Fountain–Gould left orders and "fractional" idempotents

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ABSTRACT

In this paper we relate Fountain-Gould left orders to Litoff's Theorem. This fact allows us to obtain a characterization of Fountain-Gould left orders which are Fountain-Gould orders in terms of some type of idempotents.

2000 Mathematical Subject Classification: 16S90.

1. §1. Preliminaries

In 1990 J. Fountain and V. Gould [FG_1] introduced a notion of order in a ring which need not have a unit, and gave [FG_2] a Goldie-like characterization of two-sided orders in semiprime rings with descending chain condition on principal one-sided ideals (equivalently, coinciding with their socle). In 1991 P.N. Ánh and L. Márki [AM_1] extended this result to one-sided orders. More recently the same authors have developed a general theory of Fountain-Gould left quotient rings (AM_2]).

Let a be an element of a ring R. We say that b in R is a **group inverse** of a if the following conditions hold: $aba = a, \quad bab = b, \quad ab = ba$.

It is easy to see that a has a group inverse b in R if and only if there exists a unique idempotent e ($e = ab$) in R such that a is invertible in the ring eRe (with inverse b), hence the group inverse is unique and a is said to be **locally**

[1] Partially supported by the MCYT, BFM2001-1938-C02-01, and by the "Plan Andaluz de Investigación y desarrollo Tecnológico", FQM-264.

invertible. Denote by $a^\sharp$ the group inverse of a and by $LocInv(R)$ the set of all locally invertible elements of R.

An element $a \in R$ is **left semiregular** (or **left square cancellable**) if $a^2x = a^2y$ implies $ax = ay$ for $x, y \in R \cup \{1\}$. A **right semiregular** (**right square cancellable**) element is defined analogously, and **semiregular** (**square cancellable**) means both left and right semiregular. We denote by $SemiReg_l(R)$, $SemiReg_r(R)$ and $SemiReg(R)$ the sets of all left, right and two-sided semiregular elements of R, respectively.

We recall that a subring R of a ring Q is a **Fountain-Gould left order** in Q if:

(i) Every element of $SemiReg(R)$ has a group inverse in Q and

(ii) every element $q \in Q$ can be written $q = a^\sharp x$, where $a \in SemiReg(R)$ and $x \in R$.

In a similar way, **Fountain-Gould right orders** are defined. If R is both a Fountain-Gould left and right order in Q, then we say that R is an **Fountain-Gould order** in Q. When (only) condition (2) is satisfied we say that R is a **weak Fountain-Gould left order** in Q.

We recall the notion of left quotient ring, due to Utumi (see [U]), and the notion of local ring at an element.

Let R be a subring of a ring Q. We say that Q is a **left quotient ring** of R if given $p, q \in Q$, with $p \neq 0$, there exists $r \in R$ such that $rp \neq 0$ and $rq \in R$. Notice that if R is a Fountain-Gould left order in a ring Q, by [AM$_2$, Theorem 1], Q is a left quotient ring of R.

Let R be a ring and let $a \in R$. Then, the abelian group of R endowed with the a-**homotope product**: $x.y = xay$ becomes a ring, the a-**homotope ring**, denoted by R^a, which has as an ideal the set $Ker(a) = \{x \in R \mid axa = 0\}$. The **local ring** of R **at** a is defined as $R^a/Ker(a)$ and it is denoted by R_a.

2. §2. The Theorem

The following theorem [GS, Theorem 4.12] gives same equivalent conditions for a subring R of a semiprime ring Q coinciding with its socle to be a Fountain-Gould left order in Q.

2.1 THEOREM 2.1. *Let R be a subring of a semiprime ring Q which coincides with its socle. The following conditions are equivalent:*

- *” (i)” R is a weak Fountain-Gould left order in Q,*
- *” (ii)” R is a Fountain-Gould left order in Q,*
- *” (iii)” $Q = RQR$ and for every nonzero element $a \in R$ we have that R_a is a classical left order in the semisimple artinian ring Q_a,*
- *” (iv)” R is semiprime, $Q = RQ$ and Q is a left quotient ring of R,*
- *” (v)” for every finite subset Y of Q there exists an element $a \in \mathcal{S}\rceil\Updownarrow\rangle\mathcal{R}\rceil\}(\mathcal{R})$ such that $Y \subseteq aQa$ and R_a is a classical left order in the semisimple artinian ring Q_a.*

Since for every semiregular element a in a ring R we have that the map $ara \mapsto \overline{r}$ between aRa and R_{a^2} is an isomorphism, we can add another equivalent condition to the previous theorem:

- *” (v′)” For every finite subset Y of Q there exists an element $a \in \mathcal{S}\rceil\Updownarrow\rangle\mathcal{R}\rceil\}(\mathcal{R})$ such that $Y \subseteq aQa$ and aRa is a classical left order in the semisimple artinian ring eQe, with $e = aa^\sharp$.*

2.2 DEFINITION 2.2. *Let R be a Fountain-Gould left (or a classical left) order in a ring Q. We say that an idempotent $e \in Q$ is a* **fractional idempotent** *if there exists $a \in SemiReg(R)$ such that $e = aa^\sharp$.*

Note that the condition (v′) provides a nice relationship between Fountain-Gould left orders and Litoff's Theorem. Namely, if R is a Fountain-Gould left order in a semiprime ring Q which coincides with its socle, then for every finite subset Y of Q there exists a fractional idempotent $e \in Q$ such that $Y \subset eQe$.

The first question to be considered here is whether or not an idempotent is fractional.

2.3 EXAMPLE 2.3. *Let D be a classical left order in a division ring Δ which is not a classical right order in Δ. Then $R := \mathcal{M}_\in(\mathcal{D})$ is a classical left order in $Q := \mathcal{M}_\in(\cdot)$ and Q has idempotents which are not fractional.*

Proof. It is well known that R and Q are semiprime and Q is artinian. On the other hand, it is easy to prove that Q is a left quotient ring of R. Then by [GS, Corollary 3.4], R is a classical left order in Q. Now take $\lambda \in \Delta$ such that $\lambda D \cap D = \{0\}$, (such a λ exists because D is not a classical right order in Δ).

Then $e = \begin{pmatrix} 1 & 0 \\ \lambda & 0 \end{pmatrix} \in Q$ *is a idempotent which is not fractional because, as it is easy to see,* $eR \cap R = \{0\}$. ■

The next proposition follows the ideas of [FGGS, Proposition 7.6].

2.4 Proposition 2.4. *Let* R *be a semiprime associative ring which is a weak Fountain-Gould order in a ring* Q*, and let* $0 \neq q \in Q$*. Then we have*

" *(i)*" $qRq \cap R \neq 0$.

Moreover, if Q *coincides with its socle and* $e^2 = e \in Q$*, then*

" *(ii)*" *there exists* $b \in SemiReg(R)$ *such that* $eQe = bQb$.

Proof.

(i) If we write $q = a^\sharp b = dc^\sharp$ *with* $a^\sharp ab = b$, $dcc^\sharp = d$ *then* $0 \neq aqc = bc = ad \in R$ *(otherwise* $q = a^\sharp aqc^\sharp c = 0$*), and by semiprimeness of* R, $aqcRaqc \neq 0$, *which implies* $0 \neq qcRaq = dRb \subset qRq \cap R$.

(ii) Note that by (i) Q *is semiprime. Suppose now that* Q *coincides with its socle. If* $e = 0$ *there is nothing to prove. suppose* $e \neq 0$*. Then, by [FGGS, Proposition 5.2 (i) and (v)],* eQe *is semiprime and artinian and* e *is its unit element. Write* $e = e_1 + \ldots + e_n$ *as a sum of orthogonal division idempotents in* eQe*. By (i), for each* $1 \leq i \leq n$ *there exists* $0 \neq b_i \in e_iRe_i \cap R$*. Since the* e_i *are mutually orthogonal in* eQe, $b := b_1 + \ldots + b_n \in Inv(eQe)$ *and furthermore* $eQe = bQb$. ■

Now we can formulate our main result.

2.5 Theorem 2.5. *Let* R *be a ring which is a Fountain-Gould left order in a simple ring* Q *which coincides with its socle. Suppose that* Q *is not a division ring. The following conditions are equivalent:*

" *(i)*" R *is a Fountain-Gould order in* Q.

" *(ii)*" *Every idempotent of* Q *is fractional.*

Proof.

By [GS, Theorem 4.7], R *is a prime ring.*

$(i) \Rightarrow (ii)$*. Let* e *be a nonzero idempotent. By condition (ii) of Proposition 2.4 there exists* $b \in SemiReg(R)$ *such that* $eQe = bQb$*. Now, since* Q *is von Neumann regular and* e *is the identity in* eQe, $b \in eQe$ *and* $e = bqb = ebeqebe$*. Therefore* $b \in Inv(eQe)$*, what implies* $b \in LocInv(Q)$ *with* $e = bb^\sharp$.

(ii) ⇒ (i). By hypothesis $Q = QR$. If we show that Q is a right quotient ring of R, we will have, by Theorem 2.1 (4), that R is a Fountain-Gould right order in Q.

Let $g \in Q$ such that u–dim$_Q(g) \geq 2$ (such an element g exists because Q is not a division ring). Let $p, q \in Q$, with $p \neq 0$, by (v') of Theorem 2.1, there exists $w \in SemiReg(R)$ such that $\{g, p, q\} \subset wQw$ and wRw is a classical left order in eQe, where $e = ww^\sharp$ (note that u–dim$_Q(e) \geq 2$). We know by [GS, Proposition 2.1 (i) and (v)] that eQe is simple and artinian, so there exists $e_1, e_2, \dots, e_n$ ($n \geq 2$), orthogonal division idempotents such that $e = \sum e_i$. By assumption, for every e_i there exists $x_i \in SemiReg(R)$ such that $e_i = x_i^\sharp x_i$. Notice that $x_i \in e_iQe_i \subset eQe$. Write $x = \sum x_i$. It is easy to see that $x \in Inv(eQe)$ with $x^\sharp = \sum x_i^\sharp$.

We finish the proof into several steps:

(1). For every $0 \neq p_{ij} \in e_iQe_j$, there exists $y \in x_jRx_j$ such that $0 \neq p_{ij}y \in R$.

(a). If $i \neq j$. We consider the idempotent $e_j + x_i^\sharp p_{ij} x_j \in Q$. By assumption, there exists $a \in SemiReg(R)$ such that $e_j + x_i^\sharp p_{ij} x_j = a^\sharp a$, therefore $a = (e_j + x_i^\sharp p_{ij} x_j)a = a(e_j + x_i^\sharp p_{ij} x_j)$ which implies that $(e_j + e_i)a = a = ae_j$. Denote by $y = x_j a x_j \in x_jRx_j$. Then $y \neq 0$ because $0 \neq e_j = e_j aa^\sharp = e_j a e_j a^\sharp = x_j^\sharp x_j a x_j x_j^\sharp a^\sharp$ and $p_{ij}y = p_{ij}x_j a x_j = x_i x_i^\sharp p_{ij} x_j a x_j = x_i(e_j + x_i^\sharp p_{ij} x_j) a x_j = x_i a a^\sharp a x_j = x_i a x_j \in x_jRx_j$. Moreover, $p_{ij}y \neq 0$, because y is a nonzero element in the division ring e_jQe_j.

(b). If $i = j$. Let $0 \neq x_{ik} \in x_iRx_k \subset e_iQe_k$. Since $p_{ii} \in e_iQe_i$ (which is a division ring), $0 \neq p_{ii}x_{ik}$. By (a), there exists $b_{kk} \in x_kRx_k$ such that $0 \neq p_{ii}x_{ik}b_{kk} \in R$. Now, since R is a prime ring, there exists $t \in R$ such that $0 \neq p_{ii}x_{ik}b_{kk}tx_i \in R$. Furthermore $y := x_{ik}b_{kk}tx_i \in x_iRx_i$ is our element.

(2). Now since $p, q \in eQe$, $p = \sum_{r,s=1}^n e_r p e_s$, $q = \sum_{r,s=1}^n e_r q e_s$. Let $i, j \in \{1, 2, \dots, n\}$ such that $e_ipe_j \neq 0$. If we take e_1qe_j, by (i), there exists $0 \neq x_jy_1x_j \in x_jRx_j$ such that $e_1qe_j\, x_jy_1x_j \in R$. Again by (i), if we take $e_2qe_jx_jy_1x_j$, there exists $0 \neq x_jy_2x_j \in x_jRx_j$ such that $e_2qe_j\, x_jy_1x_j\, x_jy_2x_j \in R$. We continue in this way to obtain $z = x_jy_1x_j^2y_2x_j \dots x_j^2y_nx_j$. Therefore $0 \neq z \in x_jRx_j$, $qz \in R$ and $pz \neq 0$ (because $0 \neq e_ipe_jz$ since z is a element of the division ring e_jQe_j and $e_ipe_j \neq 0$). ■

REFERENCE

[AM$_1$] **P.N. Ánh and L. Márki** *"Left orders in regular rings with mini-*

*mum condition for principal one-sided ideals" Math. Proc. Camb. Phil. Soc.***109** *(1991), 323–333.*

[AM$_2$] **P.N. Ánh and L. Márki** *"A general theory of Fountain-Gould quotient rings" Math. Slovaca***44** *(1994), 255–235.*

[FG$_1$] **J. Fountain and V. Gould** *"Orders in rings without identity" Comm. Algebra***18** *(1990), 3085–3110.*

[FG$_2$] **J. Fountain and V. Gould** *"Straight left orders in rings" Quart. J. Math (Oxford)***43** *(1992), 303–311..*

[FGGS] **A. Fernández López, E. García Rus, M. Gómez Lozano and M. Siles Molina** *"Goldie theorem for associative pairs" Commun. in Algebra***26(9)** *(1998), 2987–3020.*

[GS] **M. Gómez and M. Siles** *"Quotient rings and Fountain-Gould left orders by the local approach (preprint).*

[U] **Y. Utumi** *"On Quotient rings" Osaka Mathematical Journal***8** *(1956), 1-18.*