LOCAL RINGS OF EXCHANGE RINGS

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Abstract. We characterize the exchange property for non-unital rings in terms of their local rings at elements, and we use this characterization to show that the exchange property is Morita invariant for idempotent rings. We also prove that every ring contains a greatest exchange ideal (with respect to the inclusion).

Introduction.

Local algebras at elements were introduced by Meyberg [M], and play a fundamental role in the structure theory of Jordan systems (see [AMcC, ACM, McC]). They have also proved to be very useful in the context of associative pairs, in order to develop a Goldie-like theory [FGGS], and to study orders in semiprime (non-unital) rings with involution which coincide with their socles [S]. We will see in the present

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paper that local rings are also a useful tool for the study of non-unital exchange rings.

For an element x in a ring I we define the local ring of I at x as the additive subgroup xIx, endowed with the product $xyx \centerdot xzx := xyxzx$. We denote it by I_x. If x is a von Neumann regular element, then I_x is a unital ring and x is the unit (recall that an element $x \in I$ is said to be **von Neumann regular** if there exists $y \in I$ such that $x = xyx$). In particular, if e is an idempotent of I, then I_e is the subring eIe of I.

The exchange property for modules was introduced by Crawley and Jónsson in [CJ]. Roughly, the exchange property is what is needed to have suitable versions of the Krull-Schmidt Theorem even when the modules do not decompose as direct sums of indecomposables. Following Warfield [W_2], we say that an associative unital ring R is an **exchange ring** if R_R has the exchange property. Warfield proved that this property is left-right symmetric. A useful characterization of exchange rings was obtained independently by Goodearl [GW] and Nicholson [N]. This characterization was adapted by the first author in [A_1] to give the definition of an exchange ring without unit. Namely, a ring without unit I is said to be an **exchange ring** if for each $x \in I$ there exist an idempotent $e \in I$ and elements $r, s \in I$ such that $e = rx = s + x - sx$ [A_1, Lemma 1.1]. As in the unital case, the definition is left-right symmetric, and it gives exchange properties for suitable module decompositions (see [A_1, Theorem 1.2]).

We give in Section 1 a characterization of exchange rings in terms of their local rings at elements: A ring I is an exchange ring if and only if, for every $x \in I$, the ring I_x is an exchange ring. As a consequence, I is an exchange ring if and only if every principal left (right) ideal of I is an exchange ring.

In section 2 we establish that the exchange property for idempotent rings (those I such that $I^2 = I$) is Morita invariant in the sense that if I and J are idempotent rings such that $I-$Mod and $J-$Mod are equivalent categories, then I is an exchange ring if and only if J is an exchange ring. Here $I-$Mod is the category of "unital" nondegenerate right R-modules (see Section 2 for definitions).

Section 3 is devoted to show an equivalent condition for the exchange property: a ring I is an exchange ring if and only if every local ring of I at an idempotent is an exchange ring and I/I_0 is a radical ring, where I_0 denotes the ideal of I generated by the idempotents of I. Note that the local rings I_e, with $e = e^2 \in I$, are the unital rings eIe, and so our characterization allows to reduce the verification of the exchange property for a non-unital ring almost entirely to unital rings. Moreover we prove the existence of a maximum exchange ideal of every ring.

From now on, all our modules over unital rings will be unital (i.e. $m = m1_R$ for all $m \in M_R$ or $m = 1_R m$ for all $m \in {}_RM$). If I is a ring, we denote by I^1 the unitization of I, that is, $I^1 = I \oplus \mathbb{Z}$ with addition by componentwise and multiplication defined by $(x, n)(y, m) = (xy + ny + mx, nm)$ for all $x, y \in I$ and $n, m \in \mathbb{Z}$. Note that I is an ideal of I^1.

§1. Local rings of exchange rings.

In this section we prove that the exchange property for a ring is equivalent to the exchange property for every local ring at an element of the ring. As a consequence we obtain that a ring I is an exchange ring if and only if every principal left ideal of I is an exchange ring.

A ring without unit I is an **exchange ring** if, for every element $x \in I$, the equivalent conditions in next Lemma are satisfied. Other characterizations of the exchange property for non necessarily unital rings can be found in [A_1].

Lemma 1.1. *(*[A_1, Lemma 1.1]*) Let I be a ring without unit and let R be a unital ring containing I as an ideal. Then the following conditions are equivalent for an element $x \in I$:*

- (i) *There exists $e^2 = e \in I$ with $e - x \in R(x - x^2)$.*
- (ii) *There exist $e^2 = e \in Ix$ and $c \in R$ such that $(1-e)-c(1-x) \in J(R)$.*
- (iii) *There exists $e^2 = e \in Ix$ such that $R = Ie + R(1 - x)$.*
- (iv) *There exists $e^2 = e \in Ix$ such that $1 - e \in R(1 - x)$.*
- (v) *There exist $r, s \in I$, $e^2 = e \in I$ such that $e = rx = s + x - sx$.*

Radical rings and π-regular rings are examples of exchange rings. Many other examples can be found in [N], [Sto], [AGOP], [A_1]. Recall that a ring I is said to be **π-regular** if for each $x \in I$ there exist a positive integer n and $y \in I$ such that $x^n = x^n y x^n$. Also, recall that a **radical ring** is a ring of the form $J(R)$ for some unital ring R, where $J(\cdot)$ denotes the Jacobson radical. These rings are characterized intrinsically by the following condition: I is a radical ring if and only if for every $x \in I$ there exists $y \in I$ such that $y + x - yx = 0$ (see [J]).

It is apparent from this and Lemma 1.1(v) that the radical rings are exactly the exchange rings without nonzero idempotents.

Usually we will consider a non-unital exchange ring I which is an ideal of a unital ring R. We will shorten this situation saying that "I is an exchange ideal of R". We will make repeated use of the following result, which we state here in a form which is more suited for the present work.

Theorem 1.2. *(*[A_1, Theorem 2.2]*) Let A and B be two (possibly non-unital) rings and let $f : A \to B$ be a surjective ring homomorphism. Then A is an exchange ring if and only if B and $\ker(f)$ are exchange rings and idempotents of B lift to idempotents of A.*

Proposition 1.3. *A ring I is an exchange ring if and only if every left ideal of I is an exchange ring.*

Proof. Let L be a left ideal of I and suppose that I is an exchange ring. Take $x \in L$. By Lemma 1.1(iv), there exists $e^2 = e \in Ix \subseteq L$ such that $1 - e \in I^1(1 - x)$. Write $1 - e = (\alpha - e)(1 - x)$ for some $\alpha \in I^1$. As $1-e = \alpha(1-x) - e + ex$, we have $1 = ex + \alpha(1-x)$. By Lemma 1.1(iv) there exist orthogonal idempotents e_1, e_2 in I^1, with $1 = e_1 + e_2$, such that $e_1 \in Iex$ and $e_2 \in I^1\alpha(1 - x)$. Note that $e_1 \in Iex \subseteq ILx \subseteq Lx$.

Since $e_2 \in I^1\alpha(1-x)$, there exists $\beta \in I^1$ such that $e_2 = (1-\beta)(1-x)$, so $e_2 = 1-x-\beta+\beta x$ implies $1-e_2 = x+\beta-\beta x$. As $1-e_2, x \in L$, it follows that $\beta \in L$, so $e_2 = (1-\beta)(1-x) \in L^1(1-x)$.

We have proved that, given $x \in L$, there exists an idempotent $e_1 \in Lx$ such that $1 - e_1 \in L^1(1 - x)$, so L is an exchange ring by Lemma 1.1(iv). $\square$

If I is a ring and x an element in I, we define the **local ring of** I **at** x, and denote it by I_x, as the ring xIx with product

$$xyx \centerdot xzx := xyxzx.$$

We are now ready to prove the main result of this Section, which provides a characterization of the exchange property for a non-unital ring in terms of its local rings at elements.

Theorem 1.4. *A ring I is an exchange ring if and only if, for every $x \in I$, the local ring of I at x is an exchange ring.*

Proof. Suppose that I is an exchange ring and let $x \in I$. Consider the map

$$\begin{array}{rccc} \varphi: & Ix & \to & I_x \\ & yx & \mapsto & xyx \end{array}$$

For $y, z \in I$, we have $\varphi(yxzx) = xyxzx = xyx \centerdot xzx = \varphi(yx) \centerdot \varphi(zx)$, where the dot denotes the product in I_x, so φ is a ring homomorphism. By Proposition 1.3, Ix, which is a left ideal of I, is an exchange ring, and by Theorem 1.2, $I_x = \varphi(Ix)$ is an exchange ring.

Conversely, assume that all the local rings at elements of I are exchange rings. For every $x \in I$, denote by I^x the ring which coincides with I as an abelian group, and whose product is given by $r \circ s = rxs$.

Step 1: I^x is an exchange ring.

Consider the map

$$\begin{array}{rccc} \varphi: & I^x & \to & I_x \\ & y & \mapsto & xyx \end{array}$$

We have $\varphi(y \circ z) = \varphi(yxz) = xyxzx = xyx \centerdot xzx = \varphi(y) \centerdot \varphi(z)$, so φ is a ring homomorphism. As φ is an epimorphism, $I^x/Ker(\varphi) \cong I_x$, which is an exchange ring. It is very easy to check that $(\mathrm{Ker}(\varphi))^3 = 0$ and so, by [A_1, Corollary 2.4], I^x is an exchange ring.

Step 2: I is an exchange ring.

Take an element x in I. Denote by I^1 and I^{x1} the unitizations of I and I^x respectively. Observe that $(0, 1)$ is the unity of these two rings. We denote also by $\circ$ the product in I^{x1}. Since I^x is an exchange ring (step 1), it follows from Lemma 1.1(iv) that, given $x \in I$, there exist $e \circ e = e \in I^x \circ x$ such that $1 - e \in I^{x1} \circ (1 - x)$. Let $(r, 1)$ be in I^{x1} such that $(-e, 1) = (r, 1) \circ (-x, 1)$.

We have $(r, 1) \circ (-x, 1) = (-rx^2 - x + r, 1) = (r + rx, 1)(-x, 1)$ (this last product in the ring I^1), so $1 - e = (r + rx + 1)(1 - x)$. As $1 = e + (1 - e) = e \circ e + (1 - e) = exe + (r + rx + 1)(1 - x)$, then $I^1 = Ixe + I^1(1 - x)$, with xe idempotent of I and $xe \in Ix^2 \subseteq Ix$. By Lemma 1.1(iii), I is an exchange ring. □

The last part of the following result was already proved in [A_1, Proposition 1.3].

Corollary 1.5. *Let I be an exchange ideal of a unital ring R. Then for every element $x \in R$, I_x is an exchange ring. In particular, if e is an idempotent in R, then eIe and $(1 - e)I(1 - e)$ are exchange rings.*

Proof. Let $x \in R$. Note that as Ix is a left ideal of I, then Ix is an exchange ring by Proposition 1.3. Since I_x is a homomorphic image of Ix (see the proof of Theorem 1.4), we deduce from Theorem 1.2 that I_x is an exchange ring. □

Corollary 1.6. *A ring I is an exchange ring if and only if every principal left ideal of I is an exchange ring.*

Proof. For an exchange ring I we have seen that every left ideal of I is an exchange ring (Proposition 1.3). For the converse, take an element $x \in I$ and consider the ring epimorphism

$$\begin{array}{rccc} \varphi : & Ix & \to & I_x \\ & rx & \mapsto & xrx \end{array}$$

As $Ix/\mathrm{ker}\varphi \cong I_x$ and Ix is an exchange ring, we get that I_x is an exchange ring. Therefore, I_x is an exchange ring for all $x \in I$, and so I is an exchange ring by Theorem 1.4. □

§2. Morita invariance of the exchange property.

In this section we obtain that the exchange property is Morita invariant for idempotent rings. Our interest in this question was motivated by the observation that, for any simple ring with identity R, every local ring at a nonzero element of R is Morita equivalent to R (see Proposition 2.5). First of all, we present our key result, without speaking about Morita contexts or Morita equivalent rings. Recall that a ring R is said to be an **idempotent ring** if $R = R^2$, that is, every element of R is a finite sum of elements of the form r_1r_2, with $r_i \in R$ for $i \in \{1, 2\}$.

Theorem 2.1. *Let R be an idempotent exchange ring and let S be any ring. Assume that there is a 6-tuple $(R, S, P, Q, \varphi, \psi)$ with ${}_RP_S$, ${}_SQ_R$ bimodules such that $P = RP$, $Q = QR$, and $\varphi : Q\otimes_R P \to S$, $\psi : P\otimes_S Q \to R$ are S-bimodule and R-bimodule maps, respectively, satisfying the compatibility relations: $\varphi(q\otimes p)q' = q\psi(p\otimes q')$, $p'\varphi(q\otimes p) = \psi(p' \otimes q)p$ for every $p, p' \in P$, $q, q' \in Q$. Suppose that φ is surjective. Then S is an exchange ring.*

Proof. Let $s \in S$. There exist $p_1, \ldots, p_n \in P$, $q_1, \ldots, q_n \in Q$, $r_1, \ldots, r_n$, $r'_1, \ldots, r'_n \in R$, such that $s = \varphi(\sum_{i=1}^n q_ir_i \otimes r'_ip_i)$ (using that $QR = Q$, $RP = P$ and the surjectivity of φ).

Let ${}^nQ$ denote the S-$M_n(R)$-bimodule of n-rows with coefficients in Q, and let P^n denote the $M_n(R)$-S-bimodule of n-columns with coefficients in P. We have $\varphi^{(n)} : {}^nQ \otimes_{M_n(R)} P^n \to S$ defined by:

$$\varphi^{(n)} \left((q'_1, \ldots, q'_n) \otimes \begin{pmatrix} p'_1 \\ \vdots \\ p'_n \end{pmatrix} \right) = \varphi(\sum_{i=1}^{n} q'_i \otimes p'_i)$$

and $\psi^{(n)} : P^n \otimes_S {}^nQ \to M_n(R)$ given by:

$$\psi^{(n)} \left(\begin{pmatrix} p'_1 \\ \cdots \\ p'_n \end{pmatrix} \otimes (q'_1, \ldots, q'_n) \right) = \left(\psi(p'_i \otimes q'_j) \right).$$

Taking into account that $M_n(R)$ is an exchange ring ([A$_1$, Theorem 1.4]), it is easily seen that $(M_n(R), S, P^n, {}^n Q, \varphi^{(n)}, \psi^{(n)})$ satisfies all the same conditions as the original 6-tuple.

Consider $P_1 := M_n(R)(p_1, \ldots, p_n)^t \subseteq P^n$, and $Q_1 := (q_1, \ldots, q_n) \cdot M_n(R) \subseteq {}^nQ$.

Let $\varphi_1 : Q_1 \otimes_{M_n(R)} P_1 \to S$ be the map induced by $\varphi^{(n)}$. Write $T = Q_1 \otimes_{M_n(R)} P_1$. Then T with the following product is a ring:

$$(q_1 \otimes p_1)(q'_1 \otimes p'_1) = q_1 \psi^{(n)}(p_1 \otimes q'_1) \otimes p'_1,$$

where $p_1, p'_1 \in P_1$ and $q_1, q'_1 \in Q_1$.

Write $\mathbf{p} = (p_1, \ldots, p_n)^t$ and $\mathbf{q} = (q_1, \ldots, q_n)$. We shall see that T is an exchange ring. Let $t \in T$. Put $t = \sum_i \mathbf{q} r_i \otimes r'_i \mathbf{p}$ $(r_i, r'_i \in M_n(R))$. Set $a = \sum_i \psi^{(n)}(\mathbf{p} \otimes \mathbf{q} r_i) \psi^{(n)}(r'_i \mathbf{p} \otimes \mathbf{q}) \in M_n(R)$.

Define $\alpha : T_t \to M_n(R)_a$ by

$$\alpha(t') = \alpha(\sum_j \mathbf{q} r''_j \otimes r'''_j \mathbf{p}) = \sum_j \psi^{(n)}(\mathbf{p} \otimes \mathbf{q} r''_j) \psi^{(n)}(r'''_j \mathbf{p} \otimes \mathbf{q}).$$

<u>α is well-defined:</u>

Assume $\sum_j \mathbf{q} r''_j \otimes r'''_j \mathbf{p} = 0$ in $Q_1 \otimes_{M_n(R)} P_1$.

Then $\varphi^{(n)}(\sum_j \mathbf{q} r''_j \otimes r'''_j \mathbf{p}) = 0$ and

$$\sum_j \psi^{(n)}(\mathbf{p} \otimes \mathbf{q} r''_j) \psi^{(n)}(r'''_j \mathbf{p} \otimes \mathbf{q}) = \sum_j \psi^{(n)}(\mathbf{p} \otimes \mathbf{q} r''_j \psi^{(n)}(r'''_j \mathbf{p} \otimes \mathbf{q}))$$

$$= \sum_j \psi^{(n)}(\mathbf{p} \otimes \varphi^{(n)}(\mathbf{q} r''_j \otimes r'''_j \mathbf{p}) \mathbf{q}) = \psi^{(n)}(\mathbf{p} \otimes (\sum_j \varphi^{(n)}(\mathbf{q} r''_j \otimes r'''_j \mathbf{p})) \mathbf{q})$$

$$= \psi^{(n)}(\mathbf{p} \otimes 0) = 0.$$

∎

Now, if $t' = tt_1t = (\sum \mathbf{q}r_i \otimes r_i'\mathbf{p})(\sum \mathbf{q}r_j'' \otimes r_j'''\mathbf{p})(\sum \mathbf{q}r_k \otimes r_k'\mathbf{p})$, then

$$\begin{aligned} t' &= (\sum \mathbf{q}r_i \otimes \psi^{(n)}(r_i'\mathbf{p} \otimes \mathbf{q}r_j'')r_j'''\mathbf{p})(\sum \mathbf{q}r_k \otimes r_k'\mathbf{p}) \\ &= \sum_{i,j,k} \mathbf{q}r_i \otimes \psi^{(n)}(r_i'\mathbf{p} \otimes \mathbf{q}r_j'')\psi^{(n)}(r_j'''\mathbf{p} \otimes \mathbf{q}r_k)r_k'\mathbf{p}, \end{aligned}$$

so

$$\begin{aligned} \alpha(t') &= \sum_{i,j,k} \psi^{(n)}(\mathbf{p} \otimes \mathbf{q}r_i)\psi^{(n)}(r_i'\mathbf{p} \otimes \mathbf{q}r_j'')\psi^{(n)}(r_j'''\mathbf{p} \otimes \mathbf{q}r_k)\psi^{(n)}(r_k'\mathbf{p} \otimes \mathbf{q}) \\ &= \sum_{i,j,k} \psi^{(n)}(\mathbf{p} \otimes \mathbf{q}r_i)\psi^{(n)}(r_i'\mathbf{p} \otimes \mathbf{q})r_j''r_j'''\psi^{(n)}(\mathbf{p} \otimes \mathbf{q}r_k)\psi^{(n)}(r_k'\mathbf{p} \otimes \mathbf{q}) \\ &= a(\sum_j r_j''r_j''')a \in M_n(R)_a. \end{aligned}$$

$\blacksquare$

α is surjective:

This is clear from the above expression and the fact that $R = R^2$.

α is a ring homomorphism:

$$\begin{aligned} &\alpha((tt_1t) \centerdot (tt_2t)) = \alpha(tt_1tt_2t) \\ &= \alpha((\sum \mathbf{q}r_i \otimes r_i'\mathbf{p})(\sum \mathbf{q}r_j^{(2)} \otimes r_j^{(3)}\mathbf{p})(\sum \mathbf{q}r_k \otimes r_k'\mathbf{p}) \\ &(\sum \mathbf{q}r_l^{(4)} \otimes r_l^{(5)}\mathbf{p})(\sum \mathbf{q}r_m \otimes r_m'\mathbf{p})) \\ &= \alpha(\sum \mathbf{q}r_i \otimes \psi^{(n)}(r_i'\mathbf{p} \otimes \mathbf{q})r_j^{(2)}r_j^{(3)}\psi^{(n)}(\mathbf{p} \otimes \mathbf{q}r_k) \\ &\psi^{(n)}(r_k'\mathbf{p} \otimes \mathbf{q})r_l^{(4)}r_l^{(5)}\psi^{(n)}(\mathbf{p} \otimes \mathbf{q}r_m)r_m'\mathbf{p}) \\ &= \sum \psi^{(n)}(\mathbf{p} \otimes \mathbf{q}r_i)\psi^{(n)}(r_i'\mathbf{p} \otimes \mathbf{q})r_j^{(2)}r_j^{(3)}\psi^{(n)}(\mathbf{p} \otimes \mathbf{q}r_k) \\ &\psi^{(n)}(r_k'\mathbf{p} \otimes \mathbf{q})r_l^{(4)}r_l^{(5)}\psi^{(n)}(\mathbf{p} \otimes \mathbf{q}r_m)\psi^{(n)}(r_m'\mathbf{p} \otimes \mathbf{q}) \\ &= a(\sum_j r_j^{(2)}r_j^{(3)})a(\sum_l r_l^{(4)}r_l^{(5)})a, \end{aligned}$$

while $\alpha(tt_1t) = a(\sum_j r_j^{(2)}r_j^{(3)})a$ and $\alpha(tt_2t) = a(\sum_l r_l^{(4)}r_l^{(5)})a$. So we obtain $\alpha(tt_1t) \centerdot \alpha(tt_2t) = \alpha((tt_1t) \centerdot (tt_2t))$.

Therefore we have proved that $\alpha : T_t \to M_n(R)_a$ is a surjective ring homomorphism.

Assume $\sum \mathbf{q}r_j'' \otimes r_j'''\mathbf{p} \in \mathrm{Ker}(\alpha)$. Then

$\sum \psi^{(n)}(\mathbf{p}\otimes \mathbf{q}r_j'')\psi^{(n)}(r_j'''\mathbf{p}\otimes \mathbf{q}) = 0.$

Now compute:

$$(\sum \mathbf{q}r_j''\otimes r_j'''\mathbf{p})^3 = (\sum \mathbf{q}r_j''\otimes r_j'''\mathbf{p})(\sum \mathbf{q}r_k''\otimes r_k'''\mathbf{p})(\sum \mathbf{q}r_l''\otimes r_l'''\mathbf{p})$$
$$= \sum \mathbf{q}r_j''r_j'''\psi^{(n)}(\mathbf{p}\otimes \mathbf{q}r_k'')\psi^{(n)}(r_k'''\mathbf{p}\otimes \mathbf{q})r_l''\otimes r_l'''\mathbf{p}$$
$$= \sum_{j,l}\mathbf{q}r_j''r_j'''(\sum_k \psi^{(n)}(\mathbf{p}\otimes \mathbf{q}r_k'')\psi^{(n)}(r_k'''\mathbf{p}\otimes \mathbf{q}))r_l''\otimes r_l'''\mathbf{p} = 0.$$

It follows that every element of $\mathrm{Ker}(\alpha)$ is nilpotent. Since $M_n(R)_a$ is an exchange ring by [A_1, Theorem 1.4] and Theorem 1.4, and $\mathrm{Ker}(\alpha)$ is a nilideal, we deduce from [A_1, Corollary 2.4] that T_t is an exchange ring. Since this holds for every $t \in T$, we see from Theorem 1.4 that T is an exchange ring. Now $\varphi_1(T)$ is an exchange ring, and since $s \in \varphi_1(T)$, we conclude that there exist $e = e^2 \in \varphi_1(T)$ and $z, v \in \varphi_1(T) \subseteq S$ such that $e = zs = v + s - vs$. Since s is an arbitrary element of S, we obtain that S is an exchange ring. □

In the classical Morita theory it is shown that two rings with identity R and S are Morita equivalent if and only if there is a Morita context between R and S. The approach to the Morita theory for rings without identity by means of Morita contexts appears in a number of papers (see [GS] and the references therein), in which many consequences are obtained from the existence of a Morita context between rings I and J. In particular it is shown in [Ky, Theorem] that, if I and J are arbitrary rings such that there is a Morita context for these rings, then the categories $I-$Mod and $J-$Mod are equivalent. It is proved in [GS, Proposition 2.3] that the converse implication holds for idempotent rings.

For an idempotent ring I we denote by $I-$Mod the full subcategory of the category of all left I-modules whose objects are the "unital" nondegenerate modules. Here a left I-module M is said to be **unital** if $M = IM$, and M is said to be **nondegenerate** if, for $m \in M$, $Rm = 0$ implies $m = 0$. Note that, if I has an identity, then $I-$Mod is the usual category of left $I-$modules.

We will use the well-known definition of a Morita context in the case where the rings I and J have not necessarily an identity. Let I and J be idempotent rings. We say that $(I, J, M, N, \varphi, \psi)$ is a **Morita context** if ${}_IM_J$ and ${}_JN_I$ are unital bimodules and $\varphi : N\otimes_I M \to J$, ψ :

$M \otimes_J N \to I$ are surjective J-bimodule and I-bimodule maps, respectively, satisfying the compatibility relations: $\varphi(n \otimes m)n' = n\psi(m \otimes n')$, $m'\varphi(n \otimes m) = \psi(m' \otimes n)m$ for every $m, m' \in M$, $n, n' \in N$.

The following result can be found in [GS] (see Proposition 2.5 and Theorem 2.7).

Theorem 2.2. *Let I and J be two idempotent rings. Then $I{-}Mod$ and $J{-}Mod$ are equivalent categories if and only if there exists a Morita context $(I, J, M, N, \varphi, \psi)$.*

The following result follows from Theorem 2.1 and Theorem 2.2.

Theorem 2.3. *Let I and J be two idempotent rings, and suppose that the categories $I{-}Mod$ and $J{-}Mod$ are equivalent. Then I is an exchange ring if and only if J is an exchange ring.*

Given an idempotent ring I, consider the module $M \in Mod - I$ which is the direct sum of countably many copies of I_I. Clearly $M^*M = I$. Choosing different submodules N of M^* such that $N = IN$ and $NM = I$, we will get different rings which are Morita equivalent to I, namely the rings $J = MN$, where MN is the subring of $\mathrm{End}(M_I)$ consisting of all finite sums of endomorphisms of the form $\theta_{m,n}$, for $\theta_{m,n}(m') = m\langle n, m'\rangle$. If we choose $N = \oplus_I I$ (countably many copies), with the obvious action on M, we get $J = FM(I)$, the ring of countably infinite matrices over I with only finitely many nonzero entries. If R is unital and $N = M^*$, then we get the ring $FR(R)$, which is the ring of countably infinite matrices over R having only a finite number of nonzero rows (these constructions can be found in [A_2]). So, we see from Theorem 2.3 that, if I is an idempotent exchange ring, then $FM(I)$ is an exchange ring, and if R is a unital exchange ring, then $FR(R)$ is an exchange ring (of course, the fact that $FM(I)$ is an exchange ring when I is an exchange ring follows also from [A_1, Theorem1.4]).

It is clear from the proof of Theorem 2.1 that local rings at elements are a useful tool in the study of Morita equivalence for non-unital rings. In our next result we will establish a further connection between these subjects.

Assume that I is embedded in a ring R, but not necessarily as an ideal. Let x be an element of R such that $IxI \subseteq I$. This kind of elements is related to the notion of approximation of non-unital rings by unital rings, introduced in [AHR]. Then we can consider as in Section 1 the rings I_x and I^x. Namely $I_x = xIx$ with the product $(xyx)\centerdot(xzx) = xyxzx$, and I^x is I as abelian group, but the product is defined by $y \circ z = yxz$. We have a surjective homomorphism with nilpotent kernel $\alpha : I^x \to I_x$, defined by $\alpha(y) = xyx$. It seems that the methods of Section 1 are not powerful enough to give a proof of the following result, which we will prove by using Theorem 2.1.

Theorem 2.4. *Let I be an idempotent exchange ring and let R be a ring containing I. Let x be an element of R satisfying $IxI \subseteq I$. Then I_x and I^x are exchange rings.*

Proof. We are going to define a 6-tuple $(I, I_x, {}_I(Ix)_{I_x}, {}_{I_x}(xI)_I, \varphi, \psi)$. To start, the actions ${}_I(Ix)$ and $(xI)_I$ are defined in the usual way. Now for $xyx \in I_x$ and $z \in I$, define $(zx)\centerdot(xyx) = zxyx$, and $(xyx)\centerdot(xz) = xyxz$. Now define $\varphi : xI \otimes_I Ix \to I_x$ by

$$\varphi(xy \otimes zx) = xyzx \qquad (y, z \in I),$$

and define $\psi : Ix \otimes_{I_x} xI \to I$ by

$$\psi(yx \otimes xz) = yxz \qquad (y, z \in I).$$

It is easily checked that the 6-tuple just defined verifies all the conditions in Theorem 2.1. It follows from Theorem 2.1 that I_x is an exchange ring. By using the map $\alpha : I^x \to I_x$ defined above, it follows from [A_1, Corollary 2.4] that I^x is an exchange ring. $\square$

As a corollary of (the proof of) Theorem 2.4, we get the following result, which motivated our study of the Morita invariance of the exchange property for idempotent rings.

Proposition 2.5. *Let R be a simple ring with identity. For every $x \in R \setminus \{0\}$ the local ring R_x of R at x is Morita equivalent to the ring R.*

Proof. Since R is a simple ring with identity, it is clear that the 6-tuple

$$(R, R_x, {}_R(Rx)_{R_x}, {}_{R_x}(xR)_R, \varphi, \psi)$$

defined in the proof of Theorem 2.4 is a Morita context. $\square$

§3. The maximum exchange ideal of a ring.

In this section we show that every ring contains a maximum exchange ideal. We first prove a characterization of exchange rings: A ring I satisfies the exchange property if and only if every local ring at an idempotent is an exchange ring and the quotient of I by the ideal generated by its idempotents is a radical ring. To do this, we need a preliminary result.

Lemma 3.1. *Let I be any ring and let $\{e_\alpha\}_{\alpha\in\Lambda}$ be a nonempty family of idempotents in I. Then the ring $T = [e_\alpha I e_\beta]$ of $\Lambda\times\Lambda$ matrices with only a finite number of nonzero entries, with entry (α,β) in $e_\alpha I e_\beta$, is an exchange ring provided all $e_\alpha I e_\alpha$ are exchange rings.*

Proof. Obviously it suffices to check the case in which Λ is finite, so assume that $e_1,\dots e_n$ are idempotents in I such that $e_i I e_i$ is an exchange ring for every $i\in\{1,\dots,n\}$. Consider the ring

$$T = \begin{pmatrix} e_1Ie_1 & \dots & e_1Ie_n \\ \dots & \dots & \dots \\ e_nIe_1 & \dots & e_nIe_n \end{pmatrix}.$$

This ring is unital, with $1_T = \begin{pmatrix} e_1 & & \\ & \ddots & \\ & & e_n \end{pmatrix}$, and as e_iIe_i is an exchange ring for every $i\in\{1,\dots,n\}$, [N, Corollary 2.6] says us that T is an exchange ring. □

Theorem 3.2. *Let I be any ring, and let $\{e_\alpha\}_{\alpha\in\Lambda}$ be a nonempty family of idempotents of I such that $e_\alpha I e_\alpha$ is an exchange ring for all $\alpha\in\Lambda$. Let I' denote the ideal of I generated by $\{e_\alpha\}_{\alpha\in\Lambda}$. Then I' is an exchange ring.*

Proof. The ring $J = [e_\alpha I e_\beta]_{\alpha,\beta\in\Lambda}$ is a ring with local units (see [Ab], [AM]). If we consider $P = \oplus e_\alpha I$ and $Q = \oplus I e_\alpha$, we have natural bimodule structures ${}_JP_{I'}$, ${}_{I'}Q_J$. Let $\psi : P\otimes_{I'} Q \to J$ be given by $\psi((e_\alpha x_\alpha)\otimes(x'_\beta e_\beta)) = (e_\alpha x_\alpha x'_\beta e_\beta)$ and $\varphi : Q\otimes_J P \to I'$ given by $\varphi((x'_\beta e_\beta)\otimes(e_\alpha x_\alpha)) = \sum_\alpha x'_\alpha e_\alpha x_\alpha$. Since all the rings $e_\alpha I e_\alpha$ are exchange rings by hypothesis, we obtain from Lemma 3.1 that J is an exchange ring. Therefore we see that the 6-tuple $(J, I', {}_JP_{I'}, {}_{I'}Q_J, \varphi, \psi)$ satisfies all the conditions of Theorem 2.1 (in fact it is a Morita context). It follows from Theorem 2.1 that I' is an exchange ring. □

Theorem 3.3. *Let I be any ring. Let I_0 be the ideal of I generated by the idempotents of I. Then the following conditions are equivalent:*

(i) *I is an exchange ring,*

(ii) *(a) I/I_0 is a radical ring,*
(b) eIe is an exchange ring for all $e = e^2 \in I$.

Proof. (i) $\Rightarrow$ (ii). By Theorem 1.2, I/I_0 is an exchange ring without nonzero idempotents, so I/I_0 is a radical ring and (a) is proved. (b) is contained in Corollary 1.5.

(ii) $\Rightarrow$ (i). By Theorem 1.2, it suffices to check that I_0 is an exchange ring, what is immediate from Theorem 3.2. $\square$

Recall than an idempotent e in a ring I is said to be **local** if $eIe/J(eIe)$ is a division ring, where for an arbitrary ring I, $J(I)$ denotes its Jacobson radical.

Corollary 3.4. *Let I be any ring. Let $\mathcal{L}(I)$ be the ideal generated by all the local idempotents of I. Then $\mathcal{L}(I)$ is an exchange ring.*

Proof. It is clear from Theorem 3.2 because, for every local idempotent $e \in I$, eIe is an exchange ring. $\square$

Theorem 3.5. *Every ring contains a greatest exchange ideal (with respect to the inclusion).*

Proof. Let I be a ring. Let I' be the ideal generated by all the idempotents $e \in I$ such that eIe is an exchange ring. Then I' is an exchange ring by Theorem 3.2. Let $\epsilon(I) = \pi^{-1}(J(I/I'))$, where $\pi : I \to I/I'$ is the canonical projection. Then $\epsilon(I)$ is an exchange ring, because $\epsilon(I)_0 = I'$ is an exchange ring and $\epsilon(I)/\epsilon(I)_0 = \epsilon(I)/I' = J(I/I')$ is a radical ring.

Now, assume that L is an ideal of I and that L is an exchange ring. Then, for every $e = e^2 \in L$, $eLe = eIe$ is an exchange ring and so L_0, the ideal of L generated by its idempotents, is contained in I'. So we get the chain of ideals $L_0 \leq L \cap I' \leq L$. Moreover, L/L_0 is a radical ring.

Now $\pi(L) = (L+I')/I' \cong L/(L\cap I') \cong (L/L_0)/(L\cap I'/L_0)$ is radical, so $\pi(L) \subseteq J(I/I')$. It follows that $L \subseteq \pi^{-1}(J(I/I')) = \epsilon(I)$. So $\epsilon(I)$ is an exchange ideal of I containing all the exchange ideals of I. $\square$

We close by computing the maximum exchange ideal $\epsilon(R)$ for a semilocal ring R.

Example 3.6. *Let R be a semilocal ring. Then $\epsilon(R) = J(R) + \mathcal{L}(R)$, where $J(R)$ is the Jacobson radical of R and $\mathcal{L}(R)$ the ideal generated by all the local idempotents of R. Moreover, no minimal idempotent in $R/\epsilon(R)$ can be lifted to an idempotent of R.*

Proof. Let e be a minimal exchange idempotent of R (if there is any). Then eR_R is indecomposable and satisfies the exchange property, so eRe is local [W_1, Proposition 1]. Since every idempotent in R is an orthogonal finite sum of minimal idempotents, we conclude that the ideal generated by all the exchange idempotents is $\mathcal{L}(R)$. Since $J(R/\mathcal{L}(R)) = (J(R) + \mathcal{L}(R))/\mathcal{L}(R)$, we obtain from the proof of Theorem 3.5 that $\epsilon(R) = J(R) + \mathcal{L}(R)$.

Let e be a minimal idempotent of the semisimple ring $R/\epsilon(R)$, and assume that e lifts to an idempotent f in R. Let $\rho : R \to R/\epsilon(R)$ be the canonical projection. Then $(fRf)/(f\epsilon(R)f) \cong \rho(f)(R/\epsilon(R))\rho(f)$ is a division ring, and so fRf is an exchange ring by using [A_1, Proposition 1.3] and Theorem 1.2. By the definition of $\epsilon(R)$ in Theorem 3.5, we see that $f \in \epsilon(R)$, and so $e = \rho(f) = 0$, which is a contradiction. It follows that no minimal idempotent of $R/\epsilon(R)$ can be lifted to an idempotent of R. $\square$

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