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GOLDIE THEOREMS FOR ASSOCIATIVE PAIRS

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ABSTRACT. We develop a Goldie theory for associative pairs and characterize associative pairs which are orders in semiprime associative pairs coinciding with their socle, and those which are orders in semiprime artinian associative pairs.

§1. Introduction. Goldie's Theorem is certainly one of the fundamental results of the theory of (associative) rings. Today this theorem is usually formulated as follows: A ring $\mathcal{R}$ is a classical left order in a semisimple (equivalently, semiprime artinian) ring $\mathcal{Q}$ if and only if $\mathcal{R}$ is semiprime, left nonsingular, and does not contain infinite direct sums of left ideals. Moreover, $\mathcal{R}$ is prime if and only if $\mathcal{Q}$ is simple.

In 1990 J. Fountain and V. Gould [18] introduced a notion of order in a ring which need not have a unit, and gave [19] a Goldie-like characterization of two-sided orders in semiprime rings with descending chain condition (dcc) on principal one-sided ideals (equivalently, coinciding with their socle). Later P.N. Ánh and L. Márki [3] extended this result to one-sided orders.

The first two authors, jointly with E. Sánchez Campos, studied [16] a notion of local order for associative algebras equivalent to that of Fountain-Gould for orders in simple rings with minimal inner ideals, and proved an extension of Posner's theorem to prime algebras satisfying a generalized polynomial identity (result revisited in [4]).

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Recall that an associative pair (of the first kind) is a pair (A^+, A^-) of Φ-modules (Φ is an arbitrary unital commutative ring of scalars) equipped with trilinear maps $(x, y, z) \to xyz$ from $A^\sigma \times A^{-\sigma} \times A^\sigma$ in A^σ ($\sigma = \pm$) satisfying the following identities

$$uv(xyz) = u(vxy)z = (uvx)yz.$$

Every associative algebra $\mathcal{A}$ gives rise to an associative pair $A = (\mathcal{A}, \mathcal{A})$ under the triple product abc, where juxtaposition denotes the product of $\mathcal{A}$. A more interesting example is given by $B = (hom(M, N), hom(N, M))$, where M and N are left modules over a Φ-algebra $\mathcal{A}$, under the triple product abc where juxtaposition now denotes the mapping composition. If M and N are actually finite dimensional left vector spaces (of dimensions m and n respectively) over a division Φ-algebra Δ , then $B \cong (\mathrm{Mat}_{n\times m}(\Delta), \mathrm{Mat}_{m\times n}(\Delta))$ is a simple artinian associative pair, and conversely, every simple artinian associative pair is isomorphic to one of these (see [24] for the corresponding result for associative pairs of the second kind). Another step in the structure theory of associative pairs was the classification of prime associative pairs with minimal inner ideals [8,11].

Associative pairs play a fundamental role in the new approach (see [1]) to Zelmanov's classification of strongly prime Jordan pairs, and had been already used by O. Loos in the classification of the nondegenerate Jordan pairs of finite capacity [24].

The purpose of this paper is to develop a Goldie theory for associative pairs following the pattern of that for associative algebras (rings), but defining orders in associative pairs in the unique way which seems to be possible, namely, locally. Thus our definition of order in an associative pair is inspired by that of local order in associative algebras already cited, and even by its Jordan version introduced in a previous paper [12] by the first two authors.

There is a link between associative pairs and associative algebras. Let $A = (A^+, A^-)$ be an associative pair and b in A^-. Then the submodule bA^+b equipped with the multiplication defined by $(bxb) \cdot (byb) = bxbyb$ is an associative algebra called the *local algebra* of A at b (Section 5) and denoted by A_b. Analogous definition is given if b is in A^+. Note that if $b \in A^\pm$ is von Neumann regular, i.e., $b \in bA^\mp b$, then A_b is unital with b as the unit. By using local algebras we can define orders in associative algebras.

Suppose now that $A = (A^+, A^-)$ is a subpair of an associative pair $Q = (Q^+, Q^-)$. We say (Section 8) that A is an *order* in Q if for each $q \in Q^\pm$ there exists $b \in A^\pm$ such that (i) b is von Neumann regular in Q, (ii) $q \in bQ^\mp b$, and (iii) the local algebra A_b of A at b is a two-sided order in the unital associative algebra Q_b. This definition is consistent with the classical one for orders in associative algebras: Let $\mathcal{A}$ be a subalgebra of a unital associative algebra $\mathcal{Q}$. Then the associative pair $A = (\mathcal{A}, \mathcal{A})$ determined by

$\mathcal{A}$ is an order in the associative pair $Q = (\mathcal{Q}, \mathcal{Q})$ determined by $\mathcal{Q}$ if and only if $\mathcal{A}$ is an order in $\mathcal{Q}$. Moreover, it also extends the Fountain-Gould notion of order in simple associative algebras with minimal one-sided ideals. In a more purely pair context, if D is a two-sided order in an associative division algebra Δ, and m and n are positive integers, then the associative pair $(\mathrm{Mat}_{n\times m}(D), \mathrm{Mat}_{m\times n}(D))$ is an order in $(\mathrm{Mat}_{n\times m}(\Delta), \mathrm{Mat}_{m\times n}(\Delta))$.

We also introduce the notion of left (right) singular ideal (Section 3), and prove the following main result.

Theorem 8.10. *For an associative pair A the following conditions are equivalent:*

(i) *A is an order in a semiprime associative pair Q coinciding with its socle,*
(ii) *A is semiprime, satisfies the ascending chain condition on the left annihilators of single elements, $lan_A(x)$ for $x \in A^+$, and has finite both left and right local Goldie dimension,*
(iii) *A is a semiprime local Goldie associative pair,*
(iv) *A is semiprime and all its local algebras are Goldie.*

In this case,

(1) *A is prime if and only if Q is simple, and*
(2) *A is Goldie if and only if Q is artinian.*

A key tool in the proof of this theorem is the notion of standard imbedding (Section 4) of an associative pair (see [27]). This notion is used to prove the implication (iii) $\Rightarrow$ (i) of Theorem 8.10. We first show (Section 6) that a semiprime local Goldie associative pair A is an essential subdirect product of prime local Goldie associative pairs, which allows us to reduce the question to the case that A is prime. Then the standard imbedding $(\mathcal{A}, e)$ of A is a prime nonsingular algebra and such that the set $I(\mathcal{A})$ of those elements of $\mathcal{A}$ having finite both left and right Goldie dimension is a nonzero ideal. In this situation, by a result of P. N. Ánh and L. Márki [4], $\mathcal{A}$ can be embedded in a prime associative algebra $\mathcal{Q}$ with minimal one-sided ideals such that $I(\mathcal{A})$ is a Fountain-Gould order in the socle $Soc(\mathcal{Q})$ of $\mathcal{Q}$. Hence we obtain that $A = (eI(\mathcal{A})(1-e), (1-e)I(\mathcal{A})e)$ is an order in the simple associative pair with minimal inner ideals $Q = (eSoc(\mathcal{Q})(1-e), (1-e)Soc(\mathcal{Q})e)$.

Another important fact in the proof of this theorem is the local characterization of certain properties of associative pairs. Primeness, nonsingularity, left (right) local Goldie dimension, and coincidence with the socle are properties of an associative pair which can be characterized in terms of its local algebras. This technique of using local algebras to pass information back and forth between pairs and algebras has been usefully used in the current structure theory of Jordan systems (algebras, triples and pairs) [2, 5, 6, 7, 28].

§2. Basic notions and associative pairs with chain conditions on inner ideals. Throughout this paper Φ will denote a unital commutative associative ring of scalars. Let $A = (A^+, A^-)$ be a pair of Φ-modules, equipped with trilinear maps $(x, y, z) \to xyz$ from $A^\sigma \times A^{-\sigma} \times A^\sigma$ in A^σ $(\sigma = \pm)$. Then A is called an *associative pair* if the identities

$$(1) \qquad uv(xyz) = u(vxy)z = (uvx)yz$$

are satisfied. We define left, middle and right multiplications by

$$(2) \qquad \lambda(x,y)z = \mu(x,z)y = \rho(y,z)x = xyz.$$

It follows from (1)

$$(3) \qquad \lambda(u,v)\lambda(x,y) = \lambda(uvx, y) = \lambda(u, vxy)$$

and similarly

$$(4) \qquad \rho(x,y)\rho(u,v) = \rho(uvx, y) = \rho(u, vxy).$$

Hence it is clear that the linear span of all operators $T : A^\sigma \to A^\sigma$ of the form $T = \lambda(a,b)$ or $T = Id_{A^\sigma}$ $(a \in A^\sigma,\ b \in A^{-\sigma})$ is a unital associative algebra denoted by $\Lambda(A^\sigma, A^{-\sigma})$ and A^σ is clearly a left $\Lambda(A^\sigma, A^{-\sigma})$-module. Similarly it is defined $\Pi(A^{-\sigma}, A^\sigma)$ as the linear span of all the right multiplications plus the identity on A^σ. Then A^σ becomes a left $\Pi(A^{-\sigma}, A^\sigma)$-module. This allows us to apply results of modules to associative pairs. The left ideals $L \subset A^\sigma$ of A are precisely the left $\Lambda(A^\sigma, A^{-\sigma})$-submodules of A^σ, and the right ideals $R \subset A^\sigma$ are the left $\Pi(A^{-\sigma}, A^\sigma)$-submodules, while a two-sided ideal $B \subset A^\sigma$ is a left and right ideal.

A left ideal U of A contained in A^σ is *uniform* if it is so as $\Lambda(A^\sigma, A^{-\sigma})$-module, i.e., any nonzero left ideals L, M of A contained in U have nonzero intersection. An element $a \in A^\sigma$ is *l-uniform* if the principal left ideal $A^\sigma A^{-\sigma} a$ generated by a is uniform. Let $L \subset A^\sigma$ be a left ideal of A which does not contain infinite direct sums of nonzero left ideals. By [20, Prop. 3.19], there exists a nonnegative integer n, called the *left Goldie (or uniform) dimension* of L, such that L contains a direct sum of n nonzero left ideals and any direct sum of nonzero left ideals contained in L has at most n summands (notice that any direct sum of n nonzero left ideals is essential in the sense that it intersects any nonzero left ideal contained in L and its summands are necessarily uniform). Now a uniform nonzero left ideal is just a nonzero left ideal of left Goldie dimension one. If both A^+ and A^- have finite left Goldie dimension then we will say that the whole pair A has *finite left Goldie (or uniform) dimension*. The *left Goldie (or uniform) dimension* of an element

$a \in A^\sigma$ is the left Goldie dimension of the principal left ideal generated by a. If any element of A has finite left Goldie dimension, we will say that A has *finite left local Goldie dimension.*

An *ideal* I of A is a pair $I = (I^+, I^-)$ of two-sided ideals, $I^\sigma \subset A^\sigma$, such that $A^\sigma I^{-\sigma} A^\sigma \subset I^\sigma$ ($\sigma = \pm$). If $B = (B^+, B^-)$ and $C = (C^+, C^-)$ are ideals of A, then

$$B \star C := (B^+A^-C^+ + A^+B^-A^+C^-A^+, B^-A^+C^- + A^-B^+A^-C^+A^-)$$

is an ideal of A, and we have that the lattice $\mathcal{L}(A)$ of all ideals of A is an *algebraic lattice* in the sense of [15].

An associative pair A is *semiprime* if and only if $I^\sigma A^{-\sigma} I^\sigma = 0$, $\sigma = \pm$, implies $I = 0$ for I ideal of A, equivalently, $I \star I = 0$ implies $I = 0$, while A is *prime* if and only if $I^\sigma A^{-\sigma} J^\sigma = 0$, $\sigma = \pm$, implies $I = 0$ or $J = 0$, for I and J ideals of A, equivalently, $I \star J = 0$ implies $I = 0$ or $J = 0$.

Clearly an element a in A^+ gives rise to an ideal $I = (I^+, I^-)$ of A by taking $I^+ := \Phi a + A^+A^-a + aA^-A^+ + A^+A^-aA^-A^+$ and $I^- := A^-aA^-$. This allows us to obtain elemental characterizations of semiprimeness and primeness (see [1, 1.18]): A is semiprime if and only if A is *nondegenerate* ($aA^\sigma a = 0$ implies $a = 0$), and A is prime if and only if A is *elementally prime* ($aA^\sigma b = 0$ implies $a = 0$ or $b = 0$, $a, b \in A^{-\sigma}$).

Each associative algebra $\mathcal{A}$ gives rise to an associative pair $A = (\mathcal{A}, \mathcal{A})$ under the triple product abc where juxtaposition denotes the associative product of $\mathcal{A}$. Similarly, each associative pair A becomes a Jordan pair, denoted by A^J, with quadratic maps $Q(x)y = xyx$. This will allow us to apply Jordan-theoric results to associative pairs. We can transfer the Jordan notion of inner ideal to associative pairs. An *inner ideal* K of A contained in A^σ is a Φ-submodule of A^σ such that $xA^{-\sigma}x \subset K$ for any $x \in K$. Note that if L is a left ideal and R is a right ideal of A, both contained in the same A^σ, then $L \cap R$ is an inner ideal. Now, the elemental characterization of semiprimeness reads: an associative pair A is semiprime if and only if the Jordan pair A^J is nondegenerate.

Let $X \subset A^\sigma$. The *left annihilator* of X in A is defined to be the set

$$\{b \in A^{-\sigma} : bXA^{-\sigma} = 0 = A^\sigma bX\}$$

written $lan(X)$ or $lan_A(X)$ when it is necessary to emphasize the dependence on A. Similarly, the *right annihilator* $ran_A(X) = ran(X)$ of X is defined by

$$\{b \in A^{-\sigma} : XbA^\sigma = 0 = A^{-\sigma}Xb\}$$

We also write $ann_A(X) = ann(X) := lan(X) \cap ran(X)$ to denote the *anni-*

hilator of X. Clearly, $lan(X)$ is a left ideal of A, $ran(X)$ is a right ideal, and $ann(X)$ is an inner ideal of A [1,1.15].

Lemma 2.1. *Let A be a semiprime associative pair. For $a \in A^{\sigma}$, $b \in A^{-\sigma}$ the following conditions are equivalent:*

(i) $\lambda(a,b) = 0$,
(ii) $\rho(a,b) = 0$,
(iii) $a \in lan(b)$,
(iv) $b \in ran(a)$.

Proof. Since (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (i) + (ii) is straightforward from the definitions of $lan(b)$ and $ran(b)$, we just need to prove the equivalence (i) $\Leftrightarrow$ (ii). Suppose $\rho(a,b) \neq 0$. Then $xab \neq 0$ for some $x \in A^{-\sigma}$ implies, by nondegeneracy of A, that $xabcxab \neq 0$ for some $c \in A^{\sigma}$. Hence $\lambda(a,b)c \neq 0$. Similarly, $\lambda(a,b) \neq 0$ implies $\rho(a,b) \neq 0$. □

Now we record some properties of annihilators in semiprime associative pairs which will be used later. Note first that if $I = (I^+, I^-)$ is an ideal of a semiprime (equivalently, nondegenerate) associative pair A, then $I^+ = 0$ if and only if $I^- = 0$. Indeed, $I^+ = 0 \Rightarrow A^+I^-A^+ \subset I^+ = 0 \Rightarrow (I^-A^+I^-)A^+(I^-A^+I^-) = I^-(A^+I^-A^+)I^-A^+I^- = 0$, hence $I^-A^+I^- = 0$, and $I^- = 0$, using twice the nondegeneracy of A.

Proposition 2.2. *Let A be a semiprime associative pair, $B \subset A^+$ a two-sided ideal of A, and $I = (I^+, I^-)$ be an ideal of A. Then*

(i) *$lan(B) = ran(B) = ann(B)$ is a two-sided ideal of A.*
(ii) *$ann(B) = \{z \in A^- : zBz = 0\}$.*
(iii) *$I^{\sigma} \cap ann(I^{-\sigma}) = 0$.*
(iv) *$ann(I) := (ann(I^-), ann(I^+))$ is an ideal of A called the annihilator ideal of I.*
(v) *$A/ann(I)$ is a semiprime associative pair.*

Proof. (i) Let $z \in lan(B)$. For any $a \in A^-$ and $b \in B$, we have

$$(abz)A^+(abz) = abz(A^+ab)z \subset abzBA^+ = 0.$$

By nondegeneracy of A, $A^-Bz = 0$, which implies $z \in ran(B)$ by Lemma 2.1. This shows $lan(B) \subset ran(B)$. Similarly, $ran(B) \subset lan(B)$.
(ii) $zBz = 0$ implies, for any $a \in A^-$ and $b \in B$,

$$(zba)A^+(zba) = (z(baA^+)z)ba \subset (zBz)ba = 0,$$

and $zBA^- = 0$ by nondegeneracy of A. Hence, $z \in lan(B)$ by Lemma 2.1 and $lan(B) = ann(B)$ by (i).
Now (iii), (iv) and (v) follow from (ii) applied to the two-sided ideals $I^{\pm}$. □

As a direct consequence of the preceding proposition, we obtain that for two ideals B and C of a semiprime associative pair A, the following are equivalent: (a) $B \star C = 0$, (b) $B \cap C = 0$, and (c) $B \subset ann(C)$. Indeed, the equivalence (a) $\Leftrightarrow$ (b) follows from [15, 2.1] applied to the semiprime algebraic lattice $(\mathcal{L}(A), \star)$. Moreover, (a) $\Leftrightarrow$ (c) shows that $ann(C)$ coincides with the annihilator $C^{\perp}$ defined in $(\mathcal{L}(A), \star)$ (see [15, p.2]). Another useful property of annihilators is the following.

Lemma 2.3. *Let I be an ideal of a semiprime associative pair A. For any subset M of I^+, $lan_I(M) = lan_A(M) \cap I^-$.*

Proof. Clearly $lan_A(M) \cap I^-$ is contained in $lan_I(M)$. Let $z \in lan_I(M)$. By Lemma 2.1, we just need to show that $zMA^- = 0$. If $m \in M$ and $a \in A^-$, then $zmaA^+zma \subset zmI^- = 0$, which implies $zma = 0$ by semiprimeness of A, as required. □

Following [22, p.70], let $(X, X', <.,.>)$ and $(Y, Y', <.,.>)$ be two dual pairs of vector spaces over an associative division Φ-algebra Δ. An operator $a : X \to Y$ is *adjointable* if there exists $a^\# : Y' \to X'$, necessarily unique, such that $<xa, y'> = <x, a^\# y'>$. Notice that we write the mappings of a left vector space on the right (thus composing them from left to right), and the mappings of a right vector space on the left (thus composing them from right to left). We denote by $\mathcal{L}(X, Y)$ the set of all adjointable linear operators of X to Y, and by $\mathcal{F}(X, Y)$ the subset of those operators having finite rank.

For $x' \in X', y \in Y$, write $x' \otimes y$ to denote the adjointable linear operator from X to Y defined by $x(x' \otimes y) = <x, x'> y$ for $x \in X$ with adjoint $(x' \otimes y)^\# y' = x' <y, y'>$. Note that $(x' \otimes y)b = x' \otimes yb$ for all operator b from Y to X, and $a(x' \otimes y) = a^\# x' \otimes y$ for all adjointable $a \in \mathcal{L}(Y, X)$. Every $a \in \mathcal{F}(X, Y)$ can be expressed as $a = \sum x'_i \otimes y_i$, where both $\{x'_i\} \subset X'$ and $\{y_i\} \subset Y$ are linearly independent, which just means that $\mathcal{F}(X, Y)$ is isomorphic as Φ-module to the tensor product $X' \otimes_\Delta Y$ (see [22, Prop. 1, p.74] and [14, p.3]). We have that

$$(\mathcal{F}(X, Y), \mathcal{F}(Y, X)) = (X' \otimes_\Delta Y, Y' \otimes_\Delta X)$$

is an associative pair under the triple products abc with juxtaposition denoting the mapping composition. Actually, by [8] or [11], an associative pair is simple with minimal inner ideals if and only it is isomorphic to one of these.

Proposition 2.4. *Let $A = (\mathcal{F}(X, Y), \mathcal{F}(Y, X))$ be the associative pair defined by two dual pairs $(X, X', <.,.>)$ and $(Y, Y', <.,.>)$ of vector spaces over an associative division Φ-algebra Δ.*

(i) *For V' and W, subspaces of X' and Y respectively, we have that $V' \otimes W$ is an inner ideal of A, $X' \otimes W$ is a left ideal, and $V' \otimes Y$ is a right ideal, all of them contained in A^+. Conversely, for any inner ideal I of A contained in $A^+ = \mathcal{F}(X, Y)$, we have that $I^\# Y'$ and XI*

are subspaces of X' *and* Y *respectively, and* $I = I^{\#}Y' \otimes XI$*. Similarly, left ideals* L *of* A *contained in* A^+ *are of the form* $L = X' \otimes XL$*, and right ideals* R *of* A *contained in* A^+ *are of the form* $R = R^{\#}Y' \otimes Y$*.*

(ii) *If a left ideal* $L \subset A^+$ *is principal, then* XL *is finite dimensional and the converse, i.e.,* XL *being finite dimensional implies* L *is principal, holds when* $dimXL \leq dimX$*.*
If a right ideal $R \subset A^+$ *is principal, then* $R^{\#}Y'$ *is finite dimensional, and the converse is true when* $R^{\#}Y' \leq dimY$*.*
An inner ideal $I \subset A^+$ *is principal if and only if* XI *and* $I^{\#}Y'$ *have the same finite dimension.*

(iii) *For a left ideal* $L \subset A^+$*,* $ran(L) = (XL)^{\perp} \otimes X$*, for a right ideal* $R \subset A^+$*,* $lan(R) = Y' \otimes (R^{\#}Y')^{\perp}$*, and for an inner ideal* $I \subset A^+$*,* $ann(I) = (XI)^{\perp} \otimes (I^{\#}Y')^{\perp}$*.*

(iv) *If* L *is a principal left ideal of* A*, then* $lan(ran(L)) = L$*. Similarly,* $ran(lan(R)) = R$ *for a principal right ideal* R *of* A*, while a principal inner ideal* $I \subset A^{\pm}$ *coincides with its double annihilator if and only if* $ann(I) \neq 0$ *or* $I = A^{\pm}$*.*

Moreover, the following conditions are equivalent:

(a) *A satisfies the descending chain condition (dcc) on all inner ideals,*
(b) *A satisfies the ascending chain condition (acc) on all inner ideals,*
(c) *A satisfies the dcc on all left ideals,*
(d) *A satisfies the acc on all left ideals,*
(e) *A satisfies the dcc on all right ideals,*
(f) *A satisfies the acc on all right ideals,*
(g) *both vector spaces* X *and* Y *are finite dimensional, and hence* A *is isomorphic to* $(Mat_{n\times m}(\Delta), Mat_{m\times n}(\Delta))$ *for some positive integers* n *and* m*.*

Proof. (i) Note first that $V' \otimes W$ is an inner ideal of A for any V' and W subspaces of X' and Y respectively. Conversely, let I be an inner ideal of A contained in $A^+ = \mathcal{F}(X,Y)$. As in [14, Theorem 3], we have that the sets

$$XI = \{xa : x \in X,\, a \in I\} \text{ and } I^{\#}Y' = \{a^{\#}y' : y' \in Y',\, a \in I\}$$

are subspaces of Y and X' respectively, and $I = I^{\#}Y' \otimes XI$. The assertions concerning left and right ideals can be proved similarly.
(ii) can be shown as (17) of [14].
(iii) Let $I = I^{\#}Y' \otimes XI$ be an inner ideal of A contained in A^+. We will show $ann(I) = (XI)^{\perp} \otimes (I^{\#}Y')^{\perp}$, where as usual, $V'^{\perp}$ is the orthogonal of the subspace V' of X' relative to the dual pair (X, X'), and $W^{\perp}$ is the orthogonal of the subspace W of Y relative to (Y, Y'): Since $J = ann(I)$ is an inner ideal of A, we have by the dual of (i) that $ann(I) = J^{\#}X' \otimes YJ$. We

claim that $J^{\#}X' \subset (XI)^{\perp}$ and $YJ \subset (I^{\#}Y')^{\perp}$. Without loss of generality, we may assume that I is nonzero. Then

$$\begin{aligned}((XI)^{\perp} \otimes (I^{\#}Y')^{\perp})IA^{-} &= ((XI)^{\perp} \otimes (I^{\#}Y')^{\perp})(I^{\#}Y' \otimes XI)(Y' \otimes X) \\ &= (XI)^{\perp} \otimes < (I^{\#}Y')^{\perp}, I^{\#}Y' >< XI, Y' > X = 0\end{aligned}$$

implies $(XI)^{\perp} \otimes (I^{\#}Y')^{\perp} \subset lan(I)$ by Lemma 2.1. Analogously,

$$A^{-}I((XI)^{\perp} \otimes (I^{\#}Y')^{\perp}) = 0$$

implies $(XI)^{\perp} \otimes (I^{\#}Y')^{\perp} \subset ran(I)$. On the other hand,

$$\begin{aligned}0 &= (J^{\#}X' \otimes YJ)IA^{-} = (J^{\#}X' \otimes YJ)(I^{\#}Y' \otimes XI)(Y' \otimes X) \\ &= J^{\#}X' \otimes < YJ, I^{\#}Y' >< XI, Y' > X\end{aligned}$$

implies $YJ \subset (I^{\#}Y')^{\perp}$ since I is nonzero. Similarly, $J^{\#}X' \subset (XI)^{\perp}$.

The corresponding results for left and right ideals are easier.
(iv) follows from (iii), using the fact that any finite dimensional subspace coincides with its double orthogonal.

Suppose now that $I_1 = I_1^{\#}Y' \otimes XI_1$ and $I_2 = I_2^{\#}Y' \otimes XI_2$ are inner ideals of A contained in A^{+}. It is obvious, using (i), that $I_1 \subset I_2$, $I_1 \neq I_2$ if and only if $I_1^{\#}Y' \subset I_2^{\#}Y'$, $XI_1 \subset XI_2$ and either $I_1^{\#}Y' \neq I_2^{\#}Y'$ or $XI_1 \neq XI_2$. Hence A satisfies the dcc (equivalently, acc) on all inner ideals if and only if both vector spaces X and Y are finite dimensional. This proves the equivalences (a)$\Leftrightarrow$(b)$\Leftrightarrow$(g). Similarly, by using the geometrical representation given in (i) of left (right) ideals of A contained in A^{+}, and of left (right) ideals of A contained in A^{-}, we obtain the remaining equivalences. □

For a semiprime associative pair A the *socle* $Soc(A) = (Soc(A)^{+}, Soc(A)^{-})$ is by definition the socle of A^{J}, i.e., $Soc(A)^{\sigma}$ is the sum of all minimal inner ideals of A^{σ}, $\sigma = \pm$. By [9, Theorem 1] $Soc(A)$ is a direct sum of simple ideals of A each of which contains a minimal inner ideal. It follows from [25] that a semiprime associative pair A coincides with its socle if and only if it satisfies the dcc on principal inner ideals. Moreover, $Soc(A)$ is a von Neumann regular ideal. If $\mathcal{A}$ is a semiprime associative algebra, then $Soc(\mathcal{A})^{+} = Soc(\mathcal{A})^{-}$ coincides with the usual associative socle of $\mathcal{A}$ (see [10]).

Let $A = \oplus M_i$ be a semiprime associative pair coinciding with its socle, where the M_i are its simple components. By using the von Neumann regularity of A, it is easy to see that any inner (left, right) ideal K of A is a direct sum $K = \oplus K_i$ of inner (left, right) ideals K_i of the M_i. Moreover, if K is principal then only finitely many K_i are nonzero.

The above results yield the following theorem we include for completeness.

Theorem 2.5. *Let Q be a semiprime associative pair satisfying the dcc on principal inner ideals (equivalently coinciding with its socle). Then*

(1) *Q satisfies the following chain conditions:*

(i) *the dcc on principal left ideals,*
(ii) *the dcc on principal left ideals,*
(iii) *the acc on $ann(x)$, $lan(x)$, and $ran(x)$ for $x \in A^\sigma$ ($\sigma = \pm$).*

(2) *Q is a direct sum of simple ideals each of which is isomorphic to an associative pair $(\mathcal{F}(X,Y), \mathcal{F}(Y,X))$ defined by two dual pairs of vector spaces over a division Φ-algebra Δ.*

(3) *The following conditions are equivalent for Q*

(i) *Q satisfies the dcc on all inner ideals,*
(ii) *Q satisfies the dcc on all left ideals,*
(iii) *Q satisfies the dcc on all right ideals,*
(iv) *Q satisfies the acc on all inner ideals.*
(v) *Q satisfies the acc on all left ideals,*
(vi) *Q satisfies the acc on all right ideals.*

Moreover, any of the conditions of (3) is equivalent to

(4) *Q is a direct sum of finitely many simple ideals each of which is isomorphic to $(Mat_{n\times m}(\Delta), Mat_{m\times n}(\Delta))$ where Δ is a division Φ-algebra and m, n are positive integers.*

Associative pairs satisfying conditions (i)-(vi) of (3) are called *artinian.*

Proposition 2.6. *Let Q be a semiprime associative pair with dcc on principal inner ideals (equivalently, coinciding with its socle). If Q is not artinian then Q has infinite both left and right Goldie dimension.*

Proof. If Q is not artinian then Q contains infinitely many simple ideals or Q contains a simple ideal M which is not artinian. In the first case it is clear that Q has infinite direct sums of left ideals and of right ideals. In the second case, by Proposition 2.4, $M = (\mathcal{F}(X,Y), \mathcal{F}(Y,X))$ where some of the vector spaces X or Y is infinite dimensional. Hence, by Proposition 2.4(i) and (ii), we can construct an infinite direct sum of left (right) ideals of M, and therefore of A as well. □

§3. The singular ideal of a semiprime associative pair. The notion of nonsingularity proved particularly useful in the general theory of quotient rings initiated by Y. Utumi in 1956. Here we show the existence of a singular ideal in any semiprime associative pair A and study its properties. Set $Z_l(A^\sigma) = \{z \in A^\sigma : lan(z)$ is an essential left ideal of $A\}$.

Theorem 3.1. For a semiprime associative pair A, $(Z_l(A^+), Z_l(A^-))$ is an *ideal of A called the* left singular ideal *of A and denoted by $Z_l(A)$.*

Proof. It is easy to see that $Z_l(A^+)$ is a right ideal. Now let $z \in Z_l(A^+)$, $x \in A^+$, and $y \in A^-$. We must show that $lan(xyz)$ is an essential left ideal. Let L be a nonzero left ideal of A contained in A^-. If $Lxy = 0$ then $A^+Lxyz = 0$ and hence $L \subset lan(xyz)$ by Lemma 2.1. If $Lxy \neq 0$ then $Lxy \cap lan(z) \neq 0$. Taking $0 \neq lxy \in lan(z)$, with $l \in L$, we have $0 \neq l \in lan(xyz) \cap L$. We have shown $xyz \in Z_l(A^+)$. Therefore $Z_l(A^+)$ is a two-sided ideal. Similarly $Z_l(A^-)$ is a two-sided ideal. It remains to prove that the pair $(Z_l(A^+), Z_l(A^-))$ is invariant under middle multiplications. Let $z \in Z_l(A^+)$ and $y_1, y_2 \in A^-$, and take a nonzero left ideal L of A contained in A^+. If $A^-Ly_1 = 0$ then, by Lemma 2.1, $L \subset lan(y_1) \subset lan(y_1zy_2)$. If $A^-ly_1 \neq 0$ for some $l \in L$, then $A^-ly_1 \cap lan(z) \neq 0$. Take $0 \neq aly_1 \in lan(z)$, where $a \in A^-$. By semiprimeness of A, there exists $b \in A^+$ such that $bal \neq 0$, but $aly_1 \in lan(z)$ implies $bal \in lan(y_1zy_2) \cap L$. In both cases $L \cap lan(y_1zy_2) \neq 0$, so $lan(y_1zy_2)$ is essential, which completes the proof. □

A semiprime associative pair $A = (A^+, A^-)$ will be called *left nonsingular* if its left singular ideal $Z_l(A) = 0$. *Right nonsingular* pairs are defined similarly, while *nonsingular* means that A is both left and right nonsingular.

Proposition 3.2. *Let I be an ideal of a semiprime associative pair A. Then $Z_l(A) \cap I = Z_l(I)$.*

Proof. We may assume $I \neq 0$. Let $z \in Z_l(A)^+ \cap I^+$. For any nonzero left ideal L of I contained in I^-, I^-I^+L is a left ideal of A contained in $L \subset I^-$ which is nonzero by semiprimeness of A (otherwise $(LA^+L)A^+(LA^+L) = L(A^+LA^+LA^+)L \subset I^-I^+L = 0 \Rightarrow LA^+L = 0 \Rightarrow L = 0$). So, by Lemma 2.3, $0 \neq lan_A(z) \cap I^-I^+L \subset lan_I(z) \cap L$, which implies $z \in Z_l(I)^+$. Conversely, let $z \in Z_l(I)^+$ and $0 \neq a \in A^-$. We will show that $A^+A^-a \cap lan_A(z) \neq 0$, which obviously implies that any nonzero left ideal of A contained in A^+ hits $lan_A(z)$. If $a \in lan_A(z)$ we have finished. Suppose then $a \notin lan_A(z)$. By Lemma 2.1 $A^+az \neq 0$. Hence by semiprimeness of A, $0 \neq azA^-A^+a \subset I^-A^+a$ which is a left ideal of I. Since $z \in Z_l(I)^+$, we have $0 \neq I^-A^+a \cap lan_I(z) \subset A^-A^+a \cap lan_A(z)$ by Lemma 2.3, which completes the proof. □

An ideal $I = (I^+, I^-)$ of an associative pair A is called *essential* if $I \cap J \neq 0$ for any nonzero ideal J of A. If A is semiprime, we have by Proposition 2.2 that I is an essential ideal if and only if I^+ and I^- are essential left ideals.

Corollary 3.3. *Let A be a semiprime associative pair and I an essential ideal of A. Then A is left nonsingular if and only if I is left nonsingular.*

Proof. By Proposition 3.2, $Z_l(I) \neq 0$ implies $Z_l(A) \neq 0$. Assume, conversely, that I is left nonsingular. Now $Z_l(A) \cap I = 0$ implies $Z_l(A) = 0$ since I is an essential ideal of A. □

Proposition 3.4. *Let A be a semiprime associative pair and let $0 \neq a \in A^+$ be von Neumann regular. Then a is not in $Z_l(A)^+$.*

Proof. Suppose, on the contrary, that $a \in Z_l(A)^+$ and let $b \in A^-$ be such that $a = aba$. Thus A^-ab is a nonzero left ideal of A and hence $0 \neq xab \in$

$A^-ab \cap lan(a)$ for some $x \in A^-$ which leads to a contradiction since $xab = xabab = 0$ because $xab \in lan(a)$. □

Corollary 3.5. *Let A be a semiprime associative pair whose socle is essential. Then A is nonsingular.*

Proof. Since $Soc(A)$ is von Neumann regular [25, Theorem 1], $Z_l(Soc(V)) = 0$ by Proposition 3.4, which implies $Z_l(A) = 0$ by Corollary 3.3. □

Proposition 3.6. *Let A be a semiprime associative pair satisfying the acc on left annihilators $lan(a)$ $(a \in A^+)$. Then A is nonsingular.*

Proof. Let us first show that A is left nonsingular. Otherwise both $Z_l(A)^+$ and $Z_l(A)^-$ are nonzero by semiprimeness of A. Then we can take a nonzero element $x \in Z_l(A)^+$ with $lan(x)$ maximal in the set $\{lan(y) : 0 \neq y \in Z_l(A)^+\}$. Let $a \in A^-$ be such that $xax \neq 0$. Then $A^-xa \neq 0$ and hence there exists $0 \neq zxa \in lan(x)$ for some $z \in A^-$, which implies by Lemma 2.1 that $z \in lan(xax)$, with $z \notin lan(x)$, which contradicts the maximality of $lan(x)$.

Suppose now $Z_r(A)^+ \neq 0$ and take $0 \neq x \in Z_r(A)^+$ with $lan(x)$ maximal in the set $\{lan(y) : 0 \neq y \in Z_r(A)^+\}$. By semiprimeness of A, there exist $a, b \in A^-$ such that $xaxbxax \neq 0$. Hence axA^- is a nonzero right ideal and $axA^- \cap ran(x) \neq 0$. Let $0 \neq axz \in ran(x)$ with $z \in A^-$ and take $c \in A^+$ such that $axzcaxz \neq 0$. Then $bxaxzcA^- = 0$ implies (by Lemma 2.1 again) $bxa \in lan(xzc) = lan(x)$ since $lan(x)$ is maximal, which is a contradiction because $xaxbxax \neq 0$. □

§4. The standard imbedding of an associative pair. Let $\mathcal{A}$ be a unital associative algebra. Consider the Peirce decomposition $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$ of $\mathcal{A}$ with respect to an idempotent e, and denote by $\pi_{ij} : \mathcal{A} \to \mathcal{A}_{ij}$ the corresponding Peirce projections (see [24, p.92]). For any set $X \subset \mathcal{A}$ we put $X_{ij} := \pi_{ij}(X)$. Then $(\mathcal{A}_{12}, \mathcal{A}_{21})$ is an associative pair with the usual associative triple product. Conversely, every associative pair $A = (A^+, A^-)$ can be obtained in this way (see [27, (2.3)]), i.e., there exists a unital associative algebra $\mathcal{A}$ with an idempotent e such that A is isomorphic to the associative pair $(\mathcal{A}_{12}, \mathcal{A}_{21})$ defined above, where $\mathcal{A}_{11}$ ($\mathcal{A}_{22}$ respectively) is spanned by e and all products $x_{12}y_{21}$ ($1-e$ and all products $y_{21}x_{12}$ respectively) for $x_{12} \in \mathcal{A}_{12}$, $y_{21} \in \mathcal{A}_{21}$, and has the property that

$$(5)\qquad \begin{aligned} x_{11}\mathcal{A}_{12} = \mathcal{A}_{21}x_{11} = 0 &\text{ implies } x_{11} = 0, \text{ and} \\ x_{22}\mathcal{A}_{21} = \mathcal{A}_{12}x_{22} = 0 &\text{ implies } x_{22} = 0. \end{aligned}$$

The pair $(\mathcal{A}, e)$ is called the *standard imbedding* of A. If A is semiprime then (5) is equivalent to

$$(6)\qquad \begin{aligned} x_{11}\mathcal{A}_{12} = 0 &\text{ implies } x_{11} = 0, \text{ and} \\ x_{22}\mathcal{A}_{21} = 0 &\text{ implies } x_{22} = 0, \end{aligned}$$

or to

$$(7) \qquad \begin{aligned} \mathcal{A}_{21}x_{11} = 0 \text{ implies } x_{11} = 0, \text{ and} \\ \mathcal{A}_{12}x_{22} = 0 \text{ implies } x_{22} = 0. \end{aligned}$$

Proposition 4.1. *Let $A = (A^+, A^-)$ be an associative pair with standard imbedding $(\mathcal{A}, e)$.*

(i) *If $\mathcal{I}$ is a nonzero ideal of $\mathcal{A}$, then $(\mathcal{I}_{12}, \mathcal{I}_{21}) = (\mathcal{I} \cap A^+, \mathcal{I} \cap A^-)$ is a nonzero ideal of A.*

(ii) *Let L be a left ideal of A contained in A^+. Then $\mathcal{L} := L \oplus \mathcal{A}_{21}L$ is a left ideal of $\mathcal{A}$. Moreover, $\mathcal{L} = 0$ if and only if $L = 0$, and for L_1 and L_2 left ideals of A contained in A^+, L_1 is strictly contained in L_2 if and only if $\mathcal{L}_1$ is strictly contained in $\mathcal{L}_2$.*

Suppose now that A is semiprime.

(iii) *If $\mathcal{L}$ is a left ideal of $\mathcal{A}$ then $\mathcal{L}_{12}$ and $\mathcal{L}_{21}$ are left ideals of A. Moreover, $\mathcal{L} = 0$ if and only if $\mathcal{L}_{12} = 0$ and $\mathcal{L}_{21} = 0$.*

(iv) *Let $\mathcal{L}$ be a nonzero left ideal of $\mathcal{A}$ contained in $\mathcal{A}_{12} \oplus \mathcal{A}_{22}$. Then $\mathcal{L}_{12} = \mathcal{L} \cap \mathcal{A}_{12}$ is a nonzero left ideal of A.*

(v) *Let L and M be left ideals of A contained in A^+, and let $\mathcal{L}$ and $\mathcal{M}$ be the corresponding left ideals of $\mathcal{A}$. Then $\mathcal{L} \cap \mathcal{M} = 0$ if and only if $L \cap M = 0$.*

Proof. (i) First we note that $\mathcal{I} \cap A^+ = \mathcal{I} \cap \mathcal{A}_{12} = \mathcal{I}_{12}$ since $\mathcal{I}$ is an ideal of $\mathcal{A}$, and similarly $\mathcal{I} \cap A^- = \mathcal{I}_{21}$. If $\mathcal{I}_{12} = \mathcal{I}_{21} = 0$ then for any $x \in \mathcal{I}$, $x = x_{11} + x_{22}$; but $\mathcal{A}_{21}x_{11} = \mathcal{A}_{21}x \subset \mathcal{I} \cap \mathcal{A}_{21} = \mathcal{I}_{21} = 0$, and $x_{11}\mathcal{A}_{12} = x\mathcal{A}_{12} \subset \mathcal{I} \cap \mathcal{A}_{12} = \mathcal{I}_{12} = 0$. Hence $x_{11} = 0$ by (5), and similarly $x_{22} = 0$.
(ii) Clearly, if L is a left ideal of A contained in A^+ then $\mathcal{L} := L \oplus \mathcal{A}_{21}L$ is the left ideal of $\mathcal{A}$ generated by L.

Suppose now that A is semiprime.
(iii) Let $\mathcal{L}$ be an ideal of $\mathcal{A}$. It is easy to see that $\mathcal{L}_{12}$ and $\mathcal{L}_{21}$ are left ideals of A. Indeed,

$$A^+A^-\mathcal{L}_{12} \subset \mathcal{A}_{11}\mathcal{L}_{12} = \mathcal{A}_{11}\pi_{12}(\mathcal{L}) = \pi_{12}(\mathcal{A}_{11}\mathcal{L}) \subset \pi_{12}(\mathcal{L}) = \mathcal{L}_{12}.$$

Now, if $\mathcal{L}_{12} = \mathcal{L}_{21} = 0$ then, for any $x \in \mathcal{L}$, $x = x_{11} + x_{22}$; but $\mathcal{A}_{21}x_{11} = \mathcal{A}_{21}x \subset \mathcal{L} \cap \mathcal{A}_{21} \subset \mathcal{L}_{21} = 0$. Hence $x_{11} = 0$ by (7), and similarly $x_{22} = 0$.
(iv) Let $\mathcal{L}$ be a nonzero left ideal of $\mathcal{A}$ contained in $\mathcal{A}_{12} \oplus \mathcal{A}_{22}$. Clearly $\mathcal{L}_{21} = 0$ and hence $\mathcal{L}_{12} \neq 0$ by (i). Moreover, $\mathcal{L}_{12} = e\mathcal{L} \subset \mathcal{L} \cap \mathcal{A}_{12} \subset \mathcal{L}_{12}$, which implies $\mathcal{L}_{12} = \mathcal{L} \cap \mathcal{A}_{12}$.
(v) Clearly, $\mathcal{L} \cap \mathcal{M} \supset (L \cap M) \oplus \mathcal{A}_{21}(L \cap M)$ for L and M left ideals of A contained in A^+. Hence, $\mathcal{L} \cap \mathcal{M} = 0$ implies $L \cap M = 0$. Suppose now that $L \cap M = 0$, and take $a_{12} + a_{22} \in \mathcal{L} \cap \mathcal{M}$. By the Peirce decomposition, $a_{12} \in L \cap M = 0$ and $a_{22} \in \mathcal{A}_{21}L \cap \mathcal{A}_{21}M$. Hence $\mathcal{A}_{12}a_{22} \in L \cap M = 0$. Thus $a_{22} = 0$ by (7), and $\mathcal{L} \cap \mathcal{M} = 0$, as required. □

Proposition 4.2. *Let $A = (A^+, A^-)$ be an associative pair with standard imbedding $(\mathcal{A}, e)$. Then A is semiprime (respectively prime) if and only if $\mathcal{A}$ is semiprime (respectively prime).*

Proof. It is easy to see that if $\mathcal{A}$ is semiprime then A is nondegenerate. If $x_{12}\mathcal{A}_{21}x_{12} = 0$ then, by the Peirce relations, $x_{12}\mathcal{A}x_{12} = x_{12}\mathcal{A}_{21}x_{12} = 0$, which implies $x_{12} = 0$ by semiprimeness of $\mathcal{A}$. Similarly, $x_{21}\mathcal{A}_{12}x_{21} = 0$ implies $x_{21} = 0$. Conversely, suppose that A is semiprime. Let $\mathcal{I}$ be an ideal of $\mathcal{A}$ such that $\mathcal{I}^2 = 0$. Then $(\mathcal{I} \cap A^+, \mathcal{I} \cap A^-)$ is an ideal of A satisfying $(\mathcal{I} \cap A^\pm)(\mathcal{I} \cap A^\mp)(\mathcal{I} \cap A^\pm) \subset \mathcal{I}^2 = 0$. Hence $(\mathcal{I} \cap A^+, \mathcal{I} \cap A^-) = 0$ by semiprimeness of A, which implies $\mathcal{I} = 0$ by Proposition 4.1.

Let $\mathcal{A}$ be prime. As in the first part of the proof, if $x_{12}\mathcal{A}_{21}y_{12} = 0$ then $x_{12}\mathcal{A}y_{12} = x_{12}\mathcal{A}_{21}y_{12} = 0$, which implies $x_{12} = 0$ or $y_{12} = 0$ by primeness of $\mathcal{A}$ and similarly $x_{21}\mathcal{A}_{12}y_{21} = 0$ implies $x_{21} = 0$ or $y_{21} = 0$, showing that A is prime.

Finally, suppose that A is prime. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals of $\mathcal{A}$ such that $\mathcal{I}\mathcal{J} = 0$. Then $(\mathcal{I} \cap A^+, \mathcal{I} \cap A^-)$ and $(\mathcal{J} \cap A^+, \mathcal{J} \cap A^-)$ are ideals of A satisfying $(\mathcal{I} \cap A^\pm)A^\mp(\mathcal{J} \cap A^\pm) \subset \mathcal{I}\mathcal{J} = 0$. Hence $(\mathcal{I} \cap A^+, \mathcal{I} \cap A^-) = 0$ or $(\mathcal{J} \cap A^+, \mathcal{J} \cap A^-) = 0$ by primeness of A, which implies $\mathcal{I} = 0$ or $\mathcal{J} = 0$ by Proposition 4.1. □

Proposition 4.3. *Let $A = (A^+, A^-)$ be a semiprime associative pair with standard imbedding $(\mathcal{A}, e)$. An element $a \in A^+$ has left Goldie dimension equal to n in A if and only if a has left Goldie dimension equal to n in $\mathcal{A}$.*

Proof. Let $\{\mathcal{L}^i\}$ be a direct sum of nonzero left ideals of $\mathcal{A}$ contained in $\mathcal{A}a$. Then each $\mathcal{L}^i$ is contained in $\mathcal{A}_{12} \oplus \mathcal{A}_{22}$, and hence by Proposition 4.1(iv), $\mathcal{L}^i \cap \mathcal{A}_{12}$ is nonzero and it is contained in $\mathcal{A}a \cap \mathcal{A}_{12} = \mathcal{A}_{11}a = \Phi a + A^+A^-a$. By semiprimeness of A, $0 \neq \mathcal{A}_{12}\mathcal{A}_{21}(\mathcal{L}^i \cap \mathcal{A}_{12}) \subset A^-A^+a$, and, since $\mathcal{A}_{12}\mathcal{A}_{21}(\mathcal{L}^i \cap \mathcal{A}_{12}) \subset \mathcal{L}^i$, $\{\mathcal{A}_{12}\mathcal{A}_{21}(\mathcal{L}^i \cap \mathcal{A}_{12})\}$ is a direct sum of left ideals of A. Conversely, we must show that any direct sum of nonzero left ideals of A contained in A^+A^-a gives rise to a direct sum of left ideals of $\mathcal{A}$ contained in $\mathcal{A}a$; but this follows from Proposition 4.1(ii) and (v). □

Proposition 4.4. *Let $A = (A^+, A^-)$ be a semiprime associative pair with standard imbedding $(\mathcal{A}, e)$. For every subset $X \subset A^+$, $lan_A(X) = lan_{\mathcal{A}}(X) \cap A^-$.*

Proof. Clearly, $lan_{\mathcal{A}}(X) \cap A^- \subset lan_A(X)$. Conversely, let $z_{21} \in lan_A(X)$. Then

$$z_{21}X\mathcal{A}z_{21}X \subset z_{21}X\mathcal{A}_{21}X = z_{21}XA^-X = 0$$

implies $z_{21}X = 0$ by semiprimeness of $\mathcal{A}$ (Proposition 4.2). □

As usual (see [20, p.31]) denote by $Z_l(\mathcal{A})$ the left singular ideal of an associative algebra $\mathcal{A}$, i.e.,

$$Z_l(\mathcal{A}) = \{z \in \mathcal{A} : lan(z) \text{ is an essential left ideal of } \mathcal{A}\}.$$

Singular ideals are compatible with standard imbeddings in the following sense.

Proposition 4.5. *Let $A = (A^+, A^-)$ be a semiprime associative pair with standard imbedding $(\mathcal{A}, e)$. Then $Z_l(\mathcal{A}) \cap A^\pm \subset Z_l(A)^\pm$. Hence, if A is left nonsingular, then $\mathcal{A}$ is left nonsingular.*

Proof. Let $z_{21} \in Z_l(\mathcal{A}) \cap A^-$, and L be a nonzero left ideal of A contained in A^+. By Proposition 4.1(ii) the left ideal of $\mathcal{A}$ generated by L, $\mathcal{L} = L \oplus \mathcal{A}_{21}L$, is contained in $\mathcal{A}_{12} \oplus \mathcal{A}_{22}$. Then $\mathcal{L} \cap lan_{\mathcal{A}}(z_{21})$ is a nonzero left ideal of $\mathcal{A}$ contained in $\mathcal{A}_{12} \oplus \mathcal{A}_{22}$. Hence by Proposition 4.1(iv), $\mathcal{L} \cap lan_{\mathcal{A}}(z_{21}) \cap \mathcal{A}_{12}$ is a nonzero left ideal of A equal to $L \cap lan_A(z_{21})$ by Proposition 4.4. We have shown $Z_l(\mathcal{A}) \cap A^- \subset Z_l(A)^-$. Similarly $Z_l(\mathcal{A}) \cap A^+ \subset Z_l(A)^+$.

Suppose now that A is left nonsingular. Then $Z_l(A)^\pm = 0$ implies $Z_l(\mathcal{A}) \cap A^\pm = 0$ and hence $Z_l(\mathcal{A}) = 0$ by Proposition 4.1(i). □

Now we compute the standard imbedding of a simple associative pair coinciding with its socle.

Proposition 4.6. *Let $A = (\mathcal{F}(X,Y), \mathcal{F}(Y,X))$ be a simple associative pair with minimal inner ideals, where (X, X') and (Y, Y') are two dual pairs of vector spaces over a division associative Φ-algebra Δ. Then the standard imbedding $(\mathcal{A}, e)$ of A is given by $\mathcal{A} = \mathcal{F}(V,V) + \Phi Id_V$ relative to a dual pair of vector spaces (V, V'), where*

$$V := X \oplus Y, \quad V' := X' \oplus Y', \quad < x+y, x'+y' > = < x, x' > + < y, y' >,$$

and e is the projection of V onto X, i.e., $(x+y)e = x$. In particular, if A is artinian then $\mathcal{A} = \mathcal{F}(V,V) = End_\Delta(V) \cong Mat_{n \times n}(\Delta)$ with $n = dim_\Delta X + dim_\Delta Y$.

Proof. It follows from the construction of the standard imbedding (see [27]) and from the fact that $\mathcal{F}(X,Y)\mathcal{F}(Y,X) = \mathcal{F}(X,X)$ and $\mathcal{F}(Y,X)\mathcal{F}(X,Y) = \mathcal{F}(Y,Y)$. □

Proposition 4.7. *Let $A = (A^+, A^-)$ be a semiprime associative pair with standard imbedding $(\mathcal{A}, e)$. Then*

(i) *$Soc(A)^\sigma = Soc(\mathcal{A}) \cap A^\sigma$, $(\sigma = \pm)$, and*
(ii) *A is artinian if and only if $\mathcal{A}$ is artinian.*

Proof. (i) By Proposition 4.2, $\mathcal{A}$ is semiprime. Now let $x_{12} \in \mathcal{A}_{12} = A^+$. Clearly $x_{12}\mathcal{A}x_{12} = x_{12}A^-x_{12}$. Hence the minimal inner ideals of A contained in A^+ are precisely the minimal inner ideals of $\mathcal{A}$ contained in $\mathcal{A}_{12}$, which implies, via the Jordan characterization of the associative socle [10, Prop. 2.6(i)] that $Soc(A)^+ = Soc(\mathcal{A}) \cap A^+$, and the same is true for $Soc(A)^-$.

(ii) If A is artinian then it is a direct sum of finitely many ideals M_i each of which is a simple artinian associative pair. It is easy to see that the standard imbedding $(\mathcal{A}, e)$ is the direct sum of the standard imbeddings $(\mathcal{M}_i, e_i)$ of the M_i. Hence $\mathcal{A}$ is artinian by Proposition 4.6. Conversely, if $\mathcal{A}$ is artinian then A is artinian by Proposition 4.1(ii). □

§5. The local algebras of an associative pair. Let A be an associative pair and let $b \in A^-$. Then the submodule bA^+b under the multiplication given by $(bxb)\cdot(byb) := bxbyb$ is an associative algebra called the *local algebra* of A at b, and denoted A_b. Note that if b is von Neumann regular then A_b is unital with b as a unit element.

Local algebras, which were introduced by K. Meyberg [29] and which play an important role in the current structure theory of Jordan systems [2, 5, 6, 7, 28], are usually presented in a different way [5, 0.4]. Recall that A^+, endowed with the b-homotope product $x \cdot_{(b)} y = xby$, becomes an associative algebra $A^+_{(b)}$ which has as an ideal the set $Ker(U_b) := \{x \in A^+ : bxb = 0\}$. Moreover, the mapping $x + Ker(U_b) \to bxb$ is an isomorphism from $A^+_{(b)}/Ker(U_b)$ onto our local algebra A_b (see [2, Example 1.6]).

However, our definition is more suitable for our purposes than the usual one, for instance, if A is a subpair of an associative pair Q and b is in A^-, then the local algebra A_b is a subalgebra of Q_b with our definition. We will also need an extension of the notion of local algebra which we define as follows.

Let A be a subpair of an associative pair Q and let $b \in Q^-$ be such that A^+ is a subalgebra of the homotope $Q^+_{(b)}$. Then bA^+b can be regarded as a subalgebra of the local algebra Q_b of Q at b which will be called the *generalized local algebra* of A at b, and will also be denoted by A_b. If b is actually in A^-, then the definition of generalized local algebra agrees, of course, with that given above.

Sometimes we will consider local algebras of associative algebras. This is nothing new since, as pointed out above, every associative algebra gives rise to an associative pair. With this in mind we state the following generalized transitivity of local algebras.

Lemma 5.1. *Let A be a subpair of an associative pair Q.*

(i) *If $b \in A^-$ and $x \in A^- \cap bQ^+b$, then bA^+b is a subalgebra of the x-homotope $(Q_b)_{(x)}$ of the local algebra of Q at b, and the generalized local algebra $(A_b)_x$ of A_b coincides with A_x.*

(ii) *If $q \in Q^-$ is such that A^- is a subalgebra of the q-homotope $Q_{(q)}$ then, for each $y \in A_q$, A^- is a subalgebra of the y-homotope $Q_{(y)}$, and the generalized local algebra A_y of A at y agrees with the local algebra $(A_q)_y$ of the generalized local algebra A_q.*

Proof. Straightforward. □

Some of the results stated in the next proposition were previously proved for Jordan pairs [2, 6, 7]. Nevertheless, they are included here for the sake of completeness.

Proposition 5.2. *Let A be a semiprime associative pair. Then*

(i) *All the local algebras of A are semiprime.*
(ii) *A is prime if and only all the local algebras of A at nonzero elements are prime.*
(iii) *If A is simple then all the local algebras of A at nonzero elements are simple.*
(iv) *$b \in A^-$ has left Goldie dimension equal to m in A if and only if A_b has left Goldie dimension equal to m.*
(v) *$Soc(A_b) = Soc(A)^- \cap bA^+b$. Hence A* coincides with its socle if and *only if A_b is artinian for each $b \in A^-$.*
(vi) *If A coincides with its socle, then A has finite both left and right local Goldie dimension.*

Proof. (i) We just need to prove that A_b is semiprime for any nonzero $b \in A^-$. Now $0 = bab \cdot A_b \cdot bab = babA^+bab$ implies $bab = 0$ by semiprimeness of A.
(ii) If A is prime, we have as in (i), that $0 = bab \cdot A_b \cdot bcb = babA^+bcb$ implies $bab = 0$ or $bcb = 0$. Suppose conversely that A_b is prime for each $0 \neq b \in A^-$, and let $I = (I^+, I^-)$ and $J = (J^+, J^-)$ be ideals of A such that $I^\sigma A^{-\sigma} J^\sigma = 0$. Then bI^+b and bJ^+b are ideals of A_b such that $bI^+b \cdot bJ^+b = bI^+bJ^+b = 0$. Hence, by primeness of A_b, $bI^+b = 0$ or $bJ^+b = 0$, equivalently $b \in ann(I^+)$ or $b \in ann(J^+)$ by Proposition 2.2(ii) since A is semiprime, which implies $A^- = ann(I^+) \cup ann(J^+)$ and therefore A^- is equal to one of them, say $ann(I^+)$. In this case $I^+ = 0$ by Proposition 2.2(iii), hence $I = 0$ by semiprimeness of A.
(iii) If A is simple and $0 \neq b \in A^-$, then A_b is simple. Indeed, given $0 \neq bxb \in A_b$ we have, by simplicity of A, that $A^+ = A^+bxbA^+$, and hence $A_b = bA^+b = A_b \cdot bxb \cdot A_b$.
(iv) Let L be a nonzero left ideal of A contained in A^-A^+b. By semiprimeness of A, we have that bA^+L is a nonzero left ideal of A_b contained in L. Hence every direct sum of n nonzero left ideals of A contained in A^-A^+b provides a direct sum of n nonzero left ideals of A_b.

Conversely, let $\{\mathcal{L}_i\}_{1 \leq i \leq n}$ be a direct sum of nonzero left ideals of A_b. For each $1 \leq i \leq n$ take a nonzero element $y_i \in \mathcal{L}_i$. Then $\{A^-A^+y_i\}_{1 \leq i \leq n}$ is a direct sum of nonzero left ideals of A contained in A^-A^+b. Indeed, each $A^-A^+y_i$ is nonzero by semiprimeness of A, and if $x_1 + \ldots + x_n = 0$ with $x_i \in A^-A^+y_i$ and, say $x_1 \neq 0$, then by semiprimeness again,

$$0 \neq bA^+x_1 \subset bA^+y_1 \cap (\sum_{i=2}^{n} bA^+y_i) = bA^+b \cdot y_1 \cap (\sum_{i=2}^{n} bA^+b \cdot y_i) \subset \mathcal{L}_1 \cap (\sum_{i=2}^{n} \mathcal{L}_i),$$

which is a contradiction.
(v) Since the inner ideals of A_b are precisely those inner ideals of A contained in bA^+b, $Soc(A_b) = Soc(A)^- \cap bA^+b$. Now it follows by [26, Prop.3(2)] that if $b \in Soc(A)^-$ then b has finite rank and it is von Neumann regular. Hence A_b is artinian with capacity equal to the rank of b (see Corollary 1 of [26]). Conversely, if A_b is artinian then A_b has bounded length for the chains of inner ideals, i.e, there is a bound for the lengths of chains of inner ideals of A^J contained in bA^+b, hence for chains of principal inner ideals xA^+x, with x in the inner ideal of A^J generated by b. Thus $b \in Soc(A)^-$ by [26, Prop.3(2)] again.
(vi) By (v), any local algebra of A is artinian and hence it has finite both left and right Goldie dimension. Then, by (iv), A has finite both left and right local Goldie dimension. □

Proposition 5.3. *Let A be a semiprime associative pair, and $b \in A^-$. Then*

(i) $Z_l(A_b) \subset Z_l(A^-)$,
(ii) *if* $b \in Z_l(A^-)$ *then* $Z_l(A_b) = A_b$.

Hence A is left nonsingular if and only if A_b is left nonsingular for all $b \in A^-$.

Proof. (i) Let $bab \in Z_l(A_b)$ and L be a nonzero left ideal of A contained in A^+. We may consider two possibilities: If $A^-Lb = 0$ then, by Lemma 2.1, $L \subset lan_A(b) \subset lan_A(bab)$, so $L \cap lan_A(bab) \neq 0$. On the other hand, if $A^-Lb \neq 0$ then, by semiprimeness of A, $0 \neq bA^+A^-Lb \subset bLb$ where bLb is actually a left ideal of A_b. Since $bab \in Z_l(A_b)$, there exists $0 \neq blb \in lan_{A_b}(bab)$ with $l \in L$, equivalently, $blb \neq 0$ with $blbab = blb \cdot bab = 0$. Hence, by semiprimeness of A, $A^+bl \neq 0$ with $A^-A^+blbab = 0$, which implies by Lemma 2.1 again that $A^+bl \subset lan_A(bab) \cap L$. In any case $lan_A(bab)$ hits L, hence $lan_A(bab)$ is an essential left ideal of A and $bab \in Z_l(A^-)$, as required.
(ii) Suppose now that $b \in Z_l(A^-)$. We must prove that $lan_{A_b}(bab)$ is an essential left ideal of A_b for any $bab \in A_b$. We may assume $bab \neq 0$. Let $0 \neq bcb \in A_b$. Since A_b is semiprime (Proposition 5.2(i)), the left ideal $A_b \cdot bcb$ of A_b generated by bcb is nonzero. If $bcb \in lan_{A_b}(bab)$ then $A_b \cdot bcb \subset lan_{A_b}(bab)$. Thus we may assume $bcb \notin lan_{A_b}(bab)$. Then $bcbab = bcb \cdot bab \neq 0$ implies $A^+bcba \neq 0$. Since $b \in Z_l(A^-)$, there exists $0 \neq xbcba \in A^+bcba \cap lan_A(b)$. Hence, by semiprimeness of A, $0 \neq baA^-xbcb = baA^-xb \cdot bcb \subset A_b \cdot bcb \cap lan_{A_b}(bab)$ since $baA^-xbcb \cdot bab = baA^-xbcbab = 0$. We have shown that $lan_{A_b}(bab)$ is an essential left ideal of A_b, for any $bab \in A_b$, i.e., $Z_l(A_b) = A_b$.

For the last part of the proof, if $Z_l(A^-) = 0$ then, by (i), $Z_l(A_b) = 0$ for every $b \in A^-$. Conversely, suppose that $Z_l(A_b) = 0$ for every $b \in A^-$. Then, by (ii), $b \in Z_l(A^-)$ implies $A_b = Z_l(A_b) = 0$, so $bA^+b = 0$, and hence $b = 0$ by semiprimeness of A. Since $(Z_l(A^+), Z_l(A^-))$ is an ideal of A, $Z_l(A^-) = 0$ implies $Z_l(A^+) = 0$ by semiprimeness of A. □

Every semiprime associative pair A which is left nonsingular and such that every element $x \in A^\sigma$, $\sigma = \pm$ has finite left Goldie dimension will be called a *left local Goldie associative pair.* If additionally A has finite left (global) Goldie dimension then A will be called a *left Goldie associative pair.* Right and two-sided corresponding notions are defined dually. It follows from [25, Cor.1], Proposition 5.2(vi) and Corollary 3.5 that semiprime associative pairs with dcc on principal inner ideals are local Goldie.

Proposition 5.4. *A semiprime associative pair A is a left local Goldie associative pair if and only if A_b is a left Goldie associative algebra for every $b \in A^{\pm}$.*

Proof. It follows from Proposition 5.2(iv) and Proposition 5.3. □

Corollary 5.5. *Every left local Goldie associative pair A having finite right local Goldie dimension is local Goldie.*

Proof. By Proposition 5.4 all the local algebras of A are left Goldie. Since they have also finite right Goldie dimension by Proposition 5.2(iv), they are also right Goldie [21, Lemma 7.2.2]. Hence A is local Goldie by the dual of Proposition 5.4. □

§6. Local Goldie associative pairs. The study of semiprime local Goldie associative pairs can be reduced to the prime case via the notions of uniform ideal and essential subdirect product. We refer to the reader to [20,13] were similar notions were considered for associative and Jordan algebras respectively.

A nonzero ideal I of an associative pair A will be called *uniform* if for any nonzero ideals B and C of A inside I, $B \cap C \neq 0$.

Proposition 6.1. *Let A be a semiprime associative pair. Then every l-uniform element $u \in A^+$ generates a uniform ideal.*

Proof. Let $I = I(u)$ be the ideal of A generated by u, and let B and C be nonzero ideals of A contained in I. Then both left ideals A^+B^-u and A^+C^-u are nonzero. Otherwise $A^+B^-u = 0$ would imply $u \in ran(B^-) = ann(B^-)$. Hence $B^+ \subset I^+ \subset ann(B^-)$ since $ann(B)$ is an ideal of A by Proposition 2.2(iv), which is a contradiction by semiprimeness of A, using Proposition 2.2(iii). Then, by l-uniformity of u, $0 \neq A^+B^-u \cap A^+C^-u \subset B^+ \cap C^+$. Therefore I is uniform. □

As pointed out before, the lattice $\mathcal{L}(A)$ of all ideals of an associative pair A is an algebraic lattice relative to the $\star$-product. Hence, as a particular case of [15, Prop. 3.1], using that $\mathcal{L}(A)$ is a modular lattice, we obtain:

Proposition 6.2. *Let A be a semiprime associative pair. Then*

(i) *a nonzero ideal $I = (I^+, I^-)$ of A is uniform if and only if the annihilator ideal $ann(I)$ is maximal among all annihilators ideals $ann(B)$ with*

$B = (B^+, B^-)$ *being a nonzero ideal of* A*, equivalently,* $A/ann(I)$ *is a prime associative pair,*

(ii) *for each uniform ideal* $I = (I^+, I^-)$ *of* A *there exists a unique maximal uniform ideal* $U = (U^+, U^-)$ *of* A *containing* I*, actually* $U = (ann(ann(I^+)), ann(ann(I^-)))$*,*

(iii) *the sum of all maximal uniform ideals of* A *is direct.*

A subdirect product of associative pairs $A \leqslant \prod A_\alpha$ will be called an *essential subdirect product* if A contains an essential ideal of the full product $\prod A_\alpha$. If A is actually contained in the direct sum of the A_α, then A will be called an *essential subdirect sum*. An ideal I of a semiprime associative A is called a *closed ideal* if $I = ann(ann(I))$. Since the third annihilator coincides with the first one, an ideal is closed if and only if it is the annihilator of an ideal. Notice that by Proposition 6.2(ii) maximal uniform ideals are closed.

Theorem 6.3. *For an associative pair* A *the following conditions are equivalent:*

(i) A *is an essential subdirect product of prime associative pairs* A_α*,*

(ii) A *is semiprime and every nonzero ideal of* A *contains a uniform ideal,*

(iii) A *is semiprime and every nonzero closed ideal of* A *contains a uniform ideal.*

Actually we can take $A_\alpha = A/ann(M_\alpha)$ *where* $\{M_\alpha\}$ *is the family of all maximal uniform ideals of* A*.*

Proof. (i) $\Rightarrow$ (ii). In general, any subdirect product A of a family $\{A_\alpha\}$ of semiprime associative pairs is also semiprime. Indeed, if B is an ideal of A such that $B \star B = 0$ then, for each index α, $\pi_\alpha(B)$ is an ideal of A_α with $\pi_\alpha(B) \star \pi_\alpha(B) = 0$ which implies $\pi_\alpha(B) = 0$ by semiprimeness of the A_α, so $B = 0$. Let $M \subset A$ be an essential ideal of the full direct product $\prod A_\alpha$, and set $M_\alpha := M \cap A_\alpha$, where we are regarding A_α as an ideal of $\prod A_\alpha$. Then M_α is a nonzero ideal of A_α contained in A since M is an essential ideal of $\prod A_\alpha$. Actually M_α is a uniform ideal of A since M_α is uniform in A_α because A_α is prime, and any ideal of A contained in M_α is an ideal of A_α. Now if I is a nonzero ideal of A then $\pi_\alpha(I)$ is a nonzero ideal of A_α for some index α. Hence, by primeness of A_α, $0 \neq \pi_\alpha(I) \star M_\alpha \subset I \cap M_\alpha$. Therefore I contains the nonzero ideal $I \cap M_\alpha$, which is uniform since it is contained in M_α.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i). Let $\sum M_\alpha$ be the sum of all maximal uniform ideals of A, which is direct by Proposition 6.2(iii). Since $ann(\sum M_\alpha)$ is a closed ideal, it must be zero: otherwise $ann(\sum M_\alpha)$ would contain a uniform ideal, and therefore a maximal uniform ideal because it is closed, which leads to contradiction. Hence, by a standard argument, $\cap ann(M_\alpha) = ann(\sum M_\alpha) = 0$ implies that

A is a subdirect product of the associative pairs $A_\alpha := A/ann(M_\alpha)$ each of which is prime by Proposition 6.2(i). Finally, the homomorphic image of $\oplus M_\alpha$ in $\prod A_\alpha$ is an essential ideal of $\prod A_\alpha$ since if B is an ideal of $\prod A_\alpha$ such that

$$B \star (\oplus M_\alpha) = \oplus(B \star M_\alpha) = \oplus(\pi_\alpha(B) \star M_\alpha) = 0$$

then, for each α, $\pi_\alpha(B) \star M_\alpha = 0$, so $\pi_\alpha(B) = 0$ by primeness of A_α, and hence $B = 0$, which completes the proof. □

Lemma 6.4. *Let A be a semiprime associative pair and I an ideal of A. Denote by $\overline{A}$ the quotient pair $A/ann(I) = (A^+/ann(I^-), A^-/ann(I^+))$. We have*

- (i) *any direct sum of nonzero left ideals of $\overline{A}$ can be lifted to a direct sum of nonzero left ideals of A. Hence if A has finite left Goldie dimension, then $\overline{A}$ has also finite left Goldie dimension,*
- (ii) *if $a \in A^+$ has finite left Goldie dimension in A, then $\overline{a} := a + ann(I)$ has finite left Goldie dimension in $\overline{A}$,*
- (iii) *if A is left nonsingular and I is a uniform ideal, then $\overline{A}$ is a prime left nonsingular associative pair.*

Proof. (i) Let $\sum \overline{L}_\alpha$ be a direct sum of nonzero left ideals of $\overline{A}$ contained in $\overline{A}^+$. Denoting by $\pi : A \to \overline{A}$ the canonical projection of A onto $\overline{A}$, we have that $L_\alpha := \pi^{-1}(\overline{L}_\alpha) \cap I^+$ is a nonzero left ideal of A, contained in A^+, for each index α. Let us now show that the sum $\sum L_\alpha$ is direct. Indeed, since the sum of the $\overline{L}_\alpha$ is direct, for each index β, the intersection $L_\beta \cap (\sum_{\alpha\neq\beta} L_\alpha)$ is contained in $ann(I^-)$, but this intersection is also contained in I^+, so $L_\beta \cap (\sum_{\alpha\neq\beta} L_\alpha) = 0$ by Proposition 2.2(iii), as required.
(ii) Let $\sum \overline{L}_\alpha$ be a direct sum of nonzero left ideals of $\overline{A}$ inside the principal left ideal $\overline{A}^+\overline{A}^-\overline{a}$. By taking $L_\alpha := \pi^{-1}(\overline{L}_\alpha) \cap A^+A^-a \cap I^+$, we can obtain as above a direct sum $\sum L_\alpha$ of nonzero left ideals of A, contained in $\cap A^+A^-a$.
(iii) Assume now that A is left nonsingular and I is uniform. By Proposition 6.2(i), $\overline{A}$ is a prime associative pair. Since A is left nonsingular, I is also left nonsingular, using Proposition 3.2. But I can be regarded as an ideal of $\overline{A}$ via the isomorphism $x \mapsto x + ann(I^\mp)$, for $x \in I^\pm$, and moreover, I is essential in $\overline{A}$ since $\overline{A}$ is prime. Hence, Corollary 3.3 applies, showing that $\overline{A}$ is left nonsingular. □

Theorem 6.5. *Let A be a semiprime left local Goldie associative pair. Then A is an essential subdirect sum of prime left local Goldie associative pairs. More precisely,*

$$\oplus M_\alpha \triangleleft A \leq \oplus A/ann(M_\alpha),$$

where M_α ranges over all maximal uniform ideals of A.

If A is actually left Goldie then A is an essential subdirect sum of finitely many prime left Goldie associative pairs.

Proof. Since A has finite left local Goldie dimension, any nonzero ideal of A contains an l-uniform element, and hence a uniform ideal by Proposition 6.1. Then, by Proposition 6.3, A is an essential subdirect product of prime associative pairs $A_\alpha = A/ann(M_\alpha)$, with M_α a maximal uniform ideal of A, each of which is a prime left local Goldie associative pair by Proposition 6.4(ii)(iii). Let us see that $A \subset \oplus A_\alpha$: Otherwise, there exists $x \in A^\pm$ such that $x \notin ann(M_\alpha)$ for an infinite number of α's. Say $x \notin ann(M_\alpha)$ for every $\alpha \in \Lambda$, where Λ is an infinite subset of indexes. Then $xM_\alpha^\mp x \neq 0$ for every $\alpha \in \Lambda$, which implies $0 \neq A^\pm M_\alpha^\mp x =: I_\alpha \subset M_\alpha$, where I_α is a left ideal of A contained in $A^\pm A^\mp x$, for every $\alpha \in \Lambda$, and the sum $\sum_{\alpha\in\Lambda} I_\alpha$ is direct. This implies that x has infinite left Goldie dimension, hence A has infinite left local Goldie dimension, which is impossible.

Suppose additionally that A has finite left Goldie dimension. Then it follows from Proposition 6.2(iii) that A contains only a finite number of maximal uniform ideals, and hence A is an essential subdirect sum of a finite number of A_α. Moreover, each A_α has now finite left Goldie dimension by Lemma 6.4(i). □

§7. **Orders in associative algebras.** We record in this section some results on orders in associative algebras which will be used later. Recall that a subalgebra $\mathcal{A}$ of a unital associative algebra $\mathcal{Q}$ is a *left order* in $\mathcal{Q}$ (relatively to a multiplicatively closed set S of $\mathcal{A}$) if (i) $S \subset Inv(\mathcal{Q})$, and (ii) for every $q \in \mathcal{Q}$, $q = s^{-1}a$ where $s \in S$ and $a \in \mathcal{A}$. Right orders and (two-sided) orders are defined similarly. In the particular case that S coincides with the set $Reg(\mathcal{A})$ of those regular elements of $\mathcal{A}$, the orders will be said *classical*. The reader is referred to [20, 21, 30] for general results on orders in associative algebras (rings).

Lemma 7.1. *Let $\mathcal{A}$ be an associative algebra which is a left order in a unital associative algebra $\mathcal{Q}$ relative to a multiplicatively closed set S of $\mathcal{A}$ and $s \in S$. Then*

(i) *$s\mathcal{A}s$ is a left order in $\mathcal{Q}$ relative to sSs, and*
(ii) *the local algebra $\mathcal{A}_s$ of $\mathcal{A}$ at s is a left order in the local algebra $\mathcal{Q}_s$ relative to sSs.*

Proof. (i) Clearly sSs is contained in $Inv(\mathcal{Q})$. Now let $q \in \mathcal{Q}$ and write $sqs^{-1} = r^{-1}a$ for some $r \in S$ and $a \in \mathcal{A}$. Then $q = s^{-1}r^{-1}s^{-1}sas = (srs)^{-1}sas$.
(ii) Note that $sts \in sSs$ belongs to $Inv(\mathcal{Q}_s)$ if and only if $(sts)\cdot\mathcal{Q}_s\cdot(sts) = \mathcal{Q}_s$; but

$$(sts)\cdot\mathcal{Q}_s\cdot(sts) = sts\mathcal{Q}sts = \mathcal{Q} = s\mathcal{Q}s = \mathcal{Q}_s$$

because s, and hence also sts are invertible in $\mathcal{Q}$. Now let $sqs \in \mathcal{Q}_s$ and set $sq = r^{-1}a$ for some $r \in S$ and $a \in \mathcal{A}$. Then $(srs) \cdot (sqs) = srsqs = sas$. □

There is a natural geometric notion of order in a dual pair of vector spaces which is compatible with the algebraic one as we next show.

Definition 7.2. Let $(X, X', <.,.>)$ be a dual pair of vector spaces over Δ. We shall say that (N, N', D) is a *left (right) order* in $(X, X', <.,.>)$ if

(i) D is a left (right) order in the division associative algebra Δ,
(ii) N is a left D-submodule of X such that $Dx \cap N \neq 0$, for each $0 \neq x \in X$,
(iii) N' is a right D-submodule of X' such that $x'D \cap N' \neq 0$ for each $0 \neq x' \in X'$, and
(iv) $<N, N'> \subset D$.

If (N, N', D) is both left and right order in $(X, X', <.,.>)$, we will say that (N, N', D) is an *order* in $(X, X', <.,.>)$.

Proposition 7.3. *Let (N, N', D) be a left order in the canonical dual pair (X, X') defined by a finite dimensional left vector space X over a division associative algebra Δ. Then any subalgebra $\mathcal{A}$ of $End_\Delta(X) = X' \otimes_\Delta X$ containing $N' \otimes_D N$ is a left order in $End_\Delta(X)$.*

Proof. Let us first see that any $a \in Reg(\mathcal{A})$ is invertible in $End_\Delta(X)$. Since a is invertible if and only if a is injective, we just need to show that the kernel of a is zero; but $ker(a) \neq 0$ implies $ker(a) \cap N \neq 0$ by 7.2(ii), now taking $0 \neq v \in ker(a) \cap N$ and $0 \neq v' \in N'$, we obtain $(v' \otimes v)a = v' \otimes va = 0$ with $0 \neq v' \otimes v \in N' \otimes_D N \subset \mathcal{A}$, which is a contradiction.

Let $q = x'_1 \otimes y_1 + \dots x'_r \otimes y_r$ where both $\{x'_i\} \subset X'$ and $\{y_i\} \subset X$ are linearly independent sets. We can write each $x'_i \otimes y_i = v'_i \otimes \lambda_i^{-1} w_i$ for some $v'_i \in N' \cap x'_i D$, $w_i \in N \cap Dy_i$, and $\lambda_i \in D$. Let $m = dim_\Delta X$. We construct a left denominator $b \in N' \otimes_D N$ for q as follows: Complete $\{v'_1, \dots, v'_r\}$ to a basis $\{v'_1, \dots, v'_m\}$ of X'. Then there exist $\{\rho_1, \dots, \rho_m\} \subset D$ and $\{v_1, \dots, v_m\} \subset N$ linearly independent such that, for each $1 \leq i \leq m$, $<v_i, v'_i> = \rho_i$ and $<v_i, v'_j> = 0$ for $j \neq i$. Now, for each $1 \leq i \leq r$, we have

$$v_i(v'_i \otimes \lambda_i^{-1} w_i) = <v_i, v'_i> \lambda_i^{-1} w_i = \alpha_i^{-1} \beta_i w_i$$

for some α_i and β_i in D, $\alpha_i \neq 0$. For $r < j \leq m$, let $\alpha_j = \beta_j = 1$. By multiplying the elements of a basis of X' by suitable elements of D, we obtain a basis $\{w'_j\}_{1 \leq j \leq m}$ of X' contained in N'. Putting $b =: w'_1 \otimes \alpha_1 v_1 + \dots + w'_m \otimes \alpha_m v_m \in N' \otimes_D N$, we have that b is invertible in $End_\Delta(X)$ and $bq = w'_1 \otimes \beta_1 w_1 + \dots w'_r \otimes \beta_1 w_r \in N' \otimes_D N$. □

Let $\mathcal{A}$ be a subalgebra of an associative algebra $\mathcal{Q}$. We will say that $\mathcal{Q}$ is a *tight left cover* of $\mathcal{A}$ if for each nonzero $q \in \mathcal{Q}$, $\mathcal{A}q \cap \mathcal{A} \neq 0$. Clearly, if $\mathcal{A}$ is a left order in $\mathcal{Q}$ then $\mathcal{Q}$ is a tight left cover of $\mathcal{A}$.

Proposition 7.4. *Let $\mathcal{A}$ be a (semiprime) left Goldie algebra which is a subalgebra of a semiprime artinian algebra $\mathcal{Q}$. If $\mathcal{Q}$ is a tight left cover of $\mathcal{A}$ then $\mathcal{A}$ is a left order in $\mathcal{Q}$.*

Proof. (1) $Reg(\mathcal{A}) \subset Inv(\mathcal{Q})$. Indeed, let $r \in Reg(\mathcal{A})$. By [30, Lemma 1.10, p. 54], $r \in Inv(\mathcal{Q})$ if and only if $lan_{\mathcal{Q}}(r) = 0$. Now if $lan_{\mathcal{Q}}(r) \neq 0$ then (by tightness of $\mathcal{A}$ in $\mathcal{Q}$), $0 \neq lan_{\mathcal{Q}}(r) \cap \mathcal{A} \subset lan_{\mathcal{A}}(r)$, which is a contradiction because $r \in Reg(\mathcal{A})$.
(2) Given $0 \neq q \in \mathcal{Q}$, set $(\mathcal{A} : q) := \{x \in \mathcal{A} : xq \in \mathcal{A}\}$. We claim that $(\mathcal{A} : q)$ is an essential left ideal of $\mathcal{A}$. Let $\mathcal{L}$ be a nonzero left ideal of $\mathcal{A}$. If $\mathcal{L}q = 0$ then $\mathcal{L} \subset (\mathcal{A} : q)$, so we may suppose $\mathcal{L}q \neq 0$. Then, by tightness of $\mathcal{A}$ in $\mathcal{Q}$ again, $0 \neq \mathcal{A}\mathcal{L}q \cap \mathcal{A} \subset \mathcal{L}q \cap \mathcal{A}$, so $\mathcal{L} \cap (\mathcal{A} : q) \neq 0$ which proves the claim. Since any essential left ideal of $\mathcal{A}$ contains a regular element [21, Lemma 7.2.5], there exists $r \in Reg(\mathcal{A})$ such that $rq \in \mathcal{A}$. Hence by (1) $q = r^{-1}a$ for some $a \in \mathcal{A}$, which completes the proof. □

Proposition 7.5. *Let $\mathcal{A}$ be a left order in a semiprime artinian associative algebra $\mathcal{Q}$. Then every essential ideal $\mathcal{I}$ of $\mathcal{A}$ is also a left order in $\mathcal{Q}$.*

Proof. By the classical Goldie theorem, $\mathcal{A}$ is left nonsingular and has finite left Goldie dimension. Hence $\mathcal{I}$ is also left nonsingular (see Corollary 3.3 for the analogous result for associative pairs) and has finite left Goldie dimension. Let $0 \neq q \in \mathcal{Q}$. Since $\mathcal{I}$ is essential in $\mathcal{A}$, and $\mathcal{Q}$ is a tight left cover of $\mathcal{A}$, we have $\mathcal{I}q \cap \mathcal{I} \supset \mathcal{I}(\mathcal{A}q \cap \mathcal{A}) \neq 0$, which proves that $\mathcal{Q}$ is a tight left cover of $\mathcal{I}$. Hence by Proposition 7.4, $\mathcal{I}$ is a left order in $\mathcal{Q}$, as required. □

We warn the reader that it is necessary to consider (two-sided) orders in the next lemma. Merely left orders seem not to be enough.

Proposition 7.6. *Let $\mathcal{A}$ be a semiprime associative algebra which is an order in a unital associative algebra $\mathcal{Q}$, and let $0 \neq q \in \mathcal{Q}$. We have*

(i) $q\mathcal{A}q \cap \mathcal{A} \neq 0$.

Moreover, if $\mathcal{Q}$ is artinian then

(ii) *there exists $b \in \mathcal{A}$ such that $q\mathcal{Q}q = b\mathcal{Q}b$.*

Proof. (i) Writing $q = a^{-1}b = da^{-1}$ we have, by semiprimeness of A, that $aqa\mathcal{A}aqa \neq 0$, which implies $0 \neq qa\mathcal{A}aq = d\mathcal{A}b \subset q\mathcal{A}q \cap \mathcal{A}$.
(ii) Note that by (i) $\mathcal{Q}$ is semiprime. Suppose now that $\mathcal{Q}$ is artinian. Then, by Proposition 5.2(i) and (v), the local algebra $\mathcal{Q}_q$ of $\mathcal{Q}$ at q is also semiprime and artinian (since inner ideals of $\mathcal{Q}_q$ are just those inner ideals of Q contained in qQq) and q is its unit element (since q is von Neumann regular in $\mathcal{Q}$). Write $q = q_1 + \ldots + q_n$ as a sum of orthogonal division idempotents in $\mathcal{Q}_q$. By (i), for each $1 \leq i \leq n$ there exists $0 \neq b_i \in q_i\mathcal{A}q_i \cap \mathcal{A}$. Since any element $x \in \mathcal{Q}_q$ generate the same principal inner ideal in $\mathcal{Q}_q$ as in $\mathcal{Q}$, and since

the q_i are mutually orthogonal in $\mathcal{Q}_q$, we have, for $b := b_1 + \ldots + b_n$, that $q\mathcal{Q}q = b\mathcal{Q}b$. □

As pointed out in the introduction, there exists a notion of order in rings which need not have a unit. We begin with some definitions. An element $a \in \mathcal{A}$ is called *semiregular* if

$$a^2x = 0 \Rightarrow ax = 0, \quad \text{and} \quad xa^2 = 0 \Rightarrow xa = 0$$

for $x \in \mathcal{A}'$ (the unitization of $\mathcal{A}$). We denote by $SemiReg(\mathcal{A})$ the set of all semiregular elements of $\mathcal{A}$. Certainly $SemiReg(\mathcal{A}) \supseteq Reg(\mathcal{A})$. We remark that if $a \in a^2\mathcal{Q} \cap \mathcal{Q}a^2$ for some over-algebra $\mathcal{Q} \supset \mathcal{A}$, then $a \in SemiReg(\mathcal{A})$.

Let $LocInv(\mathcal{A})$ denote the set of elements $a \in \mathcal{A}$ which are *locally invertible* in the sense that there exists an idempotent $e \in \mathcal{A}$ such that a is invertible in the unital algebra $e\mathcal{A}e$. Then the *local inverse* $a^\# \in e\mathcal{A}e$ is precisely the *group inverse* of a, and it is characterized by the conditions:

$$aa^\# = a^\# a, \quad a = aa^\# a, \quad a^\# = a^\# aa^\#.$$

The idempotent e is also unique, $e = aa^\# = a^\# a$. Moreover, a is locally invertible if and only if $a \in a^2\mathcal{A}a^2$ (see [17]). Thus, by the above remark, if $a \in \mathcal{A}$ is locally invertible in some over-algebra $\mathcal{Q}$, then a is semiregular in $\mathcal{A}$.

A subalgebra $\mathcal{A}$ of a not necessarily unital associative algebra $\mathcal{Q}$ is said to be a *Fountain-Gould left (right) order* in $\mathcal{Q}$ whenever

(1) $SemiReg(\mathcal{A}) \subset LocInv(\mathcal{Q})$, and
(2) every element $q \in \mathcal{Q}$ can be written in the form $q = a^\# b$ ($q = dc^\#$) where $a, c \in SemiReg(\mathcal{A})$ and $b, d \in \mathcal{A}$.

Left and right Fountain-Gould orders will be simply called *Fountain-Gould orders*. If condition (2) alone is satisfied, then $\mathcal{A}$ is a *weak left (right) order* in $\mathcal{Q}$. By [18, Proposition 2.6], weak left orders in semiprime associative algebras coinciding with their socles are actually Fountain-Gould left orders.

§8. Orders in associative pairs. In this section we introduce a notion of order in associative pairs and give a Goldie-like characterization of orders in semiprime associative pairs with dcc on principal inner ideals, and in semiprime artinian associative pairs.

Definition 8.1. Let A be a subpair of an associative pair Q. We will say that A is a *left (right) order* in Q if for any $q \in Q^\sigma$ there exists $x \in A^\sigma$ such that

(i) x is von Neumann regular in Q,
(ii) $q \in xQ^{-\sigma}x$, and
(iii) the local algebra A_x of A at x is a left (right) order in the unital associative algebra Q_x.

As usual, order will mean left and right order. Next we will see that orders in associative algebras can be characterized as orders in the pair sense. The reason for this fact is that isotopy and isomorphism are equivalent notions in unital associative algebras.

Proposition 8.2. *Let $\mathcal{A}$ be a subalgebra of a unital associative algebra $\mathcal{Q}$. Then $\mathcal{A}$ is a left order in $\mathcal{Q}$ if and only if the associative pair $A = (\mathcal{A}, \mathcal{A})$ is a left order in the associative pair $Q = (\mathcal{Q}, \mathcal{Q})$.*

Proof. Let $\mathcal{A}$ be a left order in $\mathcal{Q}$ relative to a multiplicatively closed set S of $\mathcal{A}$. Then given $q \in \mathcal{Q}$ there exists $s \in S$ such that $q \in \mathcal{Q} = s\mathcal{Q}s$, s is von Neumann regular in Q, and by Lemma 7.1(ii), A_s is a left order in Q_s, which proves that the associative pair $(\mathcal{A}, \mathcal{A})$ is a left order in $(\mathcal{Q}, \mathcal{Q})$

Suppose now that the associative pair $A = (\mathcal{A}, \mathcal{A})$ is a left order in $Q = (\mathcal{Q}, \mathcal{Q})$. Then given the unit element 1 of $\mathcal{Q}$, there exists $x \in \mathcal{A}$ such that $1 \in x\mathcal{Q}x$ (so x is invertible in $\mathcal{Q}$), and A_x is a left order in Q_x relative to a multiplicatively closed set xSx of A_x. We claim that xS is a multiplicatively closed set of $\mathcal{A}$ and that $\mathcal{A}$ is a left order in $\mathcal{Q}$ relative to xS. Indeed, (i) given xs_1 and xs_2 in xS, we have that $(xs_1x)\cdot(xs_2x) = xs_1xs_2x \in xSx$, and hence $xs_1xs_2 \in xS$ since x is invertible in $\mathcal{Q}$, (ii) given $xs \in xS$, we have that xsx is invertible in Q_x with inverse xwx, so $x = (xsx)\cdot(xwx) = xsxwx$ which implies $xsxw = 1$, and similarly, $xwxs = 1$, i.e., xs is invertible in $\mathcal{Q}$ with inverse xw, and (iii), given $q \in \mathcal{Q}$, since $q \in Q_x$, there exists $xsx \in xSx$ such that $(xsx) \cdot q \in x\mathcal{A}x$, so $xsq \in \mathcal{A}$, as claimed. □

There exists also a relationship between Fountain-Gould orders in associative algebras and orders in associative pairs given by the next proposition.

Proposition 8.3. *Let $\mathcal{A}$ be a subalgebra of an associative algebra $\mathcal{Q}$.*

(i) *If the associative pair $A = (\mathcal{A}, \mathcal{A})$ is a left order in the associative pair $Q = (\mathcal{Q}, \mathcal{Q})$, then $\mathcal{A}$ is a weak left order in $\mathcal{Q}$.*
(ii) *If $\mathcal{Q}$ is a simple associative algebra coinciding with its socle, then $\mathcal{A}$ is a Fountain-Gould order in $\mathcal{Q}$ if and only if $A = (\mathcal{A}, \mathcal{A})$ is an order in $Q = (\mathcal{Q}, \mathcal{Q})$.*

Proof. (i) Given $q \in \mathcal{Q}$, there exists $x \in \mathcal{A}$ such that $q \in x\mathcal{Q}x$ and A_x is a left order in Q_x. Take $xsx \in A_x \cap Inv(Q_x)$ such that $(xsx) \cdot q \in A_x$. Then

(1) $xsq = (xsx) \cdot q \in \mathcal{A}$ and
(2) $q \in Q_x = (xsx) \cdot Q_x = xsx\mathcal{Q}x \subset xs\mathcal{Q}$, with $xs \in LocInv(\mathcal{Q})$.

To see the last part note that $x \in (xsx)\cdot(xsx)\cdot Q_x \cdot (xsx) = xsxsx\mathcal{Q}xsx$ implies $xs \in (xs)^2\mathcal{Q}(xs)^2$. Now it follows from (1) and (2) that $q = (xs)^{\#}(xsq)$ with $xsq \in \mathcal{A}$, which proves that $\mathcal{A}$ is a weak left order in $\mathcal{Q}$.
(ii) Suppose now that $\mathcal{Q}$ is a simple associative algebra coinciding with its socle. If the associative pair $A = (\mathcal{A}, \mathcal{A})$ is an order in $Q = (\mathcal{Q}, \mathcal{Q})$, then

it follows from (i) and [18, Proposition 2.6] that $\mathcal{A}$ is a Fountain Gould order in $\mathcal{Q}$. Conversely, if $\mathcal{A}$ is a Fountain Gould order in $\mathcal{Q}$, then, by [3, Proposition 10] and its dual, for every $s \in SemiReg(\mathcal{A})$, the algebra $s\mathcal{A}s$ is a classical order in the (simple artinian) associative algebra $e\mathcal{Q}e = s\mathcal{Q}s$ with $e = ss^{\#}$. Hence, by Lemma 7.1(ii), the local algebra $\mathcal{A}_s$ of $\mathcal{A}$ at s is an order in $\mathcal{Q}_s$. Thus we just need to prove that given $q \in \mathcal{Q}$, there exists $s \in SemiReg(\mathcal{A})$ such that $q \in s\mathcal{Q}s$. By Litoff's theorem [22, p.90], there exists an idempotent $e \in \mathcal{Q}$ such that $q \in e\mathcal{Q}e$ with $e = e_1 + \ldots + e_n$ a sum of orthogonal division idempotents. By [18, Lemma 2.1], we can write each e_i, $1 \leq i \leq n$, as $e_i = a_i^{\#}b_i = d_ic_i^{\#}$ with $b_i, c_i \in \mathcal{A}$, $a_i, c_i \in SemiReg(A)$, $a_ia_i^{\#}b_i = b_i$ and $d_ic_i^{\#}c_i = d_i$. Since $\mathcal{A}$ is prime [19, Theorem 1.1] $0 \neq d_i\mathcal{A}b_i = d_ic_i^{\#}c_i\mathcal{A}a_ia_i^{\#}b_i = e_ic_i\mathcal{A}a_ie_i \subset e_i\mathcal{A}e_i \cap \mathcal{A}$. Hence any $0 \neq s_i \in e_i\mathcal{Q}e_i \cap \mathcal{A}$ is semiregular in $\mathcal{A}$ (since $e_i\mathcal{Q}e_i$ is a division associative algebra) and satisfies $e_i \in e_i\mathcal{Q}e_i = s_i\mathcal{Q}s_i$. Finally, taking $s = s_1 + \ldots + s_n$, we have that $s \in Inv(e\mathcal{Q}e)$ and hence $s \in SemiReg(\mathcal{A})$, with $q \in e\mathcal{Q}e = s\mathcal{Q}s$, as required. □

Now we give an example of orders in associative pairs which has a Faith-Utumi flavour [23]. As it will be proved in a forthcoming paper, any order in a simple associative pair coinciding with its socle is isomorphic to one of these.

Proposition 8.4. *Let (N, N', D) and (M, M', D) be left orders in dual pairs of vector spaces $(X, X', < .,. >)$ and $(Y, Y', < .,. >)$ respectively, over a division Φ-algebra Δ. Then any subpair A of $Q = (\mathcal{F}(X,Y), \mathcal{F}(Y,X))$ containing $(N' \otimes_D M,\ M' \otimes_D N)$ is a left order in $(\mathcal{F}(X,Y), \mathcal{F}(Y,X))$.*

Proof. Without loss of generality we may assume $A = (N' \otimes_D M,\ M' \otimes_D N)$. Let $q = y_1' \otimes x_1 + \ldots + y_r' \otimes x_r \in \mathcal{F}(Y,X)$ where $\{x_i\}$ and $\{y_i'\}$ are linearly independent sets. We can write each $y_i' \otimes x_i = w_i' \otimes \lambda_i^{-1}v_i$ with $w_i' \in M'$, $v_i \in N$, and $\lambda_i \in D$. Taking $b = w_1' \otimes v_1 + \ldots + w_r' \otimes v_r \in A^-$, we have, by Proposition 2.4(i), that $bQ^+b = qQ^+q$ since $Im(b) = Im(q)$ and $Im(b^{\#}) = Im(q^{\#})$. We needn't worry about the von Neumann regularity of b in Q because the whole pair Q is so. Let $p \in Q^+$ be such that $bpb = b$. To prove that A_b is a left order in Q_b we note that Q_b is isomorphic to the simple artinian associative algebra $b^{\#}X' \otimes Ybp$ under the mapping $bsb \to bsbp$, where $(Ybp, b^{\#}X')$ is a r-dimensional dual subpair of (Y, Y'). Under this isomorphism, A_b is isomorphic to $(b^{\#}N' \otimes_D Mbp)$ where $(Mbp, b^{\#}N', D)$ is a left order in $(Ybp, b^{\#}X')$. Indeed,

$$< Mbp, b^{\#}N' > = < Mbpb, N' > = < Mb, N' > \subset < N, N' > \subset D$$

and the remaining conditions can be verified easily. Hence A_b is a left order in Q_b by Proposition 7.3, which completes the proof. □

Remarks. We note that Propositions 8.4 and 8.2 provide a generalization of Proposition 7.3 to arbitrary simple algebras coinciding with their socle (i.e, with nonzero socle).

In the particular case that the dual pairs (X, X') and (Y, Y') are finite dimensional Proposition 8.4 implies the following assertion: Let D be an order in a division associative algebra Δ, and p, q two positive integers. Then any subpair A of $Q = (\mathcal{M}_{p,q}(\Delta), \mathcal{M}_{q,p}(\Delta^{op}))$ containing $(\mathcal{M}_{p,q}(D), \mathcal{M}_{q,p}(D^{op}))$ is an order in Q.

Orders in associative pairs satisfy a property that will be very useful in what follows. We stress such a property by saying that an associative pair $A = (A^+, A^-)$ is a *left triple product order* in an associative pair $Q = (Q^+, Q^-)$ if for any $q \in Q^\pm$ there exists $a \in A^\mp$ and $b \in A^\pm$ such that $baq \in A^\pm$ and $bap \neq 0$ for any nonzero p in the principal right ideal $\Pi(Q^\mp, Q^\pm)q$.

Proposition 8.5. *Let A be an associative pair which is a left order in an associative pair Q. Then A is a left triple product order in Q.*

Proof. Let $0 \neq q \in Q^-$. Then there exists $b \in A^-$ such that b is von Neumann regular in Q, $q \in bQ^+b$, and A_b is a left order in the unital algebra Q_b. Taking $bab \in A_b$ a left denominator for q, we have $baq = bab \cdot q \in A_b \subset A^-$. On the other hand, if bwb is the inverse of bab in Q_b, we have $bwbab = bwb \cdot bab = b$. Hence, for $p \in \Pi(Q^+, Q^-)q$, $bap = 0$ implies $p = 0$. □

Let us see how far we can go with the definition of left triple product order in associative pairs.

Proposition 8.6. *Let A be an associative pair which is a left triple product order in an associative pair Q. Then*

(i) *$L \cap A^- \neq 0$ for any nonzero left ideals L of Q contained in Q^-.*
(ii) *Q is semiprime if A is semiprime.*

Proof. (i) follows from the definition, and (ii) is an immediate consequence of (i). □

Proposition 8.7. *Let A be a semiprime associative pair which is a left triple product order in an associative pair Q. Then*

(i) *$lan_Q(X) \cap A^- = lan_A(X)$ for any subset X of A^+.*
(ii) *For $X, Y \subset A^+$ we have that $lan_A(X) \subset lan_A(Y)$ if and only if $lan_Q(X) \subset lan_Q(Y)$.*
(iii) *$Z_l(A) = Z_l(Q) \cap A$.*
(iv) *A is left nonsingular if and only if Q is so.*
(v) *For any $b \in A^-$ the local algebra Q_b of Q at b is a tight left cover of A_b.*

(vi) *Any direct sum* $\{L_i\}$ *of nonzero left ideals of* A *gives rise to a direct sum* $\{\widetilde{L}_i\}$ *of nonzero left ideals of* Q. *Moreover, if for some* $d \in A^-$, *all the* L_i *are contained in* A^-A^+d, *then all the* $\widetilde{L}_i$ *are contained in* Q^-Q^+d.

(vii) *If* Q *has finite left (local) Goldie dimension, then* A *has finite left (local) Goldie dimension as well.*

Proof. (i) Note that clearly $lan_Q(X) \cap A^-$ is contained in $lan_A(X)$. Conversely, let $z \in A^-$ such that z does not belong to $lan_Q(X)$, we have by Lemma 2.1 that there exists $x \in X$ and $q \in Q^+$ such that qzx is nonzero. By definition of left triple product order, there exist $b \in A^-$ and $a \in A^+$ such that $baq \in A^-$ and $baqzx \neq 0$. Hence $z \notin lan_A(X)$.
(ii) By (i), $lan_Q(X) \subset lan_Q(Y)$ implies $lan_A(X) \subset lan_A(Y)$. Conversely, suppose that there exists $q \in lan_Q(X)$ such that $q \notin lan_Q(Y)$. Then, by Lemma 2.1, $qYQ^- \neq 0$. Since A is a left triple product order in Q, we can find $b \in A^-$ and $a \in A^+$ such that $baq \in A^-$ and $baqYQ^- \neq 0$. Hence $baq \in lan_Q(X) \cap A^- = lan_A(X)$, by (i), with $baq \notin lan_A(Y)$.
(iii) Let $z \in Z_l(A)^+$ and L be a nonzero left ideal of Q contained in Q^-. By Proposition 8.6(i), $L \cap A^-$ is a nonzero left ideal of A, and hence, by (i)

$$0 \neq L \cap A^- \cap lan_A(z) \subset L \cap lan_Q(z),$$

which proves that $z \in Z_l(Q)^+$. Conversely, let $z \in Z_l(Q)^+ \cap A^+$. Since A is semiprime, to prove that z belongs to $Z_l(A)^+$, we just need to verify that $lan_A(z) \cap A^-A^+d \neq 0$ for any $0 \neq d \in A^-$. Take $0 \neq c \in A^-$ such that $Q^-cd \neq 0$, and let $0 \neq qcd \in lan_Q(z)$, where $q \in Q^-$, which there exists because $z \in Z_l(Q)^+$. Since $qcd \neq 0$, we can find $a \in A^+$ and $b \in A^-$ such that $baq \in A^-$ with $baqcd \neq 0$. Hence, $0 \neq A^-cd \cap lan_Q(z) = A^-cd \cap lan_A(z)$ by (i), which proves that $z \in Z_l(A)^+$.
(iv) It follows from (iii) that if Q is left nonsingular then A is left nonsingular. Conversely, $Z_l(Q) \cap A = Z_l(A) = 0$ implies $Z_l(Q) = 0$ by Proposition 8.6(i).
(v) Let $0 \neq bqb \in bQ^+b$. Then there exist $c \in A^-$ and $d \in A^+$ such that $cdbqb$ is a nonzero element of A^-. By semiprimeness of A, there exists $t \in A^+$ such that $cdbqbtcdbqb \neq 0$, so $dbqbt$ is also nonzero. Again applying the definition of left triple product order, we find $e \in A^+$ and $f \in A^-$ such that $efdbq \in A^+$ and $efdbqbt \neq 0$. Use again the semiprimeness of A to get that $efdbqbtxefdbqbt \neq 0$ for some $x \in A^+$. Then $0 \neq b(txefd)bqb = btx(efdbq)b \in A_b \cdot (bqb) \cap A_b$.
(vi) Let $\sum L_i$ be a direct sum of nonzero left ideals of A contained, say in A^-. Without loss of generality we may assume, for each index i, that $L_i = A^-a_ix_i$, where $a_i \in A^+$ and $x_i \in L_i$. We claim that the left ideals $Q^-a_ix_i$ of Q form a direct sum. Suppose on the contrary that $q_1a_1x_1 + \ldots + q_na_nx_n = 0$ where some of the summands, say $q_1a_1x_1$ is nonzero. Applying the definition of left

triple product order, we find $c_1 \in A^+$ and $b_1 \in A^-$ such that $b_1c_1q_1 \in A^-$ and $b_1c_1q_1a_1x_1 \neq 0$. Hence $0 = b_1c_1(q_1a_1x_1) + \ldots + b_1c_1(q_na_nx_n)$ with $0 \neq b_1c_1(q_1a_1x_1) \in L_1$ implies that some of the remaining summands, say $(b_1c_1q_2)a_2x_2$, is nonzero. Repeat the above argument with q_1 replaced by $(b_1c_1q_2)$. After repeating the above argument at most n-times, we obtain a sum $l_1 + \ldots + l_n = 0$ where each $l_i \in L_i$ and at least one of them is nonzero, which is a contradiction. In the case that all the L_i are contained in A^-A^+d for some $d \in A^-$, then the above construction provides now ideals $\widetilde{L}_i$ contained in Q^-Q^+d.
(vii) It is a direct consequence of (vi). □

While, as it will be seen later, orders in semiprime associative pairs with dcc on principal inner are semiprime, we don't know whether this remains true for left triple product orders. Nevertheless, with this additional requirement, everything works as expected.

Theorem 8.8. *Let A be a semiprime associative pair which is a left triple product order in a semiprime associative pair Q coinciding with its socle. Then*

(i) *A is left local Goldie,*
(ii) *for each $b \in A^-$ the local algebra A_b of A at b is a left order in the semiprime artinian associative algebra Q_b.*

Moreover,

(1) *A is prime if and only if Q is simple, and*
(2) *A has finite left Goldie dimension if and only if Q is artinian.*

Proof. (i) Since Q is nonsingular by Corollary 3.5, it follows from Proposition 8.7(iv) that A is left nonsingular. We also have that Q has finite left local Goldie dimension (Proposition 5.2(vi)), and hence A has finite left local Goldie dimension as well (Proposition 8.7(vii)).
(ii) Let $b \in A^-$. Since, by Proposition 5.2(i)(v), Q_b is a semiprime artinian associative algebra, by (i) together with Proposition 5.2(iv) and Proposition 5.3, A_b is left local Goldie. Now, since Q_b is a tight left cover of A_b (Proposition 8.7(v)), we can apply Proposition 7.4 to obtain that A_b is a left order in Q_b.
(1) If A is prime then Q is simple by Proposition 8.6(i) and by the structure of the socle (if Q is not simple then it contains two orthogonal nonzero ideals). Conversely, if Q is simple then all its local algebras Q_b at nonzero elements are simple (Proposition 5.2(iii)). Since A_b is a left order in Q_b by (ii), A_b is prime by the classical Goldie theorem, and hence A is prime by the local characterization of primeness (Proposition 5.2(ii)).
(2) Since semiprime artinian associative pairs satisfy the acc on all left ideals (Theorem 2.5(3)), it follows from Proposition 8.7(vii) that A has finite left

Goldie dimension whenever Q is artinian. Conversely, if Q is not artinian then it has infinite left Goldie dimension (Proposition 2.6), and hence A has also infinite left Goldie dimension (Proposition 8.6(i)). □

Let us return to orders in associative pairs. As announced, we next prove (together with other useful results) that orders in semiprime associative pairs with dcc on principal inner ideals are semiprime.

Proposition 8.9. *Let A be an associative pair which is an order in a semiprime associative pair Q equal to its socle. Then*

(i) *A is semiprime,*
(ii) *for each $q \in Q^-$, there exists $x \in A^-$ such that $qQ^+q = xQ^+x$,*
(iii) *any essential ideal I of A is also an order in Q.*

Proof. (i) Let $0 \neq x \in A^-$. Then there exists $b \in A^-$ such that $x \in bQ^+b$ and the local algebra A_b of A at b is an order in the semiprime artinian algebra Q_b (Proposition 5.2(v)). Hence A_b is a semiprime Goldie algebra by the classical Goldie theorem. In particular, Q_b is a tight left cover of A_b, so $A_b \cdot x \cap A_b \neq 0$. Take a nonzero element $bab \cdot x = bax \in A_b \cdot x \cap A_b$. By semiprimeness of A_b, $0 \neq (bax) \cdot A_b \cdot (bax) = baxA^+bax$, which implies $xA^+x \neq 0$, as required.
(ii) Given $q \in Q^-$ there exists $b \in A^-$ such that $q \in bQ^+b$ and A_b is an order in the semiprime artinian associative algebra Q_b. Then, by Lemma 7.6(ii), $q \cdot Q_b \cdot q = x \cdot Q_b \cdot x$ for some $x \in A_b$. Hence $qQq = q \cdot Q_b \cdot q = x \cdot Q_b \cdot x = xQx$.
(iii) As above, given $q \in Q^-$ we take $b \in A^-$ such that $q \in bQ^+b$ and A_b is an order in the semiprime artinian associative algebra Q_b. It is easy to see that bI^+b is an essential ideal of A_b (if $bab \in ann_{A_b}(bI^+b)$, then $bab \in ann_A(I^+)$ by Proposition 2.2(ii), and hence $bab = 0$ because I is an essential ideal of A). Now, by Proposition 7.5, the generalized local algebra $B =: bI^+b = I_b$ of I at b is also a left order in $K =: Q_b$. Take a denominator $bub \in Inv(K)$, with $u \in I^+$, for q. By Lemma 7.1(ii), B_{bub} is an order in K_{bub}. Now, by Lemma 5.1(ii), $B_{bub} = (I_b)_{bub}$ coincides with I_{bub}, and $K_{bub} = (B_b)_{bub}$ agrees with Q_{bub}. Therefore, we have proved that given $q \in Q^-$, there exists $bub \in I^-$ (von Neumann regular in Q) such that $q \in (bub)Q^+(bub)$ and I_{bub} is an order in Q_{bub}, as required. □

Everything is ready to prove the main result of this paper.

Theorem 8.10. *For an associative pair A the following conditions are equivalent:*

(i) *A is an order in a semiprime associative pair Q coinciding with its socle.*
(ii) *A is semiprime, satisfies the acc on $lan_A(x)$, $x \in A^+$, and has finite both left and right local Goldie dimension,*
(iii) *A is a semiprime local Goldie associative pair,*

(iv) *A is semiprime and all its local algebras are Goldie.*

In this case,

(1) *A is prime if and only if Q is simple, and*

(2) *A is Goldie if and only if Q is artinian.*

Proof. First we note that the equivalence (iii) $\Leftrightarrow$ (iv) follows from Proposition 5.2(iv) and Proposition 5.3, and that the implication (ii) $\Rightarrow$ (iii) is a consequence of Proposition 3.6.
(i) $\Rightarrow$ (ii). By Proposition 8.9(i), A is semiprime. Moreover, since orders are left (and right) triple product orders (Proposition 8.5), it follows from Theorem 8.8(i) that A has finite both left and right local Goldie dimensions, and from Proposition 8.7(ii) together Proposition 2.5(1)(ii) that A satisfies the acc on the left annihilators of a single element.
(iii) $\Rightarrow$ (i). By Theorem 6.5, A is an essential subdirect sum of prime local Goldie associative pairs. More precisely,

$$\oplus M_\alpha \triangleleft A \leq \oplus A/ann(M_\alpha),$$

where M_α ranges over all maximal uniform ideals of A. This allows us to reduce the question to the case that A is prime. Indeed, if we prove that each $A_\alpha := A/ann(M_\alpha)$ is an order in a (simple) associative pair Q_α coinciding with its socle, then, by Proposition 8.9(iii), M_α (regarded as an ideal of A_α) is also an order in Q_α. Since direct sums preserve orders, $\oplus M_\alpha$ is an order in the nondegenerate associative pair coinciding with its socle $\oplus Q_\alpha$, and hence A is also an order in $\oplus Q_\alpha$. Suppose then that A is a prime local Goldie associative pair. Let $(\mathcal{A}, e)$ be the standard imbedding of A. Then $\mathcal{A}$ is prime (Proposition 4.2), nonsingular (Proposition 4.5), and the ideal [4, Proposition 1] $I(\mathcal{A})$ of the elements of $\mathcal{A}$ having finite both left and right Goldie dimension is nonzero. Indeed, by Proposition 4.3, $A^+, A^- \subset I(\mathcal{A})$. In particular, $\mathcal{A}$ has uniform left and right ideals, hence by [4, Th.1 and its proof], $\mathcal{A}$ can be embedded in a prime associative algebra (indeed primitive) $\mathcal{Q}$ with nonzero socle $Soc(\mathcal{Q})$, such that $I(\mathcal{A})$ is a Fountain-Gould order in the simple associative algebra coinciding with its socle $Soc(\mathcal{Q})$. Thus, Proposition 8.3(ii) implies that the associative pair $(I(\mathcal{A}), I(\mathcal{A}))$ is an order in the associative pair $(Soc(\mathcal{Q}), Soc(\mathcal{Q}))$.

It is readily seen that $Q = (eSoc(\mathcal{Q})(1-e), (1-e)Soc(\mathcal{Q})e)$ is a simple associative pair equaling its socle (from the same properties satisfied by $Soc(\mathcal{Q})$ as an algebra). We claim that $A = (\mathcal{A}_{12}, \mathcal{A}_{21})$ is an order in Q: Let $q_{12} \in Q^+ \subset Soc(\mathcal{Q})$ ($Soc(\mathcal{Q})$ is an ideal of $\mathcal{Q}$). Since $(I(\mathcal{A}), I(\mathcal{A}))$ is an order in $(Soc(\mathcal{Q}), Soc(\mathcal{Q}))$, there exists, by Proposition 8.9(ii), $x \in I(\mathcal{A})$ such that $q_{12}Soc(\mathcal{Q})q_{12} = xSoc(\mathcal{Q})x$. This implies that

$$xSoc(\mathcal{Q})x = q_{12}Soc(\mathcal{Q})q_{12} = eq_{12}Soc(\mathcal{Q})q_{12}(1-e) \subset e\mathcal{Q}(1-e)$$

and, since $x \in I(\mathcal{A}) \subset Soc(\mathcal{Q})$ is von Neumann regular in $Soc(\mathcal{Q})$, $x \in xSoc(\mathcal{Q})x \subset e\mathcal{Q}(1-e)$. Thus $x = ex(1-e)$, i.e., $x \in \mathcal{A}_{12} = A^+$. On the other hand, q_{12} being von Neumann regular (as any other element) in $Soc(\mathcal{Q})$ implies $q_{12} \in q_{12}Soc(\mathcal{Q})q_{12} = xSoc(\mathcal{Q})x = x(1-e)Soc(\mathcal{Q})ex$, proving 8.1(ii). Now 8.1(i) follows from von Neumann regularity of x in $Soc(\mathcal{Q})$ ($x \in xSoc(\mathcal{Q})x = x(1-e)Soc(\mathcal{Q})ex$, since $x \in \mathcal{A}_{12}$) and we just need to establish 8.1(iii), i.e., A_x is an order in Q_x. Indeed, notice that

$$\begin{aligned} A_x =& x(1-e)\mathcal{A}ex = ex(1-e)\mathcal{A}ex(1-e) = x\mathcal{A}x \supset \\ & xI(\mathcal{A})x = I(\mathcal{A})_x \supset xA^-x = A_x \end{aligned}$$

and

$$Q_x = x(1-e)Soc(\mathcal{Q})ex = ex(1-e)Soc(\mathcal{Q})ex(1-e) = xSoc(\mathcal{Q})x = Soc(\mathcal{Q})_x$$

with coincidence also in their products, so that we will finish as soon as we prove that $I(\mathcal{A})_x$ is an order in $Soc(\mathcal{Q})_x$. But this follows from Proposition 8.5 and Theorem 8.8(ii) applied to $(I(\mathcal{A}), I(\mathcal{A}))$ which is an order in $(Soc(\mathcal{Q}), Soc(\mathcal{Q}))$.

Finally (1) and (2) were proved in Theorem 8.8, in view of Proposition 8.5. □

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