for Jordan algebras, and extended later to more general Jordan structures [7, 5, 14]. Here we recall that the socle is a (von Neumann) regular ideal which is a direct sum of simple ones. Moreover, by the Litoff theorem for Jordan algebras [1], for every finite subset  $\{a_1, \dots, a_n\} \subset \text{Soc}(J)$ , there exists an idempotent  $e \in \text{Soc}(J)$  such that  $\{a_1, \dots, a_n\} \subset U_e J$ , with  $U_e J$  being a nondegenerate unital Jordan algebra with finite capacity, which is simple whenever  $J$  is [17]. Then  $U_e J$  will be called a *Litoff envelope* of  $\{a_1, \dots, a_n\}$ .

Following [16], the *rank* of an element  $x \in J$ ,  $\text{rank}(x)$ , is the supremum of the lengths of all finite chains

$$[x_0] \subset [x_1] \subset \dots \subset [x_n]$$

of principal inner ideals  $[x_i] := U_{x_i}J$  where  $x_i$  belongs to the inner ideal  $(x) := \Phi x + [x]$  generated by  $x$ , and the length of such a chain is the number of strict inclusions.

Recall that any regular element  $x \in J$  gives rise to an idempotent  $(x, y)$  in the Jordan pair sense, that is,  $U_x y = x$  and  $U_y x = y$ . Following [16], two regular elements  $u$  and  $v$  are said to be *orthogonal* ( $u \perp v$ ) if  $u = c_\sigma$ ,  $v = d_\sigma$  are parts of orthogonal *Jordan pair idempotents* (see [13] for definition)  $c = (c_+, c_-)$  and  $d = (d_+, d_-)$ . Note that any usual idempotent,  $e = e^2$ , gives rise to a Jordan pair idempotent  $(e, e)$ , and that if  $e$  and  $f$  are orthogonal idempotents in the socle of a nondegenerate Jordan algebra, then for any  $x \in [e]$  and  $y \in [f]$  one has  $x \perp y$ .

The following properties of the rank function will be used in what follows.

PROPOSITION 2.1. *Let  $J$  be a nondegenerate Jordan algebra. Then*

- (1)  $a \in J$  is in the socle if and only if  $a$  has finite rank;
- (2)  $\text{rank}(a + b) \leq \text{rank}(a) + \text{rank}(b)$ , and the equality holds whenever  $a \perp b$ ;
- (3)  $\text{rank}(U_a b) \leq \min\{\text{rank}(a), \text{rank}(b)\}$ ;
- (4)  $\text{rank}(\{a, b, c\}) \leq 2 \text{rank}(b)$  for  $a, b, c \in \text{Soc}(J)$ .

*Proof.* The proof of (1) and (2) follows from [16, Proposition 3(2), (4), and (7)]; (3) is a consequence of [16, Corollary 1(a) of Theorem 3]. Finally, (4) follows from (2) and (3) together with the identity

$$\{x, y, z\} = \frac{1}{4}(U_{x+z}y - U_{x-z}y),$$

which works in linear Jordan algebras. ■

Given  $a \in \text{Soc}(J)$ , an idempotent  $e \in \text{Soc}(J)$  will be called a *support* of  $a$  if (1)  $a \in U_e J$  and (2)  $e$  is minimal among all the idempotents of the socle satisfying (1). It follows from the Litoff theorem for Jordan algebras that every  $a \in \text{Soc}(J)$  has a support  $e$ .

### 3. JORDAN CANONICAL FORM FOR NILPOTENT ELEMENTS

Let  $a$  be a nilpotent element in a Jordan algebra  $J$ . Then  $n$  is the *index of nilpotence* of  $a$  if  $a^n = 0$  but  $a^{n-1} \neq 0$ .

LEMMA 3.1. *Let  $J$  be a nondegenerate Jordan algebra and  $a \in \text{Soc}(J)$  a nilpotent element of index of nilpotence  $n$ . Then*

- (1)  $[0] = [a^n] \subset [a^{n-1}] \subset \dots \subset [a]$ , where the inclusions are strict;
- (2)  $\text{rank}(a^{k+1}) < \text{rank}(a^k)$  for  $1 \leq k \leq n - 1$ ;
- (3)  $n - 1 \leq \text{rank}(a)$ .

*Proof.* (1): Since  $[a^{k+1}] \subset [a^k]$  for  $1 \leq k \leq n - 1$ , we need only to verify that the inclusions are strict. Suppose on the contrary that  $[a^k] = [a^{k+1}]$  for some  $1 \leq k \leq n - 1$ . Since every element in the socle is regular,  $a^k = U_{a^k}b = U_{a^{k+1}}c$  for some  $b, c \in J$ . Then

$$a^{n-1} = a^k \cdot a^{n-1-k} = (U_{a^{k+1}}c) \cdot a^{n-1-k} = \{a^{k+1}, c, a^n\} = 0,$$

which is a contradiction.

Assertions (2) and (3) follow from (1). ■

DEFINITION 3.1. Let  $J$  be a nondegenerate Jordan algebra. An element  $a \in \text{Soc}(J)$  will be called *indecomposable* if it cannot be written as a sum  $a = a_1 + a_2$  where  $0 \neq a_i \in [e_i]$ ,  $i = 1, 2$ , with  $e_1, e_2$  orthogonal idempotents in the socle. Otherwise we will say that  $a$  is *decomposable*.

REMARK 3.1. Clearly, every indecomposable element  $a$  in  $J$  remains indecomposable in every Litoff envelope  $U_e J$  of  $\{a\}$ . Later it will be shown that if a nilpotent is indecomposable in some Litoff envelope, then it is indecomposable in  $J$ , and hence in every Litoff envelope.

PROPOSITION 3.1. *Let  $J$  be a nondegenerate Jordan algebra and  $a \in \text{Soc}(J)$  a nonzero nilpotent element of index of nilpotence  $n$ . If  $\text{rank}(a) = n - 1$ , then  $a$  is indecomposable.*

*Proof.* Let  $a = a_1 + a_2$  be a decomposition of  $a$  where each  $a_i$  belongs to  $[e_i]$ , for  $e_1, e_2$  orthogonal idempotents in the socle. By Peirce relations,  $0 = a^n = a_1^n + a_2^n$  implies that each  $a_i$  is nilpotent with index of nilpotence  $n_i \leq n$ . Moreover, one of them, say  $a_1$ , has index of nilpotence  $n_1 = n$ . Hence, by Proposition 2.1(2) and Lemma 3.1(3),

$$n - 1 = \text{rank}(a) = \text{rank}(a_1) + \text{rank}(a_2) \geq n - 1 + n_2 - 1$$

implies  $n_2 = 1$ , so  $a_2 = 0$  and  $a$  is indecomposable, as required. ■

Our next task, the most difficult part of the paper, will be to prove the converse of Proposition 3.1. We start with a technical result which is the key induction step in the successive straightening of an arbitrary “generalized inverse”  $b$  for  $a^{n-1}$  (where  $a$  is a nilpotent element of a nondegenerate Jordan algebra of index of nilpotence  $n$ ) into a generator  $c$  of a family of matrix units in Theorem 3.1 below.

LEMMA 3.2. *Let  $J$  be a Jordan algebra, and  $a \in J$  a nonzero nilpotent element of index of nilpotence  $n$ . Suppose that, for fixed  $0 \leq s < n - 1$ , there exists  $b \in J$  such that  $U_b a^{n-1} = b$  and  $U_b a^k = 0$  for  $s + 1 \leq k < n - 1$ . Then there exists  $c \in J$  such that  $U_c a^{n-1} = c$ ,  $U_c a^k = 0$  for  $s \leq k < n - 1$  and  $U_a^{n-1} c = U_a^{n-1} b$ .*

*Proof.* By Shirshov and Cohn’s theorem, we work in the subalgebra of  $J$  generated by  $a$  and  $b$  which has the form  $H(A, *)$ , so we may assume we are in a unital associative algebra with symmetric elements  $a, b$  such that

$$\begin{aligned} a^n &= 0, & a^{n-1} &\neq 0, & ba^{n-1}b &= b, \\ ba^k b &= 0 & (s + 1 \leq k < n - 1). \end{aligned} \quad (3.1)$$

We claim that the symmetric (therefore Jordan) element  $c = u_s b u_s^*$  satisfies the required properties, where in general for  $s \leq k < n - 1$

$$u_k = 1 - \frac{1}{2} a^{n-1-k} b a^k.$$

We have

$$a^j u_k = a^j = u_k^* a^j \quad (j > k). \quad (3.2)$$

Then we have

$$\begin{aligned} a^{n-1} c a^{n-1} &= (a^{n-1} u_s) b (u_s^* a^{n-1}) \\ &= a^{n-1} b a^{n-1} \quad [\text{by (3.2)}], \\ c a^{n-1} c &= (u_s b) u_s^* a^{n-1} u_s (b u_s^*) \\ &= (u_s b) a^{n-1} (b u_s^*) \quad [\text{by (3.2)}] \\ &= u_s b u_s^* \quad [\text{by (3.1)}] \\ &= c. \end{aligned}$$

Now for  $k > s$ ,

$$\begin{aligned} ca^k c &= (u_s b) u_s^* a^k u_s (b u_s^*) \\ &= u_s (b a^k b) u_s^* \quad [\text{by (3.2)}] \\ &= 0, \end{aligned}$$

and for  $k = s$ , by the definition of  $u_s$ ,

$$\begin{aligned} u_s^* a^s u_s &= u_s^* (a^s - \frac{1}{2} a^{n-1} b a^s) \\ &= a^s - \frac{1}{2} a^s b a^{n-1} - \frac{1}{2} a^{n-1} b a^s \quad [\text{by (3.2)}], \end{aligned}$$

so

$$\begin{aligned} ca^s c &= u_s (b u_s^* a^s u_s b) u_s^* \\ &= u_s [b a^s b - \frac{1}{2} (b a^s b a^{n-1} b + b a^{n-1} b a^s b)] u_s^* \\ &= u_s [b a^s b - \frac{1}{2} (b a^s b + b a^s b)] u_s^* = 0 \quad [\text{by (3.1)}]. \end{aligned}$$

■

As motivation for Theorem 3.5, we show the following two examples.

**EXAMPLE 3.1.** This result due to Malcev (see [12, p. 366]) provides a natural motivation for the first part.

Let  $a$  be a nilpotent linear transformation in a finite-dimensional vector space  $V$  over a field  $K$ , which is self-adjoint relative to a nondegenerate symmetric inner product  $\langle \cdot, \cdot \rangle$  on  $V$ . Then  $V$  is an orthogonal direct sum of subspaces  $V_i$  where  $V_i$  has a base  $\{z_i, a z_i, \dots, a^{n_i-1} z_i\}$  such that

$$\langle z_i, a^{n_i-1} z_i \rangle = \alpha_i \neq 0 \quad \text{and} \quad \langle z_i, a^j z_i \rangle = 0 \quad (0 \leq j < n_i - 1).$$

It is not difficult to see that the above geometric conditions are equivalent to the following algebraic ones [relative to the Jordan algebra  $J = H(\text{End}_k(V), *)$ ,  $*$  being the adjoint involution]:

$$U_{c_i} a^{n_i-1} = c_i, \quad U_{c_i} a^j = 0 \quad (0 \leq j < n_i - 1) \quad (3.3)$$

for  $c_i$  the self-adjoint operator defined by  $c_i v = \alpha_i^{-1} \langle v, z_i \rangle z_i$  ( $v \in V$ ). Note that  $c_i$  is a rank one nilpotent element. This use of rank one nilpotents is the key to our approach to the Jordan canonical form.

EXAMPLE 3.2. We compute the Jordan canonical form of an indecomposable nilpotent element in a simple special Jordan algebra  $J = A^+$ , where  $A$  is a simple associative algebra with minimal one-sided ideals. But first we need some definitions and notation.

Following [9], let  $\langle X, Y, \langle \cdot, \cdot \rangle \rangle$  be a pair of dual vector spaces over an associative division algebra  $\Delta$ , where  $X$  is a left vector space,  $Y$  a right vector space, and  $\langle \cdot, \cdot \rangle$  a nondegenerate bilinear form over  $\Delta$ . An operator  $a: X \rightarrow X$  is *adjointable* if there exists  $a^\#: Y \rightarrow Y$ , necessarily unique, such that  $\langle xa, y \rangle = \langle x, a^\#y \rangle$ . Notice that we write the mappings of a left vector space on the right (thus composing them from left to right), and the mappings of a right vector space on the left (thus composing them from right to left), so by our convention, if  $a, b$  are adjointable, so is  $ab$  with (careful!)  $(ab)^\# = a^\#b^\#$ . We denote by  $\mathcal{L}_Y(X)$  the ring of all adjointable linear operators of  $X$ , and by  $\mathcal{F}_Y(X)$  the ideal of those operators having finite rank. The rings  $\mathcal{F}_Y(X)$  are precisely those simple rings containing minimal one-sided ideals [9]. One can see that such rings are algebras over  $\Phi$  wherever  $\Delta$  is a  $\Phi$ -algebra. For  $x \in X, y \in Y$  write  $y \otimes x$  to denote the adjointable linear operator defined by

$$x'(y \otimes x) = \langle x', y \rangle x \quad (x' \in X)$$

with adjoint  $(y \otimes x)^\#y' = y \langle x, y' \rangle$ . Note that  $(y \otimes x)a = y \otimes xa$  for all operator  $a$ , and  $a(y \otimes x) = a^\#y \otimes x$  for all adjointable  $a$ .

Let  $a \in J = \mathcal{F}_Y(X)^+$  be a nilpotent of index of nilpotence  $n$ . For  $x \in X$  such that  $xa^{n-1} \neq 0$ , the vectors  $x, xa, \dots, xa^{n-1}$  are linearly independent. Take  $y \in Y$  such that  $\langle xa^{n-1}, y \rangle = 1$  and  $\langle xa^k, y \rangle = 0$  for  $0 \leq k < n - 1$ . Then

$$\{(a^\#)^{n-1}y, (a^\#)^{n-2}y, \dots, y\}$$

is a system dual to

$$\{x, xa, \dots, xa^{n-1}\}$$

$[\langle xa^i, (a^\#)^j y \rangle = \delta_{i, n-1-j}]$ . Hence, for  $c := y \otimes x$  we have

$$U_c a^{n-1} = c, \quad U_c a^k = 0 \quad (0 \leq k < n - 1).$$



If we consider the unit matrices (relative to the vectors  $x_0, \dots, x_{n-1}$  for  $x_i = xa^i$ )  $e_{rs} = a^{n-r-1}ca^s$ , then the element

$$e = \sum_{r=0}^{n-1} e_{rr} = \sum_{r=0}^{n-1} (a^\#)^{n-1-r} y \otimes xa^r = \sum_{r=0}^{n-1} \{a^{n-1-r}, c, a^r\}$$

is an idempotent in  $J$ . Moreover, if  $a$  is indecomposable then

$$a = ea = \sum_{k=1}^{n-1} (a^\#)^{n-k} y \otimes xa^k = \sum_{k=1}^{n-1} \{a^{n-k}, c, a^k\}.$$

Now we show that a similar decomposition holds in arbitrary Jordan algebras.

**THEOREM 3.1.** *Suppose that  $J$  is a nondegenerate Jordan algebra and  $a \in \text{Soc}(J)$  a nonzero nilpotent of index of nilpotence  $n$ . Then there exists  $b \in \text{Soc}(J)$  such that  $(a^{n-1}, b)$  is an idempotent, and  $U_b a^k = 0$  for  $0 \leq k < n - 1$ .*

*Moreover, given  $0 \neq c \in J$  such that  $U_c a^{n-1} = c$  and  $U_c a^k = 0, 0 \leq k < n - 1$ , we have:*

(1) *The element*

$$e = \sum_{r=0}^{n-1} \{a^{n-1-r}, c, a^r\}$$

*is a nonzero idempotent of  $\text{Soc}(J)$ , and  $a = a_1 + a_0$  (with respect to the Peirce decomposition relative to  $e$ ), with*

$$a_1 = \sum_{k=1}^{n-1} \{a^{n-k}, c, a^k\} \neq 0.$$

*If additionally  $a$  is indecomposable, then:*

(2) *One has*

$$a = \sum_{k=1}^{n-1} \{a^{n-k}, c, a^k\} \in U_e J.$$

(3) *One has*

$$\text{rank}(a^m) = n - m \quad \text{for } 1 \leq m \leq n - 1.$$

*In particular,  $\text{rank}(a) = n - 1$  if and only if  $a$  is indecomposable.*

(4)  *$e$  is a support of  $a$ , and every support of  $a$  has rank equal to  $n$ .*

*Proof.* By regularity of  $a^{n-1}$ , there exists  $b_1 \in J$  such that  $(a^{n-1}, b_1)$  is an idempotent in the Jordan pair sense. Suppose now that, for fixed  $2 \leq r \leq n$ , we have constructed  $b_{r-1} \in J$  such that  $(a^{n-1}, b_{r-1})$  is an idempotent and  $U_{b_{r-1}} a^k = 0$  for  $n - (r - 1) \leq k < n - 1$ . Then, by Lemma 3.2, there exists  $b_r \in J$  such that  $(a^{n-1}, b_r)$  is an idempotent and  $U_{b_r} a^k = 0$  for  $n - r \leq k < n - 1$ . Hence, by recurrence, there exists  $b = b_n \in J$  such that  $(a^{n-1}, b)$  is an idempotent and  $U_b a^k = 0$  for  $0 \leq k < n - 1$ .

By Shirshov and Cohn's theorem, we may assume we are in a Jordan algebra  $H(A, *)$ , where  $A$  is a unital associative algebra with symmetric elements  $a$  and  $c$ .

(1): The (nonsymmetric) elements  $e_{rs} = a^{n-r-1}ca^s$  for  $r, s \in \{0, \dots, n - 1\}$  are a family of  $n^2$  matrix units. In particular, the  $e_{rr}$  are orthogonal idempotents:  $e_{ij}e_{kl} = a^{n-1-i}ca^{j+n-1-k}ca^l$  equals 0 if  $j > k$  (by  $a^n = 0$ ) or if  $j < k$  (by  $ca^s c = 0$  for  $s < n - 1$ ), while if  $j = k$  it equals  $a^{n-1-i}ca^l = e_{il}$  (by  $ca^{n-1}c = c$ ), so the element

$$\begin{aligned} e &= \sum_{r=0}^{n-1} \{a^{n-1-r}, c, a^r\} = \frac{1}{2} \sum_{r=0}^{n-1} (a^{n-1-r}ca^r + a^rca^{n-1-r}) \\ &= \frac{1}{2} \left( \sum_{r=0}^{n-1} e_{rr} + \sum_{s=0}^{n-1} e_{ss} \right) = \sum_{r=0}^{n-1} e_{rr} \end{aligned}$$

is an idempotent of  $\text{Soc}(J)$ . Now

$$\begin{aligned} ae &= a \sum_{r=0}^{n-1} e_{rr} = a \sum_{r=0}^{n-1} a^{n-r-1}ca^r \\ &= \sum_{r=1}^{n-1} a^{n-r}ca^r = ea \quad (\text{since } a^n = 0) \end{aligned}$$

by symmetry, so  $a = a_1 + a_0$  (Peirce decomposition of  $a$  relative to  $e$ ), where

$$\begin{aligned} a_1 &= ae = ea = \frac{1}{2}(ae + ea) = \frac{1}{2} \sum_{r=1}^{n-1} (a^{n-r}ca^r + a^rca^{n-r}) \\ &= \sum_{r=1}^{n-1} \{a^{n-r}, c, a^r\}. \end{aligned}$$

Now we claim that  $a_1$  is nonzero. Indeed, computing  $ca_1$ , we get

$$\begin{aligned} ca_1 &= c \sum_{r=1}^{n-1} \{a^{n-r}, c, a^r\} = \frac{1}{2} \sum_{r=1}^{n-1} (ca^{n-r}ca^r + ca^rca^{n-r}) \\ &= \frac{1}{2}(ca^{n-1}ca + ca^{n-1}ca) \quad (\text{as } ca^k c = 0 \text{ if } k < n - 1) \\ &= ca \neq 0 \end{aligned}$$

because  $ca^{n-1}c = c$ , which is nonzero.

(2): Since  $a$  is indecomposable in every Litoff envelope (Remark 3.1), we may assume, without loss of generality, that  $J$  is unital. Hence,  $a = a_1 + a_0$  (with  $a_1 \neq 0$ ) implies  $a = a_1 \in U_e J$ , as required.

(3): We see first that the rank of  $a^{n-1}$  is one. Suppose otherwise that  $\text{rank}(a^{n-1}) > 1$ . By the first part of the theorem, there exists  $b \in \text{Soc}(J)$  such that  $(a^{n-1}, b)$  is an idempotent and  $U_b a^k = 0$  for  $0 \leq k < n - 1$ . Hence, by [16, Corollary 1 of Theorem 3],  $\text{rank}(b) = \text{rank}(a^{n-1}) > 1$ , so  $b = b_1 + b_2$ , where  $b_1$  and  $b_2$  are parts of nonzero orthogonal idempotents  $(b_1, d_1)$  and  $(b_2, d_2)$  [16, Proposition 1]. By Peirce relations relative to orthogonal Jordan pair idempotents (see [13, p. 44]),  $U_{d_1} b = U_{d_1} b_i = d_i$ ,  $U_b d_i = U_{b_i} d_i = b_i$ , and

$$U_{b_i} a^{n-1} = U_b U_{d_i} U_b a^{n-1} = U_b U_{d_i} b = U_b d_i = b_i.$$

Moreover, since  $U_b a^k = 0$  for  $0 \leq k < n - 1$ ,  $b_i = U_b d_i$  implies

$$U_{b_i} a^k = 0 \quad \text{for } 0 \leq k < n - 1.$$

Hence, by (2) applied to both  $c = b_1 \neq 0$  and  $c = b_2 \neq 0$ ,

$$\begin{aligned} a &= \sum_{k=1}^{n-1} \{a^{n-k}, b, a^k\} \\ &= \sum_{k=1}^{n-1} \{a^{n-k}, b_1, a^k\} + \sum_{k=1}^{n-1} \{a^{n-k}, b_2, a^k\} = a + a, \end{aligned}$$

which is a contradiction. Therefore  $\text{rank}(a^{n-1}) = \text{rank}(b) = 1$ , as required. We shall now prove that  $\text{rank}(a) = n - 1$ . By Lemma 3.1(3),  $\text{rank}(a) \geq n - 1$ . To prove the reverse inequality, we must distinguish two cases.

If  $n - 1 = 2m$  is even then, by (2),

$$a = \sum_{k=1}^m 2\{a^{n-k}, b, a^k\}.$$

Hence, by Proposition 2.1(2) and (4),  $\text{rank}(a) \leq 2m \text{rank}(b) = 2m = n - 1$ .

If  $n - 1 = 2m + 1$  is odd, then

$$a = \sum_{k=1}^m 2\{a^{n-k}, b, a^k\} + U_{a^{m+1}}b.$$

Again, by Proposition 2.1(2), (3), and (4),

$$\text{rank}(a) \leq 2m \text{rank}(b) + \text{rank}(b) = 2m + 1 = n - 1.$$

Finally, by Lemma 3.1(2),

$$1 = \text{rank}(a^{n-1}) < \text{rank}(a^{n-2}) < \dots < \text{rank}(a) = n - 1$$

implies  $\text{rank}(a^m) = n - m$ , for  $1 \leq m \leq n - 1$ .

(4): By (2),  $a \in U_e J$  and hence, by Proposition 2.1(3),  $\text{rank}(e) \geq \text{rank}(a) = n - 1$ . But actually  $\text{rank}(e) > \text{rank}(a)$ , since otherwise by [16, Corollary 1 of Prop. 3],  $\text{rank}(e) = \text{rank}(a)$  would imply  $a$  invertible in  $U_e J$ , which is a contradiction because  $a$  is nilpotent. So  $\text{rank}(e) \geq n$ . The reverse inequality can be verified as in (3), since by (1),

$$e = \sum_{r=0}^{n-1} \{a^{n-1-r}, b, a^r\}.$$

Hence,  $a \in U_e J$  with  $\text{rank}(a) = n - 1$  implies that  $e$  is a support of  $a$ . Otherwise let  $u$  be an idempotent such that  $a \in U_u J$  with  $U_u J$  strictly contained in  $U_e J$ . Then  $\text{rank}(u) = n - 1$  and again, by [16, Corollary 1 of Proposition 3],  $a$  is invertible in  $U_u J$ , which is a contradiction. Suppose now that  $f$  is another support of  $a$ . Since  $a$  remains indecomposable and with the same rank in  $U_f J$ , we may replace  $J$  by  $U_f J$  in the statement of the theorem and thus obtain, by (1) and (2), an idempotent  $e \in U_f J$  such that  $a \in U_e J \subset U_f J$  with  $\text{rank}(e) = n$ . Hence  $f = e$  has rank equal to  $n$ , which completes the proof. ■

**COROLLARY 3.1.** *Let  $J$  be a Jordan algebra and  $a \in \text{Soc}(J)$  a nonzero nilpotent element. For any Litoff envelope  $U_e J$  of  $a$ ,  $a$  is indecomposable in  $J$  if and only if  $a$  is indecomposable in  $U_e J$ .*

*Proof.* By Remark 3.1, we need only to show that if  $a$  is indecomposable in  $U_e J$  then  $a$  is indecomposable in  $J$ . But this is a consequence of Proposition 3.1 and Theorem 3.1(3) for  $m = 1$ , taking into account that both notions, index of nilpotence and rank, are the same in  $J$  and any Litoff envelope  $U_e J$ . The first is trivial, and the second follows from the fact that principal inner ideals of  $U_e J$  are those principal inner ideals  $[x]$  of  $J$  such that  $x \in U_e J$ . ■

Everything is ready to prove the main result of this paper, namely, the existence and uniqueness of a Jordan canonical form for any nilpotent element in the socle of a nondegenerate Jordan algebra.

**THEOREM 3.2.** *Suppose that  $J$  is a nondegenerate Jordan algebra and  $a \in \text{Soc}(J)$  a nonzero nilpotent of index of nilpotence  $n$ . Then:*

(1) *There exist  $e_1, e_2, \dots, e_r$  orthogonal idempotents in  $\text{Soc}(J)$  such that  $a = a_1 + a_2 + \dots + a_r$  (Jordan canonical form of  $a$ ), where, for each  $i$ ,  $e_i$  is a support of  $a_i$ , and  $a_i$  is a nonzero indecomposable nilpotent of index of nilpotence  $n_i$ ,  $n = n_1 \geq n_2 \geq \dots \geq n_r$ .*

(2) *If  $a = b_1 + b_2 + \dots + b_s$  is another Jordan canonical form of  $a$  with indices of nilpotence  $n = m_1 \geq m_2 \geq \dots \geq m_s$ , then  $r = s$  and  $n_i = m_i$  for all  $i = 1, \dots, r$ .*

(3) *The idempotent  $e = e_1 + e_2 + \dots + e_r$  is a support of  $a$ , and every support of  $a$  has rank equal to  $n_1 + n_2 + \dots + n_r$ .*

*Proof.* (1): *Existence of the Jordan canonical form.* We will induce on the rank  $k$  of  $a$ . If  $k = 1$ , then  $a$  is indecomposable by Proposition 2.1(2), so suppose that the result is true for any nilpotent of rank  $\leq k - 1$  and let  $\text{rank}(a) = k$ . If  $a$  is decomposable, then  $a = a_1 + a_2$  where each  $a_i$  is a nonzero element in  $U_e J$ , with  $e_1, e_2$  orthogonal idempotents in the socle of  $J$ . By Proposition 2.1(2),  $\text{rank}(a) = \text{rank}(a_1) + \text{rank}(a_2)$ , and hence  $\text{rank}(a_i) < \text{rank}(a)$ ,  $i = 1, 2$ , because  $a_i$  is nonzero. Then, by the induction hypothesis, each  $a_i$  has a Jordan canonical form  $a_i = a_{i1} + a_{i2} + \dots + a_{i r_i}$ . Since, by Corollary 3.1, every indecomposable nilpotent in  $U_e J$  is actually indecomposable in  $J$ ,

$$a = a_1 + a_2 = a_{11} + a_{12} + \dots + a_{1 r_1} + a_{21} + a_{22} + \dots + a_{2 r_2}$$

provides a Jordan canonical form for  $a$ , as required.

(2): *Uniqueness of the Jordan canonical form.* Let

$$a = a_1 + \cdots + a_r \quad (n = n_1 \geq \cdots \geq n_r > 1)$$

be a Jordan canonical form of  $a$ . We claim that the sequence  $n = n_1 \geq \cdots \geq n_r > 1$  is uniquely determined by  $a$ . This follows from the fact that there is an explicit formula for the  $n_i$ 's, or equivalently for the multiplicities  $p_k$  (the number of  $n_i$ 's which equal  $k$ ):

$$p_k = \text{rank}(a^{k-1}) - 2 \text{rank}(a^k) + \text{rank}(a^{k+1}),$$

which can be verified by using Proposition 2.1(2) and Theorem 3.1(3).

(3): Since  $e_i$  is a support of  $a_i$  for  $i = 1, \dots, n$ , taking  $e = e_1 + \cdots + e_r$ , we obtain a support of  $a$ . Indeed,  $a \in U_e J$  with  $\text{rank}(e) = \text{rank}(e_1) + \cdots + \text{rank}(e_r) = n_1 + \cdots + n_r$  [ $n_i$  being the index of nilpotence of  $a_i$  (Theorem 3.1(4))], and if  $a \in U_{e'} J$  for some  $e' \in U_e J$ , then [applying (1) to  $U_{e'} J$ ] we obtain [since the Jordan canonical form of  $a$  is the same in any Litoff envelope (Corollary 3.1)]  $f = f_1 + \cdots + f_r \in U_{e'} J$  with  $\text{rank}(e) = \text{rank}(f) \leq \text{rank}(e')$  and hence  $e = e'$ .

Finally, any other support of  $a$  has the same rank as  $e$  (by Corollary 3.1 again). ■

**REMARK 3.2.** If  $J$  is unital, we can extend the Jordan canonical form of a nilpotent element  $a = a_1 + \cdots + a_r$  in the socle of  $J$ , possibly by adding zero elements, to get  $a = a_1 + \cdots + a_r + a_{r+1} + \cdots + a_t$ , so that the unit element 1 of  $J$  is a sum  $1 = e_1 + \cdots + e_t$  of nonzero orthogonal idempotents with each  $a_j \in U_{e_j} J$ ,  $1 \leq j \leq t$ .

**REMARK 3.3.** Since the socle of a nondegenerate Jordan algebra  $J$  satisfies dcc on principal inner ideals (see [3] or [14]), any  $a \in \text{Soc}(J)$  has a *Fitting decomposition* [15, Theorem 1], that is, there exists a unique idempotent  $e \in \text{Soc}(J)$  such that  $a = a_1 + a_0 \in J_1 \oplus J_0$  in the Peirce decomposition  $J = J_1 \oplus J_{1/2} \oplus J_0$  of  $J$  with respect to  $e$ , where  $a_1$  is invertible in  $J_1 = U_e J$  and  $a_0$  is nilpotent. This result, together with Theorem 3.2, reduces the study of the elements of the socle to two particular types: (1) locally invertible elements, and (2) indecomposable nilpotent elements. The first ones, which were studied in [6], occur also in [4] in the extensions of Zel'manov's theorem for Goldie Jordan algebras to local orders.

4. REDUCED JORDAN ALGEBRAS

Let  $J$  be a Jordan algebra over  $\Phi$ . An element  $c \in J$  is called *reduced* if  $U_c J \subseteq \Phi c$ . By [19, (1.3)], the linear span  $\text{Red}(J)$  of all reduced elements is an ideal of  $J$ . We call  $J$  reduced if  $J = \text{Red}(J)$ . If  $\Phi = F$  is a field and  $J$  is nondegenerate, then a nonzero element  $c \in J$  is reduced if and only if  $U_c J = Fc$  and so a minimal inner ideal. Hence  $\text{Red}(J) \subseteq \text{Soc}(J)$  and every nonzero reduced element generates a simple ideal. By [5, (2.5)], for a nondegenerate Jordan algebra  $J$  over a field  $F$ ,  $\text{Soc}(J) = \text{Red}(J)$  if and only if every simple component of  $\text{Soc}(J)$  contains a nonzero reduced element. Moreover, in such case, every minimal inner ideal of  $J$  is generated by a reduced element. We also recall [5, (2.6)] that any nondegenerate Jordan algebra which is either finite-dimensional over an algebraically closed field or a (possibly infinite dimensional) complex normed Jordan algebra coinciding with its socle is reduced.

Although the proof of the following lemma only requires well-known standard arguments, it is included for completeness.

LEMMA 4.1. *Let  $J$  be a unital nondegenerate Jordan algebra over a field  $F$ , and let  $a \in J$  be a nonzero algebraic element with minimum polynomial  $m(x) \in F[x]$ . Suppose that  $m(x) = p(x)q(x)$  for  $p(x)$  and  $q(x)$  relatively prime monic polynomials of degree  $> 1$  in  $F[x]$ . Then*

- (1)  $1 = e + f$  where  $e$  and  $f$  are nonzero orthogonal idempotents in the associative subalgebra  $F[a]$  generated by  $a$ ;
- (2)  $a = b + c$  with  $b \in U_e J$  and  $c \in U_f J$ . Moreover,  $p(x)$  and  $q(x)$  are the minimum polynomials of  $b$  and  $c$  in the unital Jordan algebras  $U_e J$  and  $U_f J$  respectively.

*Proof.* Since any Jordan algebra is power associative, the subalgebra  $F[a]$  generated by  $a$  and  $1$  is isomorphic to  $F[x]/(m(x))$ , which is well known to be the direct sum of  $F[x]/(p(x))$  and  $F[x]/(q(x))$ , so  $F[a]$  is the direct sum of  $F[b]$  and  $F[c]$ ,  $1 = e + f$  with  $a = b + c$ , the minimum polynomials of  $b, c$  being  $p, q$  (as the cosets of  $x$  in the quotient mod  $p$  or  $q$ ). ■

Suppose now that  $J$  is a nondegenerate Jordan algebra which is reduced over an algebraically closed field  $F$ , and let  $a \in J$ . As pointed out in Remark 3.3,  $a$  has a Fitting decomposition  $a = a_1 + a_0$  associated to a unique idempotent  $e \in J$ , that is,  $a_1$  is invertible in  $U_e J$  and  $a_0 \in J_0(e)$  is nilpotent. Moreover, by Theorem 3.7,  $a_0$  has a Jordan canonical form

$a_0 = z_{01} + \dots + z_{0r_0}$  with indexes of nilpotence  $n_{01} \geq \dots \geq n_{0r_0} > 1$ . On the other hand,  $U_e J$  is a unital reduced Jordan algebra over  $F$ , and hence, by the structure theorem of these algebras [10, p. 203, Theorem 8], every element of  $U_e J$ , in particular  $a_1$ , is algebraic over  $F$ . Let  $m(x) := (x - \alpha_1)^{r_1} \dots (x - \alpha_k)^{r_k}$  be the minimum polynomial of  $a_1$  in  $U_e J$ , where  $\alpha_i \neq \alpha_j$  for  $i \neq j$  and  $r_i \geq 1$ . Since  $a_1$  is invertible in  $U_e J$ , the spectral values  $\{\alpha_1, \dots, \alpha_k\}$  are nonzero. Then, by Lemma 4.1,  $e = e_1 + \dots + e_k$  is a sum of nonzero orthogonal idempotents ( $e_i$  is the spectral idempotent relative to the spectral value  $\alpha_i$ ), and  $a_1 = b_1 + \dots + b_k$ , where each  $b_i \in U_{e_i} J$ , with  $(x - \alpha_i)^{r_i}$  the minimum polynomial of  $b_i$  in  $U_{e_i} J$ . Thus  $z_i := b_i - \alpha_i e_i$  is nilpotent of index of nilpotence  $r_i$ . Now we have by Remark 3.2 that each  $e_i = e_{i1} + \dots + e_{ir_i}$  is a sum of nonzero orthogonal idempotents and  $z_i = z_{i1} + \dots + z_{ir_i}$  with  $z_{ij} \in U_{e_{ij}} J$  nilpotent of index of nilpotence  $n_{ij}$ ,  $n_{i1} \geq \dots \geq n_{ir_i} \geq 1$  ( $i = 1, \dots, k$ ). Finally,  $b_i = \alpha_i e_{i1} + z_{i1} + \dots + \alpha_i e_{ir_i} + z_{ir_i}$ . Altogether, we have proved

**THEOREM 4.1.** *Let  $J$  be a nondegenerate Jordan algebra which is reduced over an algebraically closed field  $F$ . Then every element  $a \in J$  has a Jordan canonical form  $a = z_{01} + \dots + z_{0r_0} + \alpha_1 e_{11} + z_{11} + \dots + \alpha_1 e_{1r_1} + z_{1r_1} + \dots + \alpha_k e_{k1} + z_{k1} + \dots + \alpha_k e_{kr_k} + z_{kr_k}$  where*

- (1)  $a_0 = z_{01} + \dots + z_{0r_0}$  is a Jordan canonical form of the nilpotent part  $a_0$  of  $a$ .
- (2) the  $e_{ij}$  are nonzero orthogonal idempotents with  $e = \sum e_{ij}$  the Fitting idempotent of  $a$ ,
- (3)  $\{\alpha_1, \dots, \alpha_k\}$  is the spectrum of the locally invertible part  $a_1 \in U_e J$  given by the Fitting decomposition,
- (4) for each  $i = 1, \dots, k$ ,  $e_i = e_{i1} + \dots + e_{ir_i}$  is the spectral idempotent relative to the spectral value  $\alpha_i$ , and
- (5) for each  $i = 1, \dots, k$  and  $j = 1, \dots, r_i$ ,  $z_{ij} \in U_{e_{ij}} J$  is nilpotent with index of nilpotence  $n_{ij}$ ,  $n_{i1} \geq \dots \geq n_{ir_i} \geq 1$ .

**COROLLARY 4.1.** *Let  $J$  be a nondegenerate Jordan algebra which is reduced over an algebraically closed field  $F$ . If  $J$  does not contain any nonzero nilpotent element, then every  $a \in J$  is diagonalizable, that is,  $a = \alpha_1 e_1 + \dots + \alpha_k e_k$  where  $0 \neq \alpha_i \in F$  and  $\{e_1, \dots, e_k\}$  are nonzero orthogonal idempotents.*

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## REFERENCES

- 1 P. N. Ánh, Simple Jordan algebras with minimal inner ideals, *Comm. Algebra* 14:489–492 (1986).
- 2 A. Fernández López, Modular annihilator Jordan algebras, *Comm. Algebra* 13:2597–2613 (1985).
- 3 A. Fernández López and E. García Rus, A characterization of the elements of the socle of a Jordan algebra, *Proc. Amer. Math. Soc.* 108:69–71 (1990).
- 4 A. Fernández López and E. García Rus, Prime Jordan algebras satisfying local Goldie conditions, *J. Algebra* 174:1024–1048 (1995).
- 5 A. Fernández López, E. García Rus, and E. Sánchez Campos, Von Neumann regular Jordan Banach triple systems, *J. London Math. Soc.* 42:32–48 (1990).
- 6 A. Fernández López, E. García Rus, E. Sánchez Campos, and M. Siles Molina, Strong regularity and generalized inverses in Jordan systems, *Comm. Algebra* 20:1917–1936 (1992).
- 7 A. Fernández López and A. Rodríguez Palacios, On the socle of a noncommutative Jordan algebra, *Manuscripta Math.* 56:269–278 (1986).
- 8 I. N. Herstein, *Topics in Algebra*, 2nd ed., Wiley, 1975.
- 9 N. Jacobson, *Structure of Rings*, Colloq. Publ. 37, Amer. Math. Soc., Providence, 1964.
- 10 N. Jacobson, *Structure and Representations of Jordan Algebras*, Colloq. Publ. 39, Amer. Math. Soc., Providence, 1968.
- 11 N. Jacobson, *Structure Theory of Jordan Algebras*, Lecture Notes in Math. 5, Univ. of Arkansas, Fayetteville, 1981.
- 12 N. Jacobson, *Basic Algebra I*, Freeman, New York, 1985.
- 13 O. Loos, *Jordan Pairs*, Lecture Notes in Math. 460, Springer-Verlag, New York, 1975.
- 14 O. Loos, On the socle of a Jordan pair, *Collect. Math.* 40:109–125 (1989).
- 15 O. Loos, Fitting decomposition in Jordan systems, *J. Algebra* 136:92–102 (1991).
- 16 O. Loos, Diagonalization in Jordan pairs, *J. Algebra* 143:252–268 (1991).
- 17 K. McCrimmon, Peirce ideals in Jordan algebras, *Pacific J. Math.* 78:397–414 (1978).
- 18 K. McCrimmon, The Russian revolution in Jordan algebras, *Algebras Groups Geom.* 1:1–61 (1984).
- 19 K. McCrimmon, Reduced elements in Jordan triple systems, *J. Algebra* 92:540–564 (1985).
- 20 J. M. Osborn and M. L. Racine, Jordan rings with nonzero socle, *Trans. Amer. Math. Soc.* 251:375–387 (1979).
- 21 K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, and I. P. Shirshov, *Rings That Are Nearly Associative*, Pure Appl. Math. 104, Academic, New York, 1982.

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