COMMUNICATIONS IN ALGEBRA, 29(2), 907–925 (2001)

EXCHANGE MORITA RINGS

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ABSTRACT

In this paper we characterize the largest exchange ideal of a ring $\mathcal{R}$ as the set of those elements $x \in \mathcal{R}$ such that the local ring of $\mathcal{R}$ at x is an exchange ring. We use this result to prove that if $\mathcal{R}$ and $\mathcal{S}$ are two rings for which there is a quasi-acceptable Morita context, then $\mathcal{R}$ is an exchange ring if and only if $\mathcal{S}$ is an exchange ring, extending an analogue result given previously by Ara and the second and third authors for idempotent rings. We introduce the notion of exchange associative pair and obtain some results connecting the exchange property and the possibility of lifting idempotents modulo left ideals. In particular we obtain that in any exchange ring, orthogonal von Neumann regular elements can be lifted modulo any one-sided ideal.

Key Words: Exchange ring; Morita context.

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1. THE LARGEST EXCHANGE IDEAL OF A RING

The exchange property for modules was introduced in [1] by Crawley and Jónsson and, roughly speaking, it is the necessary property to have suitable versions of the Krull-Schmidt Theorem even when the modules do not decompose as direct sums of indecomposables. Following [2] we say that an associative unital ring R is an *exchange ring* if R_R has the exchange property. Warfield proved that this property is left-right symmetric. A useful characterization of exchange rings was obtained independently by Goodearl [3] and Nicholson [4]. Namely, a unital ring $\mathcal{R}$ is an exchange ring if and only if for each $x \in \mathcal{R}$ there exist an idempotent $e \in \mathcal{R}$ and elements $r, s \in \mathcal{R}$ such that $e = rx = s + x - sx$. This characterization was adapted by Ara in [5] to give the definition of an exchange ring without unit (5, Lemma 1.1). As in the unital case, the definition is left-right symmetric, and it gives exchange properties for suitable module decompositions [see (5, Theorem 1.2)].

From now on, unless specified, $\mathcal{R}$ will denote a not necessarily unital ring.

We will say that $\mathcal{R}$ is an *exchange ring* [see (5)] if for every element $x \in \mathcal{R}$ the equivalent conditions in the next lemma are satisfied.

Lemma 1.1 (5, *Lemma* 1.1). *Let $\mathcal{R}$ be a ring and let $\mathcal{R}'$ be a unital ring containing $\mathcal{R}$ as a two-sided ideal. Then the following conditions are equivalent for an element $x \in \mathcal{R}$:*

i) *There exists $e^2 = e \in \mathcal{R}$ with $e - x \in \mathcal{R}'(x - x^2)$,*
ii) *there exist $e^2 = e \in \mathcal{R}x$ and $c \in \mathcal{R}'$ such that $(1 - e) - c(1 - x) \in J(\mathcal{R}')$,*
iii) *there exists $e^2 = e \in \mathcal{R}x$ such that $\mathcal{R}' = \mathcal{R}e + \mathcal{R}'(1 - x)$,*
iv) *there exists $e^2 = e \in \mathcal{R}x$ such that $1 - e \in \mathcal{R}'(1 - x)$,*
v) *there exist $r, s \in \mathcal{R}$, $e^2 = e \in \mathcal{R}$ such that $e = rx = s + x - sx$.*

[$J(\mathcal{R}')$ *denotes the Jacobson radical of* $\mathcal{R}'$.]

Note that $\mathcal{R}$ being an exchange ring does not depend on the particular unital ring where $\mathcal{R}$ is embedded as an ideal [look at condition (v) in Lemma 1.1]. Other characterizations of the exchange property for not necessarily unital rings can be found in [5].

The following definition was introduced by the second and third authors jointly with Fernández López and García Rus in [6] in the setting of associative pairs [see Section 3 for the definition of associative pair]. The use of the local rings at elements of a ring [or associative pair] allows to exchange information between them [the ring -or the associative pair- and its local rings at elements]. For an element x in a ring $\mathcal{R}$ we define the *local ring* of $\mathcal{R}$ *at* x, and denote it by $\mathcal{R}_x$, as the additive subgroup $x\mathcal{R}x$, endowed with the product

$$xyx \cdot xzx := xyxzx.$$

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This notion had appeared in a paper of Meyberg [7], although in a different way: Let x be an element in a ring $\mathcal{R}$. The additive group $\mathcal{R}$ with the x-*homotope product*

$$a \cdot_q b = axb$$

becomes a ring called the *homotope* of $\mathcal{R}$ *at* x and is denoted by $\mathcal{R}^x$. The set

$$\mathrm{Ker}(x) = \{a \in \mathcal{R} \mid xax = 0\}$$

is an ideal of $\mathcal{R}^x$. Meyberg's local rings are the quotients $\mathcal{R}^x/\mathrm{Ker}x$ for $x \in \mathcal{R}$, but it is easy to see that the mapping $a + \mathrm{Ker}(x) \mapsto xax$ is an isomorphism from $\mathcal{R}^x/\mathrm{Ker}(x)$ onto $\mathcal{R}_x$.

Recall that an element $x \in \mathcal{R}$ is said to be *von Neumann regular* (in $\mathcal{R}$) if there exists $y \in \mathcal{R}$ such that $x = xyx$. If x is a von Neumann regular element, then $\mathcal{R}_x$ is a unital ring and x is the unit [$x \cdot_q xax = xyx \cdot_q xax = xyxaxa = xax$ and $xax \cdot_q x = xax \cdot_q xyx = xaxyx = xax$]. In particular, if e is an idempotent of $\mathcal{R}$ then $\mathcal{R}_e$ is the subring $e\mathcal{R}e$ of $\mathcal{R}$.

Let $\mathcal{I}$ be an ideal of $\mathcal{R}$ and x an element in $\mathcal{R}$. The x-homotope product is well-defined on $\mathcal{I}$ and $x\mathcal{I}x$ can be regarded as an ideal of the local ring $\mathcal{R}_x$ of $\mathcal{R}$ at x. These two rings, denoted by $\mathcal{I}^x$ and $\mathcal{I}_x$, will be called the *generalized homotope* of $\mathcal{I}$ at x and the *generalized local ring* of $\mathcal{I}$ at x, respectively [see (6) for this last definition].

An *exchange ideal* of a ring $\mathcal{R}$ is an ideal of $\mathcal{R}$ which is exchange as a ring. In (8, Theorem 1.4) the authors characterize the exchange rings as those rings $\mathcal{R}$ such that for every element $x \in \mathcal{R}$ the local ring of $\mathcal{R}$ at x, i.e. $\mathcal{R}_x$, is an exchange ring. In this section we study in depth the *exchange elements* of a not necessarily unital ring $\mathcal{R}$, understood as those $x \in \mathcal{R}$ such that $\mathcal{R}_x$ is an exchange ring. We prove that the largest exchange ideal of a ring [with respect to inclusion], denoted by $\epsilon(\mathcal{R})$, consists of the exchange elements of the ring $\mathcal{R}$. Hence $\mathcal{R}$ is an exchange ring if $\mathcal{R} = \epsilon(\mathcal{R})$ [the existence of $\epsilon(\mathcal{R})$ was proved in (8, Theorem 3.5)].

Lemma 1.2. *Let $\mathcal{R}$ be a ring, let $\mathcal{I}$ be a two-sided ideal of $\mathcal{R}$ and $y \in \mathcal{I}$. Then $\mathcal{I}_y$ is an exchange ring if and only if $\mathcal{R}_y$ is an exchange ring.*

Proof: We notice that $\mathcal{R}_y/\mathcal{I}_y$ is a nilpotent ring and since nilpotent rings are radical, the result follows from (5, Corollary 2.5). □

For a ring $\mathcal{R}$ we denote by $\mathcal{R}_1$ its unitization, that is, $\mathcal{R}_1 = \mathcal{R} \oplus \mathbb{Z}$ with componentwise addition and multiplication defined by $(x, n)(y, m) = (xy + ny + mx, nm)$ for all $x, y \in \mathcal{R}$ and $n, m \in \mathbb{Z}$. Note that $\mathcal{R}$ is an ideal of $\mathcal{R}_1$.

Proposition 1.3. *Let $\mathcal{R}$ be a ring and let $\mathcal{I}$ be a two-sided ideal of $\mathcal{R}$. For an element $x \in \mathcal{R}$ the following conditions are equivalent*:

i) *$\mathcal{I}x$ is an exchange ring,*
ii) *$x\mathcal{I}$ is an exchange ring,*

iii) *$\mathcal{I}x\mathcal{I}$ is an exchange ring,*
iv) *$\mathcal{I}_x$ is an exchange ring,*
v) *$\mathcal{I}^x$ is an exchange ring.*

If $x \in \mathcal{I}$ then the previous conditions are equivalent to each of the following ones:

i′) *$\mathcal{I}_1 x$ is an exchange ring,*
ii′) *$x\mathcal{I}_1$ is an exchange ring,*
iii′) *$\mathcal{I}_1 x\mathcal{I}_1$ is an exchange ring.*

Proof: Consider the following ring homomorphisms:

$$\begin{array}{rcl} \varphi : \mathcal{I}x & \to & \mathcal{I}_x \\ yx & \mapsto & xyx \end{array} \quad \begin{array}{rcl} \psi : x\mathcal{I} & \to & \mathcal{I}_x \\ xy & \mapsto & xyx \end{array} \quad \begin{array}{rcl} \eta : \mathcal{I}^x & \to & \mathcal{I}_x \\ y & \mapsto & xyx \end{array}$$

We have that $\mathcal{I}x/\mathrm{Ker}(\varphi)$, $x\mathcal{I}/\mathrm{Ker}(\psi)$ and $\mathcal{I}^x/\mathrm{Ker}(\eta)$ are isomorphic to $\mathcal{I}_x$. Moreover $\mathrm{Ker}(\varphi)$, $\mathrm{Ker}(\psi)$ and $\mathrm{Ker}(\eta)$ are nilpotent ideals of $\mathcal{I}x$, $x\mathcal{I}$ and $\mathcal{I}^x$ respectively. Since nilpotent ideals are π-regular, (i), (ii), and (v) are equivalent to (iv) by virtue of (5, Corollary 2.5).

(iii) $\Rightarrow$ (i). By (8, Proposition 1.3) the left ideal $(\mathcal{I}x)^2$ of the exchange ring $\mathcal{I}x\mathcal{I}$ is an exchange ring. Moreover $\mathcal{I}x/(\mathcal{I}x)^2$ is a nilpotent ring and by (5, Corollary 2.5) $\mathcal{I}x$ is an exchange ring.

(i) $\Rightarrow$(iii). Let $y = \sum_{i=1}^{n} r_i x s_i$ be an element in $\mathcal{I}x\mathcal{I}$, and let $\bar{r}$, $\bar{x}$, $\bar{s}$ be the following elements of $\mathcal{M}_n(\mathcal{R})$:

$$\bar{r} = \begin{pmatrix} r_1 & r_2 & \cdots & r_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}; \ \bar{x} = \begin{pmatrix} x & 0 & \dots & 0 \\ 0 & x & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x \end{pmatrix}; \ \bar{s} = \begin{pmatrix} s_1 & 0 & \cdots & 0 \\ s_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ s_n & 0 & \cdots & 0 \end{pmatrix}.$$

The map

$$\begin{array}{rcl} \rho : \mathcal{I}_y & \to & \mathcal{M}_n(\mathcal{I})_{\overline{rxs}} \\ u & \mapsto & uE_{11} \end{array},$$

where

$$E_{11} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

is a ring isomorphism so $\mathcal{I}_y$ is an exchange ring if and only if $\mathcal{M}_n(\mathcal{I})_{\overline{rxs}}$ is an exchange ring. By hypothesis $\mathcal{I}x$ is an exchange ring and, by (5, Theorem 1.4), $\mathcal{M}_n(\mathcal{I}x) = \mathcal{M}_n(\mathcal{I})\bar{x}$ is an exchange ring. Now by (8, Proposition 1.3) $\mathcal{M}_n(\mathcal{I})\overline{rx}$, which is a left ideal of the exchange ring $\mathcal{M}_n(\mathcal{I})\bar{x}$, is an exchange ring. If we apply (i) $\Leftrightarrow$ (ii) we obtain that $\overline{rx}\mathcal{M}_n(\mathcal{I})$ is an exchange ring and, by the right-handed version of (8, Proposition 1.3), $\overline{rxs}\mathcal{M}_n(\mathcal{I})$, which is a right ideal of $\overline{rx}\mathcal{M}_n(\mathcal{I})$, is an exchange ring. Using (i) $\Leftrightarrow$ (iv), we see that $\mathcal{M}_n(\mathcal{I})_{\overline{rxs}}$ is an exchange ring. Finally, since $(\mathcal{I}x\mathcal{I})_y$ is an ideal of $\mathcal{I}_y$, $(\mathcal{I}x\mathcal{I})_y$ is an exchange ring. Now (8, Theorem 1.4) applies to get that $\mathcal{I}x\mathcal{I}$ is an exchange ring.

Finally, notice that if $x \in \mathcal{I}$ then the rings $\mathcal{I}x$, $x\mathcal{I}$, and $\mathcal{I}x\mathcal{I}$ are ideals of $\mathcal{I}_1x$, $x\mathcal{I}_1$, and $\mathcal{I}_1x\mathcal{I}_1$ respectively. Moreover $\mathcal{I}_1x/\mathcal{I}x$, $x\mathcal{I}_1/x\mathcal{I}$, and $\mathcal{I}_1x\mathcal{I}_1/\mathcal{I}x\mathcal{I}$ are nilpotent rings, so by (5, Corollary 2.5), (i) $\Leftrightarrow$ (i′), (ii) $\Leftrightarrow$ (ii′), and (iii) $\Leftrightarrow$ (iii′). □

Theorem 1.4. *Let $\mathcal{R}$ be a ring, $x \in \mathcal{R}$ and $\mathcal{I}$ an ideal of $\mathcal{R}$. Then*:

i) $\epsilon(\mathcal{R}) = \{x \in \mathcal{R} \mid \mathcal{R}_x$ *is an exchange ring*$\}$.
ii) $\epsilon(\mathcal{I}) = \mathcal{I} \cap \epsilon(\mathcal{R})$.
iii) $\epsilon(\mathcal{I}_x) = \mathcal{I}_x \cap \epsilon(\mathcal{I}) = \mathcal{I}_x \cap \epsilon(\mathcal{R})$.

Proof:

i) If $x \in \epsilon(\mathcal{R})$, then by (8, Theorem 1.4) $\epsilon(R)_x$ is an exchange ring and by Lemma 1.2 R_x is an exchange ring. Conversely, if $\mathcal{R}_x$ is an exchange ring for an element $x \in \mathcal{R}$, then $\mathcal{R}_1x\mathcal{R}_1$ is an exchange ring by Proposition 1.3. Since $\mathcal{R}_1x\mathcal{R}_1$ is a two-sided ideal of $\mathcal{R}$ we have $\mathcal{R}_1x\mathcal{R}_1 \subseteq \epsilon(R)$ whence $x \in \epsilon(R)$.
ii) For any $y \in \mathcal{I}$ we have $y \in \epsilon(\mathcal{I})$ if and only if [by (i)] $\mathcal{I}_y$ is an exchange ring which is equivalent [by Lemma 1.2] to $\mathcal{R}_y$ being an exchange ring. This is in turn equivalent to $y \in \epsilon(\mathcal{R})$ (by (i)).
iii) By (i), $xyx \in \epsilon(\mathcal{I}_x)$ if and only if $(\mathcal{I}_x)_{xyx} = \mathcal{I}_{xyx}$ is an exchange ring, which is equivalent by (ii) to $xyx \in \epsilon(\mathcal{R}) \cap \mathcal{I}_x = \epsilon(\mathcal{I}) \cap \mathcal{I}_x$. □

Corollary 1.5. *For a ring $\mathcal{R}$*

$\epsilon(\mathcal{R}) = \{x \in \mathcal{R} \mid \mathcal{J}_x$ *is an exchange ring for some ideal $\mathcal{J}$ of $\mathcal{R}$ containing x*$\}$.

Proof: By Theorem 1.4 (i) $x \in \epsilon(\mathcal{R})$ if and only if $\mathcal{R}_x$ is an exchange ring, and, by Lemma 1.2, $\mathcal{R}_x$ is an exchange ring if and only if $\mathcal{J}_x$ is an exchange ring for some ideal $\mathcal{J}$ of $\mathcal{R}$ containing x. □

Although the following result can be obtained as a corollary of Theorem 1.4, we give here and independent proof which was suggested to us by the referee.

Proposition 1.6. *Let $\mathcal{R}$ be a ring and $\mathcal{I}$ be the ideal of $\mathcal{R}$ generated by a subset X of $\mathcal{R}$. Then $\mathcal{I}$ is an exchange ring if and only if $X \subseteq \epsilon(\mathcal{R})$.*

Proof: Assume that $X \subseteq \epsilon(\mathcal{R})$. Let $\mathcal{I} = \sum_{x\in X}\mathcal{R}x\mathcal{R}$. Since $\epsilon(\mathcal{R})$ is an ideal of $\mathcal{R}$ we have $\mathcal{I} \subseteq \epsilon(\mathcal{R})$ and in fact it is an ideal [of $\epsilon(\mathcal{R})$]. Thus, since $\epsilon(\mathcal{R})$ is an exchange ring we get that $\mathcal{I}$ is exchange.

Conversely, if $\mathcal{I} = \sum_{x\in X}\mathcal{R}x\mathcal{R}$ is exchange, then $\mathcal{R}x\mathcal{R}$ is exchange for any $x \in X$. By Proposition 1.3, $\mathcal{R}_1x\mathcal{R}_1$ is exchange. Since $\mathcal{R}_1x\mathcal{R}_1$ is an ideal of $\mathcal{R}$ [$x \in \mathcal{R}$], we get that $x \in \epsilon(\mathcal{R})$. Since x was arbitrary, this proves that $X \subseteq \epsilon(\mathcal{R})$, as desired. □

A natural extension of property (iv) in Lemma 1.1 can be proved for [not necessarily unital exchange rings], which leads to a result on lifting orthogonal idempotents modulo a left ideal in any exchange ring. This result was already known for unital rings [see (4, Proposition 1.12)].

Proposition 1.7. *Let $\mathcal{R}$ be an exchange ideal of a unital ring $\mathcal{R}'$ and suppose that [for some natural α] $x_1, \dots, x_\alpha \in \mathcal{R}$, $x_{\alpha+1} \in \mathcal{R}'$ are such that $1 = x_1 + \cdots + x_\alpha + x_{\alpha+1}$. Then there exist orthogonal idempotents $e_1, \dots, e_\alpha \in \mathcal{R}$, $e_{\alpha+1} \in \mathcal{R}'$ such that $1 = e_1 + \cdots + e_\alpha + e_{\alpha+1}$, with $e_i \in \mathcal{R}x_i$ for $i \in \{1, \dots, \alpha\}$ and $e_{\alpha+1} \in \mathcal{R}'x_{\alpha+1}$.*

Proof: The case $\alpha = 1$ follows from (iv) of Lemma 1.1.

Suppose that $\alpha \geq 2$ and that the result is true for $\alpha - 1$ and let $x_1, \dots, x_\alpha \in \mathcal{R}$, $x_{\alpha+1} \in \mathcal{R}'$ such that $1 = x_1 + \cdots + x_\alpha + x_{\alpha+1}$. Since $\mathcal{R}$ is an exchange ring, (iv) in Lemma 1.1 applied to $x = x_1 + \cdots + x_\alpha \in \mathcal{R}$ shows that there exists an idempotent $f \in \mathcal{R}(\sum_{i=1}^{\alpha} x_i)$ such that $1 - f \in \mathcal{R}'(1 - \sum_{i=1}^{\alpha} x_i) = \mathcal{R}'x_{\alpha+1}$.

Let $y \in \mathcal{R}$ be such that $f = y\sum_{i=1}^{\alpha} x_i = \sum_{i=1}^{\alpha} yx_i = \sum_{i=1}^{\alpha} yx_i f = f^2$. By (5, Proposition 1.3), $f\mathcal{R}f$ is an exchange ring and by the induction assumption, there exist orthogonal idempotents $f_1, \dots, f_\alpha$ in $f\mathcal{R}f$ such that $f = \sum_{i=1}^{\alpha} f_i$ and $f_i \in f\mathcal{R}fyx_i f$. Write $f_i = fy_i fyx_i f$ for $y_i \in \mathcal{R}$, $i \in \{1, \dots, \alpha\}$, and define $e_i = f_i y_i fyx_i$. Then, for i, j in $\{1, \dots, \alpha\}$, and using that $f_i = ff_i = f_i f$, we get

$$e_i e_j = (f_i y_i fyx_i)(f_j y_j fyx_j) = f_i fy_i fyx_i ff_j y_j fyx_j = f_i f_j y_j fyx_j,$$

which is 0 if $i \neq j$ and e_i if $i = j$. Hence, $e_1, \dots, e_\alpha$ are orthogonal idempotents and $e_i \in \mathcal{R}x_i$. Let $e = \sum_{i=1}^{\alpha} e_i$. Since $e_i f = f_i y_i fyx_i f = f_i fy_i fyx_i f = f_i f_i = f_i$ we get $(1 - e) = (1 - e)(1 - f) \in \mathcal{R}'x_{\alpha+1}$, and this completes the proof. □

As a consequence of this result we obtain the lifting theorem for orthogonal idempotents, whose proof is omitted because it is similar to that of (4, Proposition 1.12), using Proposition 1.7 in place of (4, Proposition 1.11).

Let $\mathcal{L}$ be a left ideal of a ring $\mathcal{R}$. For elements $x, y \in \mathcal{R}$ we will write $x \equiv y (\mathrm{mod}\ \mathcal{L})$ when $x - y \in \mathcal{L}$.

Proposition 1.8. *Let $\mathcal{R}$ be an exchange ring, $\mathcal{L}$ a left ideal of $\mathcal{R}$ and let $x_1, \dots, x_\alpha$ be orthogonal idempotents modulo $\mathcal{L}$, that is, $x_i \equiv x_i^2 (mod\ \mathcal{L})$ for each i and $x_i x_j \equiv 0 (mod\ \mathcal{L})$ for all $i \neq j$. Then there exist orthogonal idempotents $e_1, \dots, e_\alpha$ in $\mathcal{R}$ such that $e_i \in \mathcal{R}x_i$ and $e_i \equiv x_i (mod\ \mathcal{L})$ for each i.*

2. EXCHANGE MORITA RINGS

Let $\mathcal{R}$ and $\mathcal{S}$ be two rings, ${}_{\mathcal{R}}N_{\mathcal{S}}$ and ${}_{\mathcal{S}}M_{\mathcal{R}}$ two bimodules, and $(-,-) : N \times M \to \mathcal{R}$, $[-,-] : M \times N \to \mathcal{S}$ two maps. Then the following conditions are equivalent:

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i) $\begin{pmatrix} \mathcal{R} & N \\ M & \mathcal{S} \end{pmatrix}$ is a ring with componentwise sum and product given by:

$$\begin{pmatrix} r_1 & n_1 \\ m_1 & s_1 \end{pmatrix}\begin{pmatrix} r_2 & n_2 \\ m_2 & s_2 \end{pmatrix} = \begin{pmatrix} r_1r_2 + (n_1, m_2) & r_1n_2 + n_1s_2 \\ m_1r_2 + s_1m_2 & [m_1, n_2] + s_1s_2 \end{pmatrix},$$

ii) $[-, -]$ is $\mathcal{S}$-bilinear and $\mathcal{R}$-balanced, $(-, -)$ is $\mathcal{R}$-bilinear and $\mathcal{S}$-balanced and the following associativity conditions holds:

$$(n, m)n' = n[m, n'] \quad \text{and} \quad [m, n]m' = m(n, m').$$

$[-, -]$ being $\mathcal{S}$-bilinear and $\mathcal{R}$-balanced and $(-, -)$ being $\mathcal{R}$-bilinear and $\mathcal{S}$-balanced is equivalent to having bimodule maps $\varphi : N \otimes_{\mathcal{S}} M \to \mathcal{R}$ and $\psi : M \otimes_{\mathcal{R}} N \to \mathcal{S}$, given by

$$\varphi(n \otimes m) = (n, m) \quad \text{and} \quad \psi(m \otimes n) = [m, n]$$

so that the associativity condition above reads

$$\varphi(n \otimes m)n' = n\psi(m \otimes n') \quad \text{and} \quad \psi(m \otimes n)m' = m\varphi[m, n].$$

A *Morita context* is a sextuple $(\mathcal{R}, \mathcal{S}, N, M, \varphi, \psi)$ satisfying the conditions given above. The associated ring is called the *Morita ring* of the context. By abuse of notation we will write sometimes $(\mathcal{R}, \mathcal{S}, N, M)$ instead of $(\mathcal{R}, \mathcal{S}, N, M, \varphi, \psi)$ and will suppose $\mathcal{R}, \mathcal{S}, N, M$ contained in the Morita ring associated to the context [see also (9) for the theory of nonunital Morita rings].

Theorem 2.1. *Let $(\mathcal{R}, \mathcal{S}, N, M)$ be a context and $\mathcal{A}$ the associated Morita ring. Then*

$$\epsilon(\mathcal{A}) = \{x \in \mathcal{A} \mid \mathcal{A}_{x_{ij}} \textit{ is an exchange ring} \text{ for } i, j \in \{1, 2\}\},$$

where $\mathcal{A}_{11}, \mathcal{A}_{22}, \mathcal{A}_{12}$ and $\mathcal{A}_{21}$ denote, $\mathcal{R}, \mathcal{S}, N$, and M, respectively, and x_{ij} is the (i, j)-component of x.

Proof: By Theorem 1.4 (i), if $\mathcal{A}_{x_{ij}}$ is an exchange ring, then $x_{ij} \in \epsilon(\mathcal{A})$, hence $x = x_{11} + x_{12} + x_{21} + x_{22} \in \epsilon(\mathcal{A})$.

Conversely, let x be an element in $\epsilon(\mathcal{A})$ and write $x = x_{11} + x_{22} + x_{12} + x_{21}$. For $i, j \in \{1, 2\}$,

$$x_{ij}\mathcal{A}x_{ij}\mathcal{A}x_{ij}\mathcal{A}x_{ij} = x_{ij}\mathcal{A}_{ji}x_{ij}\mathcal{A}_{ji}x_{ij}\mathcal{A}_{ji}x_{ij} = x_{ij}\mathcal{A}_{ji}x\mathcal{A}_{ji}x\mathcal{A}_{ji}x_{ij} \subseteq \epsilon(\mathcal{A}).$$

Therefore $\mathcal{A}_{x_{ij}yx_{ij}} = x_{ij}yx_{ij}\mathcal{A}x_{ij}yx_{ij} \subseteq \epsilon(\mathcal{A})$ and $x_{ij}yx_{ij} \in \epsilon(\mathcal{A})$ by Theorem 1.4 (i) for every $y \in \mathcal{A}$.

By Theorem 1.4 (iii), $\mathcal{A}_{x_{ij}yx_{ij}} = \epsilon(\mathcal{A}_{x_{ij}yx_{ij}})$, which implies $\mathcal{A}_{x_{ij}yx_{ij}}$ is an exchange ring. By condition (i) in Theorem 1.4, $x_{ij}yx_{ij} \in \epsilon(\mathcal{A})$ and we have shown $x_{ij}\mathcal{A}x_{ij} \subseteq \epsilon(\mathcal{A})$. By Theorem 1.4 (iii), $\mathcal{A}_{x_{ij}} = \epsilon(\mathcal{A}_{x_{ij}})$, which implies that $\mathcal{A}_{x_{ij}}$ is an exchange ring. □

In classical Morita theory it is shown that two rings with identity $\mathcal{R}$ and $\mathcal{S}$ are Morita equivalent (i.e., $\mathcal{R}$-mod and $\mathcal{S}$-mod are equivalent categories) if and only if there exists a Morita context $(\mathcal{R}, \mathcal{S}, N, M, \varphi, \psi)$. The approach to Morita theory for rings without identity by means of Morita contexts appears in a number of papers [see (9) and the references therein] in which many consequences are obtained from the existence of a Morita context for two rings $\mathcal{R}$ and $\mathcal{S}$. For example, it is shown in (9, Theorem 3.10).

Theorem 2.2. *Let $(\mathcal{R}, \mathcal{S}, N, M, \varphi, \psi)$ be a Morita context. The following conditions are equivalent*:

i) $\mathrm{Hom}_{\mathcal{R}}(M_{\mathcal{R}}, -)$ *and* $Hom_{\mathcal{S}}(N_{\mathcal{S}}, -)$ *are inverse category equivalences, with the transformations given by the context, between the categories* $CMod-\mathcal{R}$ *and* $CMod-\mathcal{S}$,

ii) $M \otimes_{\mathcal{R}} -$ *and* $N \otimes_{\mathcal{S}} -$ *are inverse category equivalences, with the transformations given by the context, between the categories* $\mathcal{R}-DMod$ *and* $\mathcal{S}-DMod$,

iii) *The context $(\mathcal{R}, \mathcal{S}, N, M, \varphi, \psi)$ is left acceptable.*

Here, the context is said to be *left acceptable* if:

For every $(r_n)_{n\in\mathbb{N}} \in \mathcal{R}^{\mathbb{N}}$ there exists $n_0 \in \mathbb{N}$ such that $r_1 \ldots r_{n_0} \in \mathrm{Im}\varphi$, and for every $(s_m)_{m\in\mathbb{N}} \in \mathcal{R}^{\mathbb{N}}$ there exists $m_0 \in \mathbb{N}$ such that $s_1 \ldots s_{m_0} \in \mathrm{Im}\psi$.

For the following result we can weaken the hypothesis assumed on the Morita context.

A Morita context $(\mathcal{R}, \mathcal{S}, N, M)$ is called *quasi-acceptable* if:

For every $r \in \mathcal{R}$ there exists $\alpha \in \mathbb{N}$ such that $r^{\alpha} \in \mathrm{Im}\varphi$, and for every $s \in \mathcal{S}$ there exists $\beta \in \mathbb{N}$ such that $s^{\beta} \in \mathrm{Im}\psi$.

Theorem 2.3. *Let $(\mathcal{R}, \mathcal{S}, N, M)$ be a quasi-acceptable Morita context and $\mathcal{A}$ the associated Morita ring. Then the following conditions are equivalent*:

i) *$\mathcal{A}$ is an exchange ring,*

ii) *$\mathcal{R}$ is an exchange ring,*

iii) *$\mathcal{S}$ is an exchange ring,*

iv) *for every element $m \in M$ the ring $N_m := \mathcal{A}_m$ is an exchange ring,*

v) *for every element $n \in N$ the ring $M_n := \mathcal{A}_n$ is an exchange ring,*

Proof: Since for every element $x \in B$, where $B = \mathcal{R}, \mathcal{S}, M$ or N, we have $\mathcal{A}_x = B_x$, by Theorem 2.1, (ii)–(iv) together imply (i). By (8, Theorem 1.4), (i) implies (ii)–(iv).

Suppose that $\mathcal{R}$ is an exchange ring. Then $\mathcal{R} = \epsilon(\mathcal{R})$, which implies that $\mathcal{R}_x = \mathcal{A}_x$ is an exchange ring for every $x \in \mathcal{R}$. By condition (i) in Theorem 1.4, $\mathcal{R} \subseteq \epsilon(\mathcal{A})$ and since $\epsilon(\mathcal{R})$ is an ideal, the ideal generated by $\mathcal{R}$ in $\mathcal{A}$ is contained in $\epsilon(\mathcal{A})$. In particular, for any $m \in M$ and $n \in N$, $m\mathcal{A}m = mNm \subseteq \mathcal{A}\mathcal{R} \subseteq \epsilon(\mathcal{A})$ and $n\mathcal{A}n = nMn \subseteq \mathcal{R}\mathcal{A} \subseteq \epsilon(\mathcal{A})$. This implies, using conditions (iii) and (i) in Theorem 1.4, that (iv) and (v) follow from (ii). Analogously, (iii) implies (iv) and (v).

Now we prove (iv) $\Rightarrow$ (ii). By Theorem 1.4 (i), $M \subseteq \epsilon(\mathcal{A})$ and since $\epsilon(\mathcal{A})$ is an ideal of $\mathcal{A}$ containing the ideal NM, then NM is an exchange ring by condition (ii) in Theorem 1.4. Since the context is quasi-acceptable, $\mathcal{R}/NM$ is a nil ring. Hence, and applying (5, Corollary 2.5), we get (ii). Analogously (iii) follows from (iv).

(v) $\Rightarrow$ (ii) and (iii) are proved similarly. □

3. EXCHANGE ASSOCIATIVE PAIRS

Recall that an *associative pair* over $\mathbb{Z}$ is a pair (A^+, A^-) of $\mathbb{Z}$-modules together with $\mathbb{Z}$-trilinear maps

$$\begin{array}{ccc} A^\sigma \times A^{-\sigma} \times A^\sigma & \rightarrow & A^\sigma \\ (x, y, z) & \mapsto & xyz \end{array}$$

satisfying the following identities:

$$uv(xyz) = u(vxy)z = (uvx)yz \tag{1}$$

for all $u, x, z \in A^\sigma$, $y, v \in A^{-\sigma}$, and $\sigma = \pm$. [See (10) for other definitions and results on associative pairs].

The first example of an associative pair is given by $(\mathcal{A}, \mathcal{A})$, for any associative algebra $\mathcal{A}$, under the triple product $(x, y, z) = xyz$, where juxtaposition denotes the product of $\mathcal{A}$.

Given an associative pair $A = (A^+, A^-)$, recall that *left*, *middle*, and *right multiplications* are defined by:

$$\lambda(x, y)z = \mu(x, z)y = \rho(y, z)x = xyz. \tag{2}$$

It follows from equation (1)

$$\lambda(x, y)\lambda(u, v) = \lambda(xyu, v) = \lambda(x, yuv) \tag{3}$$

and similarly

$$\rho(u, v)\rho(x, y) = \rho(x, yuv) = \rho(xyu, v). \tag{4}$$

Hence it is clear that the linear span of all operators $T : A^\sigma \rightarrow A^\sigma$ of the form $T = \lambda(x, y)$ or $T = Id_{A^\sigma}$ is a unital associative algebra, denoted by $\Lambda(A^\sigma, A^{-\sigma})$. Clearly A^σ is a left $\Lambda(A^\sigma, A^{-\sigma})$-module. Similarly, we define $\Pi(A^{-\sigma}, A^\sigma)$ as the linear span of all the right multiplications plus the identity on A^σ. Then A^σ becomes a left $\Pi(A^{-\sigma}, A^\sigma)$-module. We define the *left ideals* $L \subset A^\sigma$ of A as the left $\Lambda(A^\sigma, A^{-\sigma})$-submodules of A^σ, and the *right ideals* $R \subset A^\sigma$ as the left $\Pi(A^{-\sigma}, A^\sigma)$-submodules. A *two-sided ideal* $B \subset A^\sigma$ is both a left and a right ideal. For some results we will consider pairs $L = (L^+, L^-)$ of left [right, two-sided] ideals of A, which, by the definition of left [right, two-sided] ideal, are themselves associative pairs. An *ideal* $I = (I^+, I^-)$ of A is a pair of two-sided ideals of A such that $A^\sigma I^{-\sigma} A^\sigma \subseteq I^\sigma$.

Associative pairs are really "abstract off-diagonal Peirce spaces" of associative algebras in the following sense: Let $\mathcal{E}$ be a unital associative algebra. Consider the Peirce decomposition $\mathcal{E} = \mathcal{E}_{11} \oplus \mathcal{E}_{12} \oplus \mathcal{E}_{21} \oplus \mathcal{E}_{22}$ of $\mathcal{E}$ with respect to an idempotent $e \in \mathcal{E}$, i.e.,

$$\mathcal{E}_{11} = e\mathcal{E}e, \quad \mathcal{E}_{12} = e\mathcal{E}(1-e), \quad \mathcal{E}_{21} = (1-e)\mathcal{E}e \quad \text{and} \quad \mathcal{E}_{22} = (1-e)\mathcal{E}(1-e),$$

and denote by $\pi_{ij} : \mathcal{E} \to \mathcal{E}_{ij}$ the corresponding Peirce projections. For any set $X \subset \mathcal{E}$ we put $X_{ij} := \pi_{ij}(X)$. Then $(\mathcal{E}_{12}, \mathcal{E}_{21})$ is an associative pair if we define $\langle xyz \rangle = xyz$ and $\langle yzw \rangle = yzw$, for $x, z \in \mathcal{E}_{12}$ and $y, w \in \mathcal{E}_{21}$. Conversely, every associative pair $A = (A^+, A^-)$ can be obtained in this way [see (11), Section (2.3)].

Let $\mathcal{C}$ be the $\mathbb{Z}$-subalgebra of $\mathcal{B} = \mathrm{End}_{\mathbb{Z}}(A^+) \times \mathrm{End}_{\mathbb{Z}}(A^-)^{op}$ spanned by $e_1 = (Id, Id)$ and all $(\lambda(x, y), \rho(x, y))$, and similarly, let $\mathcal{D}$ the subalgebra of $\mathcal{B}^{op}$ spanned by $e_2 = (Id, Id)$ and all $(\rho(y, x), \lambda(y, x))$ where $(x, y) \in (A^+, A^-)$. The associativity condition (1) ensures that these $\mathbb{Z}$-linear span are really subalgebras. Again by equation (1), A^+ is an $(\mathcal{C}, \mathcal{D})$-bimodule if we set

$$cx = c_+(x), \quad xd = d_+(x)$$

for $x \in A^+$ and $c = (c_+, c_-) \in \mathcal{C}$, $d = (d_+, d_-) \in \mathcal{D}$. Similarly, A^- is a $(\mathcal{D}, \mathcal{C})$-bimodule. Now we define multiplications on $A^\pm \times A^\mp$ with values in $\mathcal{C}$, respectively $\mathcal{D}$, by

$$xy = (\lambda(x, y), \rho(x, y)), \quad yx = (\rho(y, x), \lambda(y, x)).$$

Then it is easy to check that $\mathcal{E} = (\mathcal{C}, \mathcal{D}, A^+, A^-)$ is a Morita context and if we set $e = e_1$, then the pair $(\mathcal{E}, e)$ is called the *standard imbedding* of A. In other words, given an associative pair $A = (A^+, A^-)$, there exists a unital associative algebra $\mathcal{E}$ with an idempotent e such that A is isomorphic to the associative pair $(\mathcal{E}_{12}, \mathcal{E}_{21})$, where $\mathcal{E}_{11}$ [resp. $\mathcal{E}_{22}$ is spanned by e and all products $x_{12}y_{21}$ resp. $1-e$ and all products $y_{21}x_{12}$] for $x_{12} \in \mathcal{E}_{12}$, $y_{21} \in \mathcal{E}_{21}$, and has the property that

$$x_{11}\mathcal{E}_{12} = \mathcal{E}_{21}x_{11} = 0 \text{ implies } x_{11} = 0,$$

and

$$x_{22}\mathcal{E}_{21} = \mathcal{E}_{12}x_{22} = 0 \text{ implies } x_{22} = 0.$$

Now let $\mathcal{A}$ be the subalgebra of $\mathcal{E}$ generated by the elements x_{12} and x_{21}. It is immediate that $\mathcal{A}$ is an ideal of $\mathcal{E}$. We will call $\mathcal{A}$ the *envelope* of the associative pair A. Notice that $\mathcal{A}_{12} := e\mathcal{A}(1-e) = \mathcal{E}_{12}$ and $\mathcal{A}_{21} := (1-e)\mathcal{A}e = \mathcal{E}_{21}$, hence the associative pair A is isomorphic to the associative pair $(\mathcal{A}_{12}, \mathcal{A}_{21})$.

Let $A = (A^+, A^-)$ be an associative pair and $a \in A^\sigma$. Recall that the module $A^{-\sigma}$ endowed with the *a-homotope product*

$$x \underset{a}{\cdot} y = xay,$$

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becomes an associative ring A^a, called the *homotope* of A *at* a, which has as a [nilpotent] ideal the set

$$\mathrm{Ker}(a) = \{x \in A^{-\sigma} \mid axa = 0\}.$$

The ring $A^a/\mathrm{Ker}(a)$ is called the *local ring* of A *at* a. There is another way to introduce local rings at elements [see (6)]: The submodule $aA^{-\sigma}a$, equipped with the multiplication defined by

$$(axa) \cdot (aya) = axaya,$$

is an associative ring denoted by A_a. An element $a \in A^\sigma$ is said to be *von Neumann regular* if $a \in aA^{-\sigma}a$. Observe that if a is von Neumann regular, then A_a is unital with a as its unit. Moreover, it is easy to see that in any case the mapping $x + \mathrm{Ker}(a) \mapsto axa$ is an isomorphism from $A^a/\mathrm{Ker}(a)$ onto the ring A_a.

Let B be an associative pair. A pair of submodules $A = (A^+, A^-)$ of B is called a *subpair* if $A^\sigma A^{-\sigma} A^\sigma \subseteq A^\sigma$. Suppose that A is a subpair of an associative pair B and let $b \in B^-$ be such that A^+ is a subalgebra of the homotope B^b. Then bA^+b can be regarded as a subalgebra of the local algebra B_b of B at b which will be called the *generalized local algebra* of A *at* b and will also be denoted by A_b. If b is actually in A^-, then the definition of generalized local algebra agrees, of course, with that given above.

An associative pair A is said an *exchange associative pair* if for every element $x \in A^\sigma$ the local ring of A at x, A_x, is an exchange ring. Our definition is inspired by (8, Theorem 1.4), which asserts that a ring $\mathcal{R}$ is an exchange ring if and only if every local ring $\mathcal{R}_x$ is an exchange ring. Thus, the first example of an exchange associative pair is the associative pair $(\mathcal{R}, \mathcal{R})$ for any exchange ring $\mathcal{R}$. We point out that in the case of an associative pair $(\mathcal{R}, \mathcal{R})$ coming from an associative ring $\mathcal{R}$, the definitions of homotopes and local rings at elements coincide with the ones given in Section 1.

Remark. Notice that by the isomorphism between A_x and $A^x/\mathrm{Ker}(x)$ and by (5, Corollary 2.4), the ring A_x is exchange if and only if A^x is an exchange ring. This reasoning can also be applied when A_x is a generalized local ring.

Hence, Lemma 1.1 provides a characterization of an exchange associative pair in terms of its elements: An associative pair A is an exchange associative pair if and only if every element $x \in A^\sigma$ satisfies the equivalent conditions in the following lemma for every $y \in A^{-\sigma}$.

Lemma 3.1. *Let A be an associative pair, $x \in A^\sigma$ and $\mathcal{S}$ be a unital ring containing A^x as an ideal. Then the following conditions are equivalent for an element $y \in A^{-\sigma}$:*

(i) *There exists $e \underset{q}{\cdot} e = e \in A^{-\sigma}$ with $e - y \in \mathcal{S}(y - y \underset{q}{\cdot} y)$,*

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(ii) *there exists* $e \cdot_q e = e \in A^{-\sigma} \cdot_q y$ *and* $c \in \mathcal{S}$ *such that* $(1_{\mathcal{S}} - e) - c(1_{\mathcal{S}} - y) \in J(\mathcal{S})$,

(iii) *there exists* $e \cdot_q e = e \in A^{-\sigma} \cdot_q y$ *such that* $\mathcal{S} = A^x \cdot_q e + \mathcal{S}(1_{\mathcal{S}} - y)$,

(iv) *there exists* $e \cdot_q e = e \in A^{-\sigma} \cdot_q y$ *such that* $1_{\mathcal{S}} - e \in \mathcal{S}(1_{\mathcal{S}} - y)$,

(v) *there exist* $e \cdot_q e = e \in A^{-\sigma}$ *and* $r, s \in A^{-\sigma}$ *such that* $e = r \cdot_q y = s + y - s \cdot_q y$.

Let $A = (A^+, A^-)$ be an associative pair. We say that a pair $(x, y) \in A$ is *quasi-invertible* if x is quasi-invertible in the associative algebra A^y. The [*Jacobson*] *radical* of the associative pair A is defined as $\mathrm{Rad}A = (\mathrm{Rad}A^+, \mathrm{Rad}A^-)$, where $\mathrm{Rad}A^\sigma$ is the set of properly quasi-invertible elements of A^σ [see (10) for the definitions and some characterizations of the radical of an associative pair].

Let A be an associative pair. An element $(a, b) \in A$ is said to be an *idempotent* if $aba = a$ and $bab = b$.

If $A = (\mathcal{R}, \mathcal{R})$ is the pair associated to a ring $\mathcal{R}$, then every von Neumann regular element $a = aba \in \mathcal{R}$ gives rise to an idempotent (a, bab) in the associative pair A. Conversely, if (a, b) is an idempotent in A, then a and b are von Neumann regular elements of $\mathcal{R}$.

Example 3.2. The radical associative pairs are exactly the exchange associative pairs without nonzero idempotents.

Recall that an associative pair A is said to be *von Neumann regular* if for every element $x \in A^\sigma$ there exists $y \in A^{-\sigma}$ such that $x = xyx$.

Example 3.3. If an associative pair A is von Neumann regular, then A_x is von Neumann regular for every element $x \in A^\sigma$. By (5, Example 3), A_x is an exchange ring, hence every von Neumann regular associative pair is an exchange associative pair.

Proposition 3.4. *Let $\mathcal{R}$ be a ring and $\alpha, \beta \in \mathbb{N}$. Then $\mathcal{R}$ is an exchange ring if and only if $(\mathcal{M}_{\alpha\times\beta}(\mathcal{R}), \mathcal{M}_{\beta\times\alpha}(\mathcal{R}))$ is an exchange associative pair.*

Proof: Consider the quasi-acceptable Morita context

$$(\mathcal{M}_\alpha(\mathcal{R}), \mathcal{M}_\beta(\mathcal{R}), \mathcal{M}_{\alpha\times\beta}(\mathcal{R}), \mathcal{M}_{\beta\times\alpha}(\mathcal{R}))$$

whose associated Morita ring is $\mathcal{M}_{\alpha+\beta}(\mathcal{R})$ and apply (5, Theorem 1.4) and Theorem 2.3. □

Given an associative pair $A = (A^+, A^-)$, denote by $\mathcal{A}$ its envelope. Since $(\mathcal{A}_{11}, \mathcal{A}_{22}, \mathcal{A}_{12}, \mathcal{A}_{21})$ defines a quasi-acceptable Morita context, Theorem 2.3 can be applied to obtain the following result:

Theorem 3.5. *Let $A = (A^+, A^-)$ be an associative pair and $\mathcal{A}$ its envelope. Then the following conditions are equivalent*:

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i) *A is an exchange associative pair,*
ii) *$\mathcal{A}$ is an exchange associative ring,*
iii) *for every element $a \in A^+$ the local ring of A at a, A_a, is an exchange ring,*
iv) *for every element $b \in A^-$ the local ring of A at b, A_b, is an exchange ring,*
v) *$\mathcal{A}_{11}$ is an exchange ring,*
vi) *$\mathcal{A}_{22}$ is an exchange ring,*

Lemma 3.6. *Every pair $L = (L^+, L^-)$ of left ideals of an exchange associative pair $A = (A^+, A^-)$ is an exchange associative pair.*

Proof: Take $l \in L^\sigma$. Then L_l is a left ideal of the exchange ring A_l [because $L^{-\sigma}$ is a left $\Lambda(A^{-\sigma}, A^\sigma)$-submodule of $A^{-\sigma}$] and, by (8, Proposition 1.3), L_l is an exchange ring, hence L is an exchange associative pair. □

We will say that an ideal I of an associative pair A is an *exchange ideal* if I is an exchange associative pair. We will prove that every associative pair has a largest exchange ideal.

Proposition 3.7. *Let A be an associative pair and $\mathcal{A}$ its envelope.*

i) *If $\mathcal{I}$ is an exchange ideal of $\mathcal{A}$, then $I = (I^+, I^-) := (\mathcal{I} \cap \mathcal{A}_{12}, \mathcal{I} \cap \mathcal{A}_{21})$ is an exchange ideal of A.*
ii) *If $I = (I^+, I^-)$ is an exchange ideal of A and $\mathcal{I}$ denotes the ideal generated by I in $\mathcal{A}$, i.e., $\mathcal{I} = I^+ + I^- + \mathcal{A}_{21}I^+ + I^+\mathcal{A}_{21} + I^-\mathcal{A}_{12} + \mathcal{A}_{12}I^-$, then $\mathcal{I}$ is an exchange ideal of $\mathcal{A}$.*

Proof:

i) It is obvious that I is an ideal of A. Take $y \in I^\sigma$. By (8, Theorem 1.4), $\mathcal{I}_y$ is an exchange ring. Since I_y is a two-sided ideal of $\mathcal{I}_y$, then I_y is an exchange ring again by (8, Theorem 1.4).
ii) Let $I = (I^+, I^-)$ be an exchange ideal of A and denote by $\mathcal{I}$ the ideal generated by I in $\mathcal{A}$. Since for every $y \in I^\sigma$ we have

$$\begin{aligned} y\mathcal{I}y &= y(I^\sigma + I^{-\sigma} + A^{-\sigma}I^\sigma + I^{-\sigma}A^\sigma + A^{-\sigma}I^\sigma + I^{-\sigma}A^\sigma)y \\ &= yI^\sigma y, \end{aligned}$$

we see that $\mathcal{I}_y = I_y$ is an exchange ring. By Corollary 1.5, $y \in \epsilon(\mathcal{A})$ and applying Proposition 1.6 we obtain that $\mathcal{I}$ is an exchange ideal of $\mathcal{A}$. □

As an immediate consequence of this proposition and (8, Theorem 3.5) we obtain:

Theorem 3.8. *Every associative pair A contains a largest exchange ideal $\epsilon(A)$ [with respect to the inclusion]. Moreover, if $\mathcal{A}$ is the envelope of A, then*

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$$\epsilon(A) = (\epsilon(\mathcal{A}) \cap \mathcal{A}_{12}, \epsilon(\mathcal{A}) \cap \mathcal{A}_{21}).$$

The following easy proposition generalizes (5, Proposition 1.3).

Proposition 3.9. *Let $I = (I^+, I^-)$ be an exchange ideal of an associative pair $A = (A^+, A^-)$. Then for every element $a \in A^\sigma$ the generalized local algebra I_a is an exchange ring.*

Proof: Let $\mathcal{A}$ be the envelope of A, and let $\mathcal{I}$ be the ideal generated by I in $\mathcal{A}$. By Proposition 3.7 (ii), $\mathcal{I}$ is an exchange ideal of $\mathcal{A}$. Now consider the generalized local algebra of $\mathcal{I}$ at a, $\mathcal{I}_a$. For any element $aya \in \mathcal{I}_a \subseteq \mathcal{I}$ we have that $(\mathcal{I}_a)_{aya} = \mathcal{I}_{aya}$ is an exchange ring by (8, Theorem 1.4). Hence $I_a = \mathcal{I}_a$ is an exchange ring. □

Now we obtain a theorem about lifting orthogonal idempotents modulo a pair of left ideals in any exchange associative pair.

Let $L = (L^+, L^-)$ be a pair of left ideals of A. We say that *idempotents can be lifted modulo L* if, for every element $(a, b) \in A$ such that $a - aba \in L^+$ and $b - bab \in L^-$, there exists an idempotent $(e, f) \in A$ such that $e - a \in L^+$ and $f - b \in L^-$.

Theorem 3.10. *An associative pair A is an exchange associative pair if and only if idempotents can be lifted modulo every pair $L = (L^+, L^-)$ of left ideals of A.*

Proof: Assume that A is an exchange associative pair, and let $L = (L^+, L^-)$ be a pair of left ideals of A. Take an element $(a, b) \in A$ such that $a - aba \in L^+$ and $b - bab \in L^-$. Denote by A_1^b the unitization of A^b. By Lemma 3.1 (i) there exist $e = ebe \in A^+$ such that $e - a \in A_1^b(a - aba)$. Thus $e - a = (r, n)(a - aba, 0)$ for some $r \in A^b$ and $n \in \mathbb{Z}$. Now, computing, we see that

$$e - a = (rb(a - aba) + n(a - aba), 0) \in L^+$$

since L is a pair of left ideals, $r \in A^+$ and $b \in A^-$. Then (e, beb) is an idempotent of A and

$$\begin{aligned} beb - b &= b(a + rb(a - aba) + n(a - aba))b - b \\ &= bab - b + (brb)a(b - bab) + nba(b - bab) \in L^-. \end{aligned}$$

Conversely, assume that idempotents can be lifted modulo every pair of left ideals. Let (x, y) be in A. For $a, b \in A^-$, we will write $a \equiv b$ when $a - b$ is in the principal left ideal of A^x generated by $y - yxy$. Consider the left ideals L^+ and L^- of A generated by $xyx - xyxyxyxyx$ and $yxy - yxyxyxyxy$ respectively. Then (xyx, yxy) satisfies

$$\begin{aligned} &xyx - xyxyxyxyx \in L^+ \\ &yxy - yxyxyxyxy \in L^-. \end{aligned}$$

By hypothesis there exists an idempotent $(u, v) \in A$ such that $u - xyx \in L^+$ and $v - yxy \in L^-$. Let α and β be in $\Lambda(A^+, A^-)$, $\Lambda(A^-, A^+)$, respectively, such that

$$u = xyx + \alpha(xyx - xyxyxyxyx)$$
$$v = yxy + \beta(yxy - yxyxyxyxy).$$

Note also that

$$\begin{aligned} v = vuv &= v[xyx + \alpha(xyx - xyxyxyxyx)]v \\ &= [vxy + v(\alpha xy - \alpha xyxyxyxy)]xv. \end{aligned}$$

Define $e = vxy + v(\alpha xy - \alpha xyxyxyxy)$. Then $e \underset{q}{\cdot} e = e \in A^-$ and

$$\begin{aligned} e - yxyxy &= vxy - yxyxy + v\alpha xy - v\alpha xyxyxyxy \\ &= (v - yxy)xy + v\alpha xy - v\alpha xyxyxyxy \\ &= \beta(yxy - yxyxyxyxy)xy + v\alpha xy - v\alpha xyxyxyxy \\ &= (\beta yxyx + v\alpha x)(y - yxyxyxy) \equiv 0 \end{aligned}$$

because

$$\begin{aligned} y - yxyxyxy &= y - yxy + yxy - yxyxy + yxyxy - yxyxyxy \\ &= (y - yxy) + yx(y - yxy) + yxyx(y - yxy) \equiv 0. \end{aligned}$$

Hence

$$\begin{aligned} e - y &= e - yxyxy + yxyxy - yxy + yxy - y \\ &= (e - yxyxy) + yx(yxy - y) + (yxy - y) \equiv 0 \end{aligned}$$

which implies, by Lemma 3.1 (i), that A is an exchange associative pair. □

Corollary 3.11. *A ring $\mathcal{R}$ is an exchange ring if and only if von Neumann regular elements can be lifted modulo every left ideal.*

Proof: If $\mathcal{R}$ is an exchange ring, then $R = (\mathcal{R}, \mathcal{R})$ is an exchange associative pair. Let $\mathcal{L}$ be a left ideal of $\mathcal{R}$ and let $a \in \mathcal{R}$ be a von Neumann regular element modulo $\mathcal{L}$, i.e., $a - aba \in \mathcal{L}$ for some $b \in \mathcal{R}$. We have that (a, b) is an idempotent of R modulo the left ideal $(\mathcal{L}, \mathcal{R})$ of R. By Theorem 3.10 there exists an idempotent $(e, f) \in \mathcal{R}$ and, consequently, a von Neumann regular element $e \in \mathcal{R}$ such that $a \equiv e \pmod{\mathcal{L}}$.

For the converse, take an element $x \in \mathcal{R}$ and consider any unital ring $\mathcal{R}'$ containing $\mathcal{R}$ as an ideal. Denote by $\mathcal{L}$ the left ideal of $\mathcal{R}$ generated by $x - x^2$ and containing it, i.e., $\mathcal{L} = \mathcal{R}'(x - x^2)$. Then x is a von Neumann regular element of $\mathcal{R}$ modulo $\mathcal{L}$ because $x - x^3 = x - x^2 + x^2 - x^3 = (x - x^2) + x(x - x^2) \in \mathcal{L}$. By hypothesis there exists an idempotent $e \in \mathcal{R}$ such that $e - x \in \mathcal{L} \subseteq \mathcal{R}'$, which implies, by condition (i) in Lemma 1.1 that $\mathcal{R}$ is an exchange ring. □

If I is an ideal of an associative pair A then $A/I = (A^+/I^+, A^-/I^-)$ is an associative pair in the obvious way.

With the following result we describe the exact conditions needed for an extension of associative pairs to be an exchange associative pair

Theorem 3.12. *Let $I = (I^+, I^-)$ be an ideal of an associative pair A. Then A is an exchange associative pair if and only if I and A/I are exchange associative pairs and idempotents can be lifted modulo I.*

Proof: Assume that A is an exchange associative pair. By Lemma 3.1 (v), A/I is an exchange associative pair; by Lemma 3.6, I is an exchange associative pair and by Theorem 3.10, idempotents can be lifted from A/I to A.

Conversely, assume that I and A/I are exchange associative pairs and that idempotents lift from A/I to A. Let x be an element in A^+. By Proposition 3.9, I_x is an exchange ring. This implies, as we pointed out in the previous remark to Lemma 3.1, that I^x is an exchange ring. Denote by $\bar{t}$ the image of $t \in A^\sigma$ under the canonical map $A^\sigma \to A^\sigma/I^\sigma$ and by $\tilde{t}$ the image of t under the canonical map $A^x \to A^x/I^x$. The map

$$\begin{array}{rccc} \varphi : & A^x/I^x & \to & (A/I)_{\bar{x}} \\ & \tilde{t} & \mapsto & \bar{x}\bar{t}\bar{x} \end{array}$$

is a ring epimorphism whose kernel is a nilpotent ideal of A^x/I^x. By (8, Theorem 1.4), $(A/I)_{\bar{x}}$ is an exchange ring, hence A^x/I^x is an exchange ring, by (5, Corollary 2.4).

Now we prove that idempotents can be lifted in A^x modulo I^x.

Let $a \in A^-$ be an idempotent of A^x modulo I^x, i.e., $axa - a \in I^-$. Then (xax, a) is an idempotent in A modulo I:

$$a(xax)a - a = ax(axa) - axa + axa - a = ax(axa - a) + axa \\ - a \in I^-;\ (xax)a(xax) - xax = x(axaxa - a)x \in I^+.$$

By hypothesis, there exists an idempotent (v, u) in A such that $\tilde{u} = \tilde{a}$ and $v - xax \in I^+$. Notice that the ideal $\mathcal{I}$ generated by I in $\mathcal{A}$, the envelope of A, is an exchange ring by Proposition 3.7 (ii). Denote by $[\alpha]$ the image of $\alpha \in \mathcal{A}$ under the canonical map $\mathcal{A} \to \mathcal{A}/\mathcal{I}$ and let $\mathcal{E}$ be the standard embedding of A. The element $p = uv$ is an idempotent in $\mathcal{E}$; moreover $[p] = [uv] = [axax] = [ax]$. We can apply (5, Lemma 2.1) to get $\alpha \in p\mathcal{E}a \subseteq A^-$, $\beta \in xax\mathcal{E}p \in A^+$ such that (β, α) is an idempotent in A and $[\alpha] = [pa]$, $[\beta] = [xaxp]$. Let α' and β' be in $\mathcal{E}$ such that $\alpha = p\alpha' a$ and $\beta = xax\beta' p$. Then, since p is an idempotent, we have that

$$p\alpha' a = \alpha = \alpha\beta\alpha = p\alpha' axax\beta' pp\alpha' a = p\alpha' axax\beta' p\alpha' a$$

and therefore the element $e = ax\beta' p\alpha' a$ is an idempotent of A^x. Moreover

$$[e] = [ax\beta' p\alpha' a] = [axax\beta' p\alpha' a] = [a\beta\alpha] = [axaxppa] = [axaxpa] \\ = [axpa] = [axaxa] = [axa] = [a],$$

viewing $e, a \in \mathcal{A}$ and taking into account that $a - axa \in I^- \subseteq \mathcal{I}$. Hence $\tilde{e} = \tilde{a}$, as required.

Apply (5, Theorem 2.2) to the ideal I^x of A^x to get that A^x is an exchange ring. Consequently, A_x is an exchange ring. □

Two nonzero idempotents (e, f) and (u, v) in an associative pair A are called *orthogonal* if and only if $evA^+ = ufA^+ = 0$ and $veA^- = fuA^+ = 0$. In view of (10, (5.11)), this is a symmetric relation. By an *orthogonal system of idempotents* we mean an ordered set of pairwise orthogonal idempotents. [See (10) for the definitions].

Theorem 3.13. *Let A be an exchange associative pair, let $L = (L^+, L^-)$ be a pair of left ideals of A, and let $(x_1, y_1), \ldots, (x_\alpha, y_\alpha)$ be orthogonal idempotents modulo L, that is, $x_i \equiv x_i y_i x_i (mod\ L^+)$, $y_i \equiv y_i x_i y_i (mod\ L^-)$ for each i, and $x_i y_j A^+ \equiv 0\,(mod\ L^+)$, $y_j x_i A^- \equiv 0\,(mod\ L^-)$ for all $i \neq j$. Then there exists an orthogonal system of idempotents $(e_1, f_1), \ldots, (e_\alpha, f_\alpha)$ in A such that $e_i \equiv x_i (mod\ L^+)$ and $f_i \equiv y_i (mod\ L^-)$ for each i.*

Proof: Let $y = y_1 + \cdots + y_\alpha$ and consider the exchange ring A^y. Then L^y is a left ideal of A^y and $x_1, \ldots, x_\alpha$ are orthogonal idempotents in A^y modulo L^y. Indeed, for any i

$$x_i y x_i = \sum_{k=1}^{\alpha} x_i y_k x_i \equiv x_i y_i x_i \equiv x_i (\text{mod } L^+)$$

and for $i \neq j$

$$x_i y x_j = \sum_{k=1}^{\alpha} x_i y_k x_j \equiv x_i y_i x_j \equiv x_i y_i (x_j y_j x_j) \equiv x_i (y_i x_j y_j) x_j \equiv 0 (\text{mod } L^+).$$

Let $x = x_1 + \cdots + x_\alpha$ and let $\mathcal{S} = A_1^y$. By Proposition 1.7, there exist orthogonal idempotents $e_1, \ldots, e_\alpha$ in A^y such that $e_i \in A^y \underset{y}{\cdot} x_i$ for each i and $1 - e \in \mathcal{S}(1 - x)$, where $e = e_1 + \cdots + e_\alpha$. Define $f_i = y e_i y \in A^-$. idempotents in A^y where we write unitization of the associative

Then for $i = 1, \cdots, \alpha$,

$$e_i f_i e_i = y f_i y f_i y f_i y f_i y = y(f_i \underset{y}{\cdot} f_i \underset{y}{\cdot} f_i) y = y f_i y = e_i,$$

$$f_i e_i f_i = f_i y f_i y f_i = f_i \underset{y}{\cdot} f_i \underset{y}{\cdot} f_i = f_i$$

and for $i \neq j$

$$e_i f_j A^+ = e_i y e_j y A^+ = (e_i \underset{y}{\cdot} e_j) y A^+ = 0,$$

$$f_j e_i A^- = y e_j y e_i A^- = y(e_j \underset{y}{\cdot} e_i) A^- = 0,$$

that is, $(e_1, f_1), \ldots, (e_\alpha, f_\alpha)$ are orthogonal idempotents in A. Moreover, for each i, $xyx_i \equiv x_i (\text{mod } L^+)$ and, since $e_i \in A^y \cdot_y x_i$, then $eyx_i \equiv e_i (\text{mod } L^+)$. If $1 - e = s(1 - x)$ for some $s \in \mathcal{S}$, this implies

$$x_i - e_i \equiv (1 - e)x_i \equiv s(1 - x)x_i \equiv 0\,(\text{mod } L^+).$$

Finally,

$$yx_i y = \sum_{j,k=1}^{\alpha} y_j x_i y_k \equiv y_i x_i y_i \equiv y_i (\text{mod } L^-)$$

implies

$$y_i - f_i = y_i - ye_i y \equiv y(x_i - e_i)y \equiv 0\,(\text{mod } L^-). \qquad \square$$

Corollary 3.14. *Let $\mathcal{R}$ be a ring, $\mathcal{L}$ a left ideal of $\mathcal{R}$ and $x_1, \ldots, x_\alpha$, orthogonal von Neumann regular elements of $\mathcal{R}$ modulo $\mathcal{L}$, i.e., $x_i y_i x_i \equiv x_i$ (mod $\mathcal{L}$) for some $y_1, \ldots, y_\alpha \in \mathcal{R}$ and $x_i x_j \equiv 0$ (mod $\mathcal{L}$) for every $i \neq j$, $i, j \in \{1, \ldots, \alpha\}$. Then there exists an orthogonal system of von Neumann regular elements $e_1, \ldots, e_\alpha \in \mathcal{R}$ such that*

$$e_i \in \mathcal{R}x_i \quad \text{and} \quad e_i \equiv x_i \ (\text{mod } \mathcal{L}).$$

Proof: The result follows from the proof of Theorem 3.14. $\square$

ACKNOWLEDGMENTS

It is a pleasure to thank the referee for the careful reading of the paper and his helpful suggestions. This work was supported by the DGI, Grant PB097-1069-C02-01 and by the "Plan andaluz de Investigación y Desarrollo Tecnológico".

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Received September 27, 1999
Revised February 25, 2000 and May 31, 2000