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QUOTIENT RINGS AND FOUNTAIN–GOULD LEFT ORDERS BY THE LOCAL APPROACH

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Abstract. We study Fountain–Gould left orders in semiprime rings coinciding with their socles by means of local rings at elements.

§1. Introduction

Based on ideas from semigroup theory, Fountain and Gould [3] introduced a notion of order in a ring which need not have an identity and gave [4] a Goldie-like characterization of two-sided orders in semiprime rings with descending chain condition on principal one sided ideals (equivalently, coinciding with their socles). Later Ánh and Márki [1] extended this result to one-sided orders and more recently the same authors developed a general theory of Fountain–Gould left quotient rings [2].

In this paper we study Fountain–Gould left orders in semiprime rings coinciding with their socles. On the one hand, we show that local rings at elements are a useful tool for the study of left quotient rings. On the other hand, as previously noticed by Ánh and Márki in [2], the maximal ring of quotients give us an appropriate framework where to settle the different left quotient rings we investigate (general, Fountain–Gould and classical), although we go further, proving, among other results, that if R is a semiprime left local Goldie ring, then R is a Fountain–Gould left order in RQ, where Q denotes the maximal left quotient ring of R, and that RQ is a semiprime ring which coincides with its socle (Theorem 4.9). Focusing attention on the ring of quotients, we describe some equivalent conditions for a semiprime ring coinciding with its socle Q to be a Fountain–Gould left quotient ring of a subring (Theorem 4.11).

Finally, we would like to stress Theorem 4.6, which provides a more algebraic proof of the fact that semiprimeness and primeness are inherited by Fountain–Gould left orders ([4, Proposition 2.4]).

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Now, we recall the notion of singularity, due to R. E. Johnson for modules (the reader is referred to the books [5, 7] for basic results on ring theory). For a ring R, set

$$\mathcal{Z}_l(R) = \big\{ x \in R \mid \operatorname{lan}(x) \text{ is an essential left ideal of } R \big\},$$

which is an ideal of R (see [5, Corollary 7.4]) called the *left singular ideal* of R. The ring R will be called *left nonsingular* if its left singular ideal $\mathcal{Z}_l(R)$ is zero. *Right nonsingular* rings are defined similarly, while *nonsingular* means that R is both left and right nonsingular.

In [4, Proposition 2.3] the authors proved that every semiprime ring R satisfying the ascending chain condition (acc) on the left annihilators of the form $\operatorname{lan}(x)$, with $x \in R$, is left nonsingular. In fact, we will obtain in Corollary 1.2 that R is nonsingular, since the left and the right singular ideal of a semiprime ring have no pseudo-uniform elements. The proof of the following result is straightforward.

LEMMA 1.1. *For a nonzero element a in a ring R, the following conditions are equivalent:*
(i) $\operatorname{lan}(a) = \operatorname{lan}(ax)$ *for every $x \in R$ such that $ax \neq 0$.*
(ii) $\operatorname{ran}(a) = \operatorname{ran}(xa)$ *for every $x \in R$ such that $xa \neq 0$.*

An element a in a ring R satisfying the equivalent conditions in the previous proposition is said to be *pseudo-uniform*.

COROLLARY 1.2. *Let R be a semiprime ring.*
(i) *$Z_l(R)$ and $Z_r(R)$ have no nonzero pseudo-uniform elements.*
(ii) *If R satisfies the acc on the left annihilators of the form* $\operatorname{lan}(x)$ *with $x \in R$, then it is nonsingular.*

PROOF. (i) Suppose $0 \neq x \in Z_l(R)$ is a pseudo-uniform element. Since R is semiprime, $xrx \neq 0$ for some $r \in R$. Again by semiprimeness of R, $Rxr \neq 0$ and since $\operatorname{lan}(x)$ is an essential left ideal of R, there exists $0 \neq txr \in \operatorname{lan}(x)$, that is, $txrx = 0$, which implies $t \in \operatorname{lan}(xrx) = \operatorname{lan}(x)$ (by Lemma 1.1), a contradiction. The other assertion follows analogously by taking into account Lemma 1.1.

(ii) Suppose $Z_l(R)$ (or $Z_r(R)$) is nonzero and denote it by I. Choose a nonzero element $x \in I$ with $\operatorname{lan}(x)$ maximal in the set $\big\{ \operatorname{lan}(y) : 0 \neq y \in I \big\}$. By maximality of $\operatorname{lan}(x)$ we have $\operatorname{lan}(x) = \operatorname{lan}(xr)$ for every nonzero $xr \in R$, which implies that x is a pseudo-uniform element in I, a contradiction by (i). □

§2. The local ring at an element of a ring

This section is devoted to state the relationship among some properties of a ring and the corresponding ones of its local rings at elements. Bearing in mind this aim, we recall some definitions.

Usually, local rings at elements are presented as follows: for a ring R and an element $a \in R$, the abelian group of R endowed with the *a-homotope product* $x \underset{a}{\cdot} y = xay$ becomes a ring, the *a-homotope ring,* denoted by R^a, which has as an ideal the set $\operatorname{Ker}(a) = \{x \in R \mid axa = 0\}$.

The *local ring* of R *at* a is defined as $R^a / \operatorname{Ker}(a)$ and is denoted by R_a. The product of two elements $\overline{x}, \overline{y} \in R_a$ is denoted by $\overline{x} \cdot \overline{y}$ $(:= \overline{xay})$.

We notice that R_a is isomorphic to the additive subgroup aRa of R equipped with the multiplication $axa \circ aya := axaya$. In particular, if a is an idempotent then R_a is a subring of R, which shows clearly the relevance of the notion and explains its name.

A nonzero left ideal I of a ring R will be called *uniform* if for any nonzero left ideals B and C of R inside I we have $B \cap C \neq 0$. An element $a \in R$ is said to be *l-uniform* if the principal left ideal it generates is uniform. A nonzero ideal I of R will be called *uniform* if the intersection of any two nonzero ideals of R contained in I is nonzero.

Let L be a left ideal of R which does not contain infinite direct sums of nonzero left ideals. By [5, (6.1)] there exists a nonnegative integer n called the *left Goldie* (or *uniform*) *dimension* of L, denoted by $\text{u-dim}_R(L)$ or simply by $\text{u-dim}\,(L)$, such that L contains a direct sum of n nonzero left ideals and any direct sum of nonzero left ideals contained in L has at most n summands. Now a uniform nonzero left ideal is just a nonzero left ideal of left Goldie dimension one. If no such integer n exists (i.e., if L contains an infinite direct sum of nonzero submodules), we write $\text{u-dim}_R(L) = \infty$.

For an element a in a ring R we denote by $(a]$ the principal left ideal of R generated by a.

The *left Goldie* (or *uniform*) *dimension* of an element $a \in R$, denoted by $\text{u-dim}_R(a)$ or simply by $\text{u-dim}\,(a)$, is the left Goldie dimension of $(a]$. If any element of R has finite left Goldie dimension, we will say that R has *finite left local Goldie dimension.*

PROPOSITION 2.1. *Let R be a semiprime ring. Then:*

(i) *All the local rings of R at nonzero elements are semiprime.*

(ii) *R is prime if and only if all the local rings of R at nonzero elements are prime.*

(iii) *If R is simple, then all the local rings of R at nonzero elements of R are simple.*

(iv) *For any element $a \in R$, $\text{u-dim}_R(a) = \text{u-dim}\,(R_a)$.*

(v) *For every $a \in R$, $\overline{x} \in \operatorname{Soc}(R_a)$ if and only if $axa \in \operatorname{Soc}(R)$. Hence R coincides with its socle if and only if R_a is artinian for each $a \in R$.*

(vi) *If R coincides with its socle, then R has finite both left and right local Goldie dimension.*

(vii) *If $\overline{x} \in \mathcal{Z}_l(R_a)$ then $axa \in \mathcal{Z}_l(R)$.*

(viii) *If $a \in \mathcal{Z}_l(R)$ then $\mathcal{Z}_l(R_a) = R_a$.*

(ix) *R is left nonsingular if and only if R_a is left nonsingular for all $a \in R$.*

(x) *A nonzero element $a \in R$ is pseudo-uniform if and only if R_a has no zero divisors.*

Proof. (i) Let a be an element in R and suppose $\overline{x} \in R_a$ such that $\overline{x} \,.\, \overline{y} \,.\, \overline{x} = \overline{0}$ for every $\overline{y} \in R_a$. Then $0 = axaRaxa$. By nondegeneracy of R we have $axa = 0$, that is, $\overline{x} = \overline{0}$.

(ii) Assume that R is prime. Take $a \in R$ and let $\mathcal{I}, \mathcal{J}$ be two ideals of R_a such that $\mathcal{I} \,.\, \mathcal{J} = \overline{0}$. If $\overline{x}$ and $\overline{y}$ were nonzero elements of $\mathcal{I}$ and $\mathcal{J}$ respectively, then for every $r \in R$, $\overline{0} = \overline{x} \,.\, \overline{r} \,.\, \overline{y}$, which implies $0 = (axa)r(aya)$ and since R is prime, $axa = 0$ or $aya = 0$, i.e., $\overline{x} = \overline{0}$ or $\overline{y} = \overline{0}$.

Conversely, let I and J be two nonzero ideals of R of zero product. Choose nonzero elements x and y in I and J respectively. Then the ideals $\overline{I}$ and $\overline{J}$ of R_{x+y} are nonzero because $(x+y)I(x+y) = xIx \neq 0$, and analogously for J. Moreover $\overline{I} \,.\, \overline{J} = \overline{0}$, which contradicts the primeness of R_{x+y}.

(iii) Suppose R simple and let a be a nonzero element of R. Then R_a is a simple ring because if $\overline{0} \neq \overline{x}$, by simplicity of R, $RaxaR = R$ and hence $R_a = R_a \,.\, \overline{x} \,.\, R_a$.

(iv) (1) By semiprimeness of R, for every nonzero left ideal L of R contained in $(a]$ we have $L \cap aRa \neq 0$. (2) Let $\{L_\alpha\}$ be a family of nonzero left ideals of R contained in $(a]$ and whose sum is direct. By (1), $L_\alpha \cap aRa$ is nonzero and it is clear that if we define $\mathcal{L}_\alpha = \{\overline{x} \in R_a \,|\, axa \in L_\alpha\}$, then $\{\mathcal{L}_\alpha\}$ is a family of nonzero left ideals of R_a whose sum is direct, which proves that $\text{u-dim}_R(a) \leqq \text{u-dim}\,(R_a)$.

Now, suppose that $\{\mathcal{L}_i\}_{i=1}^n$ is a direct sum of nonzero left ideals of R_a. Choose $\overline{0} \neq \overline{l}_i \in \mathcal{L}_i$. Since the element $t_i := al_ia$ is nonzero for every $i = 1, \ldots, n$, and using the nondegeneracy of R we have that $\{Rt_i\}_{i=1}^n$ is a family of nonzero left ideals of R contained in Ra. We claim that the sum of these ideals is direct because if $r_1t_1 + \cdots + r_nt_n = 0$ with, for example, $r_1t_1 \neq 0$, by nondegeneracy of R, there exists $s \in R$ such that $asr_1al_1a \neq 0$, that is, $\overline{0} \neq \overline{sr}_1 \,.\, \overline{l}_1 \in \sum_{i=2}^n \overline{R} \,.\, \overline{l}_i \subseteqq \sum_{i=2}^n \mathcal{L}_i$, which is a contradiction.

(v) An additive subgroup K of a ring R is said to be an *inner ideal* of R if $kRk \subseteqq K$ for every $k \in K$. In particular, every generalized bi-ideal of R (an additive subgroup B of $(R, +)$ such that $BRB \subseteqq B$) is an inner ideal (the notion of generalized bi-ideal was introduced by Szász [8]). Hence, by [8, Theorem 6], the socle of a semiprime ring is the sum of all minimal inner ideals of the ring. Since $\overline{x} \,.\, \overline{y} \,.\, \overline{x} = \overline{xayax}$, $\mathcal{I}$ is an inner ideal of R_a if and only if $I = \{axa \,|\, \overline{x} \in \mathcal{I}\}$ is an inner ideal of R, hence $\overline{x} \in \text{Soc}\,(R_a)$ if and only if $axa \in \text{Soc}\,(R)$. Now, if $a \in \text{Soc}\,(R)$ then a has finite Goldie dimension and it is von Neumann regular. This implies that R_a is an artinian ring.

Conversely if R_a is artinian then R_a has bounded length for the chains of inner ideals of the form $\overline{I}$ with I an inner ideal of R contained in aRa, hence for chains of principal inner ideals of the form $\overline{bRb}$ with b in the inner ideal of R generated by a. Thus, $a \in \operatorname{Soc}(R)$.

(vi) By (v) any local ring of R at an element of R is artinian and hence it has finite both left and right Goldie dimension. Then, by (iv), R has finite both left and right local Goldie dimension.

(vii) Take $\overline{x} \in \mathcal{Z}_l(R_a)$ and let L be a nonzero left ideal of R. If $Laxa = 0$ then $L \subseteqq \operatorname{lan}_R(axa)$. Suppose $Laxa \neq 0$. By semiprimeness of R, $(Laxa)^2 \neq 0$, so $\overline{L}$ is a nonzero left ideal of R_a. Since $\operatorname{lan}_{R_a}(\overline{x})$ is an essential left ideal of R_a we can choose an element $y \in L$ such that $\overline{0} \neq \overline{y} \in \operatorname{lan}_{R_a}(\overline{x}) \cap \overline{L}$, that is, $\overline{yax} = \overline{0}$ and we have $0 \neq ay \in L \cap \operatorname{lan}_R(axa)$. In any case $\operatorname{lan}_R(axa) \cap L \neq 0$, which implies that $\operatorname{lan}_R(axa)$ is an essential left ideal of R.

(viii) For $a = 0$ there is nothing to prove. Suppose $a \in \mathcal{Z}_l(R)$ and take $\overline{0} \neq \overline{x} \in R_a$ and a nonzero left ideal $\mathcal{L}$ of R_a. If $\mathcal{L} \centerdot \overline{x} = \overline{0}$ then $\mathcal{L} \subseteqq \operatorname{lan}_{R_a}(\overline{x})$. If $\mathcal{L} \centerdot \overline{x} \neq \overline{0}$, take $\overline{y} \in \mathcal{L}$ such that $\overline{y} \centerdot \overline{x} \neq \overline{0}$, that is, $0 \neq ayaxa$. Since R is semiprime, $Rayaxa \neq 0$ and hence $Rayax$ is a nonzero left ideal of R. Now take $0 \neq rayax \in \operatorname{lan}_R(a)$. Then $0 \neq aRraya$ and $aRrayaxa = 0$, that is, $\overline{0} \neq \overline{Rray} = \overline{Rr} \centerdot \overline{y} \subseteqq \operatorname{lan}_{R_a}(\overline{x}) \cap \mathcal{L}$.

(ix) R left nonsingular implies, by (vii), that for every $a \in R$, R_a is left nonsingular too. Conversely, suppose $\mathcal{Z}_l(R_a) = 0$ for every $a \in R$. By (viii), $a \in \mathcal{Z}_l(R)$ would imply $R_a = \mathcal{Z}_l(R_a) = \overline{0}$, so $aRa = 0$ and hence $a = 0$ because R is semiprime.

(x) Suppose a is pseudo-uniform and let $\overline{x}$ be a nonzero element of R_a. Then $\overline{x} \centerdot \overline{y} = \overline{0}$ for some $\overline{y} \in R_a$ would imply $axaya = 0$. By condition (ii) in Lemma 1.1, $\operatorname{ran}(a) = \operatorname{ran}(axa)$ and hence $aya = 0$, that is, $\overline{y} = \overline{0}$. Similarly we prove that $\overline{y} \centerdot \overline{x} = \overline{0}$ implies $\overline{y} = \overline{0}$.

Conversely, suppose $\operatorname{lan}(a) \subsetneq \operatorname{lan}(ax)$ for some $x \in R$ satisfying $ax \neq 0$. Then there exists an element $r \in R$ such that $ra \neq 0$ and $rax = 0$. By semiprimeness of R, we can find $s, t \in R$ verifying $axsa, atra \neq 0$. Then $\overline{tr}$ and $\overline{xs}$ are nonzero elements of R_a but $\overline{tr} \centerdot \overline{xs} = \overline{traxs} = \overline{0}$, a contradiction since we are supposing that R_a has not zero divisors. □

§3. Left quotient rings

In this section we recall the notion of left quotient ring of a ring, given by Utumi in 1956 (see [9]), and we present some results that relate the structure of R to that of any left quotient ring of it. We also establish the relationship among the various types of rings of quotients.

Let R be a subring of a (not necessarily unital) ring Q. We will say that Q is a *left quotient ring* of R if given $p, q \in Q$, with $p \neq 0$, there exists $r \in R$

such that $rp \neq 0$ and $rq \in R$. Notice that if R is a classical left order in a (unital) ring Q, then Q is a left quotient ring of R.

PROPOSITION 3.1. *Let Q be a ring which is a left quotient ring of a ring R. Then:*

(i) $L \cap R \neq 0$ *for every nonzero left ideal L of Q.*

(ii) *Q is semiprime (prime) if R is semiprime (prime).*

PROPOSITION 3.2. *Let Q be a ring which is a left quotient ring of a ring R. Then:*

(i) *For $X, Y \subseteqq R$ we have* $\operatorname{lan}_R(X) \subseteqq \operatorname{lan}_R(Y)$ *if and only if* $\operatorname{lan}_Q(X) \subseteqq \operatorname{lan}_Q(Y)$.

(ii) $\mathcal{Z}_l(R) = \mathcal{Z}_l(Q) \cap R$.

(iii) *R is left nonsingular if and only if Q is left nonsingular.*

(iv) $\text{u-dim}\,(R) = \text{u-dim}\,(Q)$. *Moreover, for every element $a \in R$,*

$$\text{u-dim}_R(a) = \text{u-dim}_Q(a).$$

(v) *If R is semiprime, then for every nonzero element $a \in R$ the local ring Q_a of Q at a is a left quotient ring of R_a.*

PROOF. (i), (ii), (iii) and (iv) are straightforward.

(v) Let apa and aqa be in aQa with $apa \neq 0$. Apply twice that Q is a left quotient ring of R to find $r \in R$ satisfying $0 \neq rapa$ and $rapa, raq \in R$. Since R is semiprime, there exists $s \in R$ such that $asrapa \neq 0$. Then $\overline{sr} \in R_a$ satisfies $\overline{sr} \cdot \overline{p} = \overline{srap} \neq 0$ and $\overline{sr} \cdot \overline{q} = asraqa \in R_a$. $\square$

Given a subring R of a ring Q, and an element $q \in Q$ we define the set $(R : q) = \{a \in R \mid aq \in R\}$, which, clearly, is a left ideal of R.

A left ideal L of a ring R is *dense* in R if R is a left quotient ring of L.

PROPOSITION 3.3. *Let Q be a left quotient ring of a ring R. Then:*

(i) $(R : q)$ *is a dense left ideal of R for every element $q \in Q$. In particular, $(R : q)$ is an essential left ideal of R.*

If R is semiprime and Q is artinian, then:

(ii) *For every element $s \in R$ we have* $\operatorname{lan}_R(s) = 0$ *if and only if* $s \in \operatorname{Inv}(Q)$.

(iii) *Every essential left ideal L of R is a classical left order in Q.*

PROOF. (i) is [6, Lemma 4.3.2].

(ii) Take $s \in R$ such that $\operatorname{lan}_R(s) = 0$. If $\operatorname{lan}_Q(s) \neq 0$, by Proposition 3.1(i), $0 \neq \operatorname{lan}_Q(s) \cap R = \operatorname{lan}_R(s)$, a contradiction. Hence $\operatorname{lan}_Q(s) = 0$ and by [7, Lemma 1.10], $s \in \operatorname{Inv}(Q)$.

(iii) $\operatorname{Reg}(L) \subseteqq \operatorname{Inv}(Q)$: Let y be in $\operatorname{Reg}(L)$. Then $\operatorname{lan}_R(y) \cap L = \operatorname{lan}_L(y) = 0$ implies, since L is essential, $\operatorname{lan}_R(y) = 0$ and by (ii), $y \in \operatorname{Inv}(Q)$.

Take $q \in Q$. By conditions (i) and (iv) in Proposition 3.2, R satisfies the acc for the left annihilators of the form $\operatorname{lan}(a)$ with $a \in R$ and does not contain infinite direct sums of nonzero left ideals, so we can apply [5, Proposition 11.14 (5)] to $(R:q)\cap L$, which by (i) is an essential left ideal of R, and find $0\neq s\in L\cap(R:q)$ with $\operatorname{lan}_R(s)=0$. By (ii), $s\in \operatorname{Inv}(Q)$, and $sq=t\in R$ implies $q=s^{-1}t$. Now, given $p\in Q$, write $p=ps^{-1}s$, with $s\in L\cap \operatorname{Inv}(Q)$, and $ps^{-1}=u^{-1}v$, with $u\in L$ and $v\in R$ (this is possible as we have proved previously). Then $p=u^{-1}(vs)$, with $u, vs\in L$, which completes the proof. □

COROLLARY 3.4. *Let Q be a (semiprime) artinian ring which is a left quotient ring of a semiprime ring R. Then R is a classical left order in Q.*

Every ring R which is left nonsingular and such that every element $a\in R$ has finite left Goldie dimension will be called a *left local Goldie* ring (equivalently, by [5, (7.5)], R satisfies the ascending chain condition on the left annihilators of the form $\operatorname{lan}(a)$, with $a\in R$, and every element $a\in R$ has finite left Goldie dimension). If additionally R has finite left (global) dimension, then R will be called a *left Goldie* ring.

PROPOSITION 3.5. *Let Q be a (semiprime) ring coinciding with its socle which is a left quotient ring of a semiprime ring R. Then:*

(i) *R is left local Goldie.*

(ii) *For each nonzero element $a\in R$ the local ring R_a of R at a is a classical left order in the semiprime artinian ring Q_a.*

Moreover,

(1) *R is prime if and only if Q is simple.*

(2) *R has finite left Goldie dimension if and only if Q is artinian.*

PROOF. (i) By [5, (7.13)], Q is nonsingular and by Proposition 3.2(iii), R is left nonsingular. Proposition 2.1(vi) says that Q has finite left local Goldie dimension. Hence and using condition (iv) in Proposition 3.2 we obtain that R has finite left local Goldie dimension.

(ii) Let $a\in R$. By Proposition 2.1(i), R_a and Q_a are semiprime and by Proposition 2.1(v), Q_a is artinian. Finally, since Q_a is a left quotient ring for R_a (by Proposition 3.2(v)) we can apply Corollary 3.4 to obtain that R_a is a classical left order in Q_a.

(1) If R is prime then Q is simple by Proposition 3.1(ii) and by the structure of the socle.

Conversely, Q simple implies (by condition (iii) in Proposition 2.1) Q_a simple for every nonzero $a\in R$. By (ii) and the classical Goldie Theorem, R_a is a prime ring, which implies (Proposition 2.1(ii)) that R is prime.

(2) Suppose Q artinian. Since Q satisfies the acc on all left ideals, from condition (iv) in Proposition 3.2 it follows that R has finite left Goldie dimension. Conversely, if Q were not artinian it would have infinite left Goldie dimension, hence R would have infinite left Goldie dimension too (applying again Proposition 3.2(iv)). □

§4. On Fountain–Gould left orders in rings

Let a be an element of a ring R. We say that b in R is the *group inverse* of a if the following conditions hold: $aba = a$, $bab = b$, $ab = ba$.

It is easy to see that a has a group inverse b in R if and only if there exists a unique idempotent e ($e = ab$) in R such that a is invertible in the ring eRe (with inverse b), hence the group inverse is unique and a is said to be *locally invertible.* Denote by $a^\sharp$ the group inverse of a.

An element $a \in R$ is *left square cancellable* if $a^2x = a^2y$ implies $ax = ay$ for $x, y \in R \cup \{1\}$ (for $x = 1$ or $y = 1$ this means that $a^2 = a^2y$ or $a^2x = a^2$ implies $a = ay$ or $ax = a$). *Right square cancellable* elements are defined analogously, and *square cancellable* means both left and right square cancellable. We denote by $\mathcal{S}_l(R)$, $\mathcal{S}_r(R)$ and $\mathcal{S}(R)$ the sets of all left, right and two-sided square cancellable elements of R, respectively.

LEMMA 4.1. *If Q is a left quotient ring of a ring R, then $\mathcal{S}_r(R) \subseteqq \mathcal{S}_r(Q)$.*

PROOF. Let s be in $\mathcal{S}_r(R)$ and suppose $p, q \in Q \cup \{1\}$ such that $ps^2 = qs^2$. If $ps - qs \neq 0$ then there would exist $a \in R$ satisfying $a(ps - qs) \neq 0$ and $ap, aq \in R$. Since $s \in \mathcal{S}_r(R)$, $(ap - aq)s^2 = 0$ with $ap, aq \in R$ implies $(ap - aq)s = 0$, a contradiction. In consequence $\mathcal{S}_r(R) \subseteqq \mathcal{S}_r(Q)$. $\square$

We recall that a subring R is a *Fountain–Gould left order* in a ring Q if:

(1) every element of $\mathcal{S}(R)$ has a group inverse in Q and

(2) every element $q \in Q$ can be written in the form $q = a^\sharp x$, where $a \in \mathcal{S}(R)$ and $x \in R$.

We also say that Q is a *ring of Fountain–Gould left quotients* of R. In a similar way *Fountain–Gould right order* and *ring of Fountain–Gould right quotients* are defined. If R is both a Fountain–Gould left and right order in Q, then we say that R is a *Fountain–Gould order* in Q and that Q is a *Fountain–Gould ring of quotients* of R. When condition (2) is satisfied we speak about *weak Fountain–Gould left order.*

Clearly, if R is a classical left order in a unital ring Q, then R is a weak Fountain–Gould left order in Q. The converse is not true in general, as it was shown by Fountain and Gould in [3, Example 3.1]. It is not difficult to see ([3, Lemma 2.1]) that if R is a weak Fountain–Gould left order in a ring Q, then every element $q \in Q$ can be written as $q = a^\sharp b$ with $a \in \mathcal{S}(R)$, $b \in R$ and $aa^\sharp b = b$. This result will often be used without mentioning it.

The following theorem was stated by Ánh and Márki for Fountain–Gould left orders, although the same proof is valid for weak Fountain–Gould left orders.

THEOREM 4.2 (common denominator property) [2, Theorem 5]. *Let R be a weak Fountain–Gould left order in Q. Then for any $p, q \in Q$ there exist $u \in \mathcal{S}(R)$, $v, w \in R$ such that $p = u^\sharp v$, $q = u^\sharp w$.*

LEMMA 4.3 [2, Theorem 1]. *Let R be a subring of a ring Q. If Q is a weak Fountain–Gould left quotient ring of R, then Q is a left quotient ring of R.*

Propositions 4.4 and 4.5 show the relationship between classical left quotient rings and (weak) Fountain–Gould left quotient rings. Later, via the local rings at elements, these results will allow us to show how close are Fountain–Gould left orders and classical left orders.

PROPOSITION 4.4. *Let R be a Fountain–Gould left order in a ring Q.*

(i) *If Q is unital then R is a classical left order in Q.*

(ii) *If R has an element which is not a right zero divisor (in R), then R is a classical left order in Q.*

(iii) *If Q has an element which is not a right zero divisor (in Q), then R is a classical left order in Q.*

PROOF. (i) is [3, Theorem 3.4], (ii) is [2, Proposition 9] and (iii) is a trivial consequence of (ii) because if $a^\sharp b$ is not a right zero divisor in Q, then a is not a right zero divisor in R. $\square$

PROPOSITION 4.5 [3, Theorems 3.4 and 3.11 and Proposition 2.10]. *Let R be a subring of a ring Q and suppose Q semiprime and coinciding with its socle.*

(i) *If R is a weak Fountain–Gould left order in Q then R is a Fountain–Gould left order in Q.*

If Q is also artinian then

(ii) *R is a classical left order in Q if and only if R is a Fountain–Gould left order in Q.*

We would like to stress the following result. It was stated in [4, Proposition 2.4] and the proof given there used the geometric properties of the ring Q. Now we give a more algebraic proof.

THEOREM 4.6. *Let R be a ring which is a Fountain–Gould left order in a semiprime (prime) ring Q which coincides with its socle. Then R is semiprime (prime).*

PROOF. Let a be a nonzero element of R. Since Q is semiprime, there is an element $q \in Q$ satisfying $aqa \neq 0$. Apply that R is a Fountain–Gould left order in Q to get $u \in \mathcal{S}(R)$, $v \in R$ such that $q = u^\sharp v$, with $v = uu^\sharp v$. Then $0 \neq aqa = au^\sharp va$. By semiprimeness of Q, $au^\sharp vaQ \neq 0$ and since $Q = \mathrm{Soc}\,(Q)$, there exists t contained in a minimal right ideal of Q, verifying $au^\sharp vat \neq 0$.

Consider the set $\Gamma = \{u^n u^\sharp vatQ : n = 1, 2, \ldots\}$, of nonzero ($u^\sharp vat \neq 0$ and Q is semiprime) right ideals of Q. Since u-dim$\,(Q_Q) < \infty$ (by Proposition 2.1(vi)) the sum of the elements of Γ cannot be direct, hence there exist elements $\alpha_1, \ldots, \alpha_m \in Q$ such that $\sum_{k=1}^m u^k u^\sharp vat\alpha_k = 0$, with some summand nonzero. We may suppose $uu^\sharp vat\alpha_1 \neq 0$. This implies $u^\sharp vat\alpha_1 \neq 0$

and

$$u^{\sharp}vat\alpha_1 + uu^{\sharp}vat\alpha_2 + \cdots + u^{m-1}u^{\sharp}vat\alpha_m = 0. \tag{4.1}$$

On the other hand, since t belongs to a minimal right ideal of Q then $t \in tQ = t\alpha_1 Q$ and so $t = t\alpha_1\alpha$ for some $\alpha \in Q$. Multiplying (4.1) by a on the left side and by α on the right side we obtain: $au^{\sharp}vat\alpha_1\alpha + auu^{\sharp}vat\alpha_2\alpha + \cdots + au^{m-1}u^{\sharp}vat\alpha_m\alpha = 0$, i.e., $au^{\sharp}vat + auu^{\sharp}vat\alpha_2\alpha + \cdots + au^{m-1}u^{\sharp}vat\alpha_m\alpha = 0$, with $au^{\sharp}vat \neq 0$. Then $0 \neq -au^{\sharp}vat = auu^{\sharp}vat\alpha_2\alpha + \cdots + au^{m-1}u^{\sharp}vat\alpha_m\alpha$. Since $uu^{\sharp}v = v$, $0 \neq -au^{\sharp}vat = avat\alpha_2\alpha + \cdots + au^{m-2}vat\alpha_m\alpha$, and some of the summands of the right side is nonzero, that is, $au^k vat\alpha_{k+2}\alpha \neq 0$ for some $k \in \mathbf{N}$ and, consequently, $0 \neq au^k va \in aRa$, which proves that R is semiprime.

If Q were prime, the primeness of R would follow analogously. □

PROPOSITION 4.7. *Let R be a ring which is a Fountain–Gould left order in a semiprime ring Q equal to its socle. Then, for every $a \in R$, the local ring R_a of R at a is a classical left order in the semiprime artinian ring Q_a.*

PROOF. We notice that R is semiprime, by Theorem 4.6, and that Q is a left quotient ring of R by Lemma 4.3. Apply condition (ii) in Proposition 3.5 to obtain the result. □

It is well known (Goldie's Theorem) that a ring R is a classical left order in a semiprime artinian ring Q if and only if satisfies the following condition: A left ideal of R is essential if and only if it contains a regular element. The following proposition plays an analogous role for Fountain–Gould left orders in semiprime rings which coincide with their socle.

PROPOSITION 4.8. *Let R be a semiprime left local Goldie ring. Then, given any finite subset X of R and any essential left ideal L of R, there exists $s \in \mathcal{S}(R) \cap L$ such that $\mathrm{lan}_R(s) = \mathrm{lan}_R(X)$.*

PROOF. (This proof follows partially the proof of [1, Proposition 7].) Let K be a nonzero left ideal of R which is maximal with respect to the property $K \cap \mathrm{lan}\,(X) = 0$. Since L is essential, K can be taken contained in L (otherwise $K \cap L$ would suit). It is not difficult to see that $R/\mathrm{lan}\,(X)$ can be embedded (as a left R-module) into the finite direct sum $\oplus_{x\in X} Rx$. Every left ideal Rx has finite Goldie dimension, so has K. By [4, Theorem 3.15], there exists $s \in \mathcal{S}(R) \cap K$ such that Rs is essential in K and $\mathrm{lan}\,(X) \subseteqq \mathrm{lan}\,(s)$. Here $\mathrm{lan}\,(s) \cap Rs = 0$ and K is maximal with respect to $K \cap \mathrm{lan}\,(X) = 0$, therefore $\mathrm{lan}\,(X)$ must be essential in $\mathrm{lan}\,(s)$, which implies $\mathrm{lan}\,(s) = \mathrm{lan}\,(X)$. □

THEOREM 4.9. *Let R be a semiprime left local Goldie ring and put $Q = Q^l_{\max}(R)$. Then:*

(i) *R is a Fountain–Gould left order in the semiprime ring RQ which coincides with its socle.*

(ii) *For every finite subset* $Y \subseteqq RQ$, *there exists* $s \in \mathcal{S}(R)$ *such that* $Y \subseteqq sQs = sRQs$ *and* R_s *is a classical left order in* Q_s.

PROOF. First we notice that under left nonsingularity, a ring has a maximal left quotient ring which is von Neumann regular (Johnson's Theorem, [5, (13.36)]).

(i) (1) $R \subseteqq \operatorname{Soc}(Q)$. Let a be a nonzero element of R. Apply conditions (i), (iv) and (ix) in Proposition 2.1 to obtain that R_a is a semiprime left Goldie ring. On the other hand, Q_a is a semiprime (Proposition 3.1(ii)) left Goldie ring (Proposition 3.2(iii) and (iv)). By the classical Goldie Theorem, Q_a is a classical left order in a semisimple artinian ring T. Since Q is von Neumann regular, the ring Q_a is unital and von Neumann regular, therefore, $\operatorname{Reg}(Q_a) = \operatorname{Inv}(Q_a)$.

Since T is generated by Q_a and the inverses of the elements of $\operatorname{Reg}(Q_a)$, we have $T = Q_a$. Finally, T $(= Q_a)$ artinian implies, by Proposition 2.1(v), $a \in \operatorname{Soc}(Q)$.

(2) Now we prove $R \subseteqq RQ$ and RQ semiprime and coinciding with its socle. By von Neumann regularity of Q, $R \subseteqq RQ$. We notice that RQ is a left quotient ring of R, which implies, by Proposition 3.1(ii), that RQ is a semiprime ring. Therefore, and since RQ is a right ideal of a semiprime ring which coincides with its socle, RQ coincides with its socle.

(3) We see R is a Fountain–Gould left order in RQ. Take $q = \sum_{i=1}^{n} r_i q_i$ for some $r_i \in R$, $q_i \in Q$, and denote $X = \{r_1, \dots, r_n\}$. Then, by Proposition 4.8, there exists $s \in \mathcal{S}(R) \cap (R : q)$ such that $\operatorname{lan}_R(s) = \operatorname{lan}_R(X)$. By Lemma 4.1 and [3, Proposition 2.6], s has a group inverse $s^\sharp \in RQ$. Now we prove $q = s^\sharp sq$. We notice that $\operatorname{lan}_R(s) = \operatorname{lan}_R(X)$ implies (by Proposition 3.2(i)) $\operatorname{lan}_{RQ}(s) = \operatorname{lan}_{RQ}(X)$. Hence $\operatorname{lan}_{RQ}(s) \subseteqq \operatorname{lan}_{RQ}(q)$. Since for every $p \in RQ$, $pss^\sharp - p \in \operatorname{lan}_{RQ}(s)$ we have $0 = (pss^\sharp - p)q = p(ss^\sharp q - q)$ and by semiprimeness of RQ, $ss^\sharp q = q$. Finally, $q = s^\sharp(sq)$ with $sq \in R$ shows that R is a Fountain–Gould left order in RQ.

(ii) Let $Y = \{q_1, \dots, q_n\}$, write $y_i = a_i^\sharp b_i$ with $a_i \in \mathcal{S}(R)$ and $b_i \in R$ and define $X = \{a_1, \dots, a_n, b_1, \dots, b_n\}$. By [1, Proposition 7] there exist $s \in \mathcal{S}(R)$ and a left ideal L of R such that $\{s\} \cup X \subseteqq L$ and Rs is essential in L. Name $B = L \cap \operatorname{ran}_R\big(\operatorname{lan}_R(s)\big)$. We have that $X \subseteqq B$. Moreover $B \cap \operatorname{lan}_R(s) \subseteqq \operatorname{ran}_R\big(\operatorname{lan}_R(s)\big) \cap \operatorname{lan}_R(s) = 0$ and $B \cap \operatorname{ran}_R(s) \subseteqq \operatorname{ran}_R\big(\operatorname{lan}_R(s)\big) \cap \operatorname{ran}_R(s) = 0$ by [1, Proposition 6]. Hence s is not a zero divisor in B.

(1) $B \subseteqq \operatorname{ran}_R\big(\operatorname{lan}_R(s)\big) \subseteqq \operatorname{ran}_Q\big(\operatorname{lan}_Q(s)\big) = sQ$. We prove the second inclusion:

Take $x \in \operatorname{ran}_R\big(\operatorname{lan}_R(s)\big)$. If $x \notin \operatorname{ran}_Q\big(\operatorname{lan}_Q(s)\big)$, there would exist $q \in \operatorname{lan}_Q(s)$ such that $qx \neq 0$. Since Q is a left quotient ring of R, there exists $a \in R$ such that $aq \in R$ and $aqx \neq 0$.

Since $0 = qs$, we have $aq \in \operatorname{lan}_R(s)$, hence $aqx = 0$, which is a contradiction.

(2) $B \subseteqq Qs$. We notice that by Lemma 4.1 and [3, Proposition 2.6], s has a group inverse in Q (since the socle of a semiprime ring satisfies the dcc on principal left ideals).

Let a be in B. By Proposition 3.5(ii), R_s is a classical left order in Q_s, so, given the element $\overline{s^\sharp a s s^{\sharp 2}} \in Q_s$, there exist $\overline{u} \in \mathcal{S}(R_s)$, $\overline{v} \in R_s$ such that, if $\overline{w}$ denotes the inverse of $\overline{u}$ in Q_s, $\overline{s^\sharp a s s^{\sharp 2}} = \overline{w} \centerdot \overline{v}$, which implies $\overline{u} \centerdot \overline{s^\sharp a s s^{\sharp 2}} = \overline{v}$, that is, $suass^\sharp = suss^\sharp ass^{\sharp 2}s = svs$, and if we multiply on the right side by s, $suas = svs^2$. Since the elements sua and svs are in B and s is not a zero divisor in B, $sua = svs$, so $a = swsvs \in Qs$. □

The following theorem is a Goldie-like characterization of Fountain–Gould left orders in semiprime rings coinciding with their socles. Conditions (i), (ii) and (iii) were established by Ánh and Márki in [1, Theorem 1] and were proved by using different techniques. In our proof we bring out the role played by the maximal ring of left quotients and by the local rings at elements (in particular their use to apply the well-known Goldie Theorem). In Theorem 4.12 we pay attention to the ring Q. We think this is the first time (at least in the literature we know) this kind of result is established.

THEOREM 4.10. *For a ring R the following conditions are equivalent:*

(i) *R is a Fountain–Gould left order in a semiprime ring which coincides with its socle,*

(ii) *R is semiprime, the set $\{\operatorname{lan}(a) : a \in R\}$ satisfies the maximum condition and R has finite left local Goldie dimension,*

(iii) *R is a semiprime left local Goldie ring,*

(iv) *R is semiprime and for each $a \in R$ the local ring R_a is semiprime and left Goldie,*

(v) *R is semiprime and has a left quotient ring Q which is semiprime and coincides with its socle.*

PROOF. (i) $\Rightarrow$ (v) follows by Theorem 4.6 and by Lemma 4.3.

(v) $\Rightarrow$ (ii). Since Q satisfies the acc on $\operatorname{lan}_Q(x)$ with $x \in Q$, by Proposition 3.2(i) the set $\{\operatorname{lan}_R(a) : a \in R\}$ satisfies the maximum condition. On the other hand, R has finite left local Goldie dimension by Proposition 3.2(iv).

(ii) $\Rightarrow$ (iii) follows by Corollary 1.2(ii).

(iii) $\Leftrightarrow$ (iv) follows by Proposition 2.1(i), (iv) and (ix).

(iii) $\Rightarrow$ (i) follows by Theorem 4.9(i). □

We notice that condition (v) in the following theorem is a generalization of the contents of condition (4) in Theorem 1 as well as Propositions 9 and 10 from [1]. In fact, if a is a square cancellable element in R, then R_a is isomorphic to a^2Ra^2 ($\overline{a} \mapsto a^2xa^2$ provides a ring isomorphism) and by condition (v) in Theorem 4.12, a^2Ra^2 is a classical left order in the semisimple artinian ring $a^2Qa^2 = a^\sharp aQa^\sharp a$.

THEOREM 4.11. *Let R be a subring of a semiprime ring Q which coincides with its socle. The following conditions are equivalent:*

(i) *R is a weak Fountain–Gould left order in Q,*

(ii) *R is a Fountain–Gould left order in Q,*

(iii) *$Q = RQR$ and for every nonzero element $a \in R$ we have that R_a is a classical left order in the semisimple artinian ring Q_a,*

(iv) *R is semiprime, $Q = RQ$ and Q is a left quotient ring of R,*

(v) *for every finite subset Y of Q there exists an element $a \in \mathcal{S}(R)$ such that $Y \subseteq aQa$ and R_a is a classical left order in the semisimple artinian ring Q_a.*

PROOF. (ii) ⇒ (i) is trivial and (i) ⇒ (ii) follows from Proposition 4.5(i).

(ii) ⇒ (iii). By Proposition 4.7, for each nonzero $a \in R$, R_a is a classical left order in the semisimple artinian ring Q_a. Now, if $q \in Q$, write $q = a^\sharp b$ with $a \in \mathcal{S}(R)$, $b \in R$. Then $q = a(a^\sharp)^2 b \in RQR$.

(iii) ⇒ (iv). Since for every nonzero element $a \in R$, R_a is a classical left order in Q_a, then $R_a \neq 0$, which implies R semiprime. In what follows we prove that Q is a left quotient ring of R.

(1) Let aqc be a nonzero element, with $a, c \in R$, $q \in Q$. Then the following is an essential left ideal of R_a: $\Lambda_{aqc} := \left\{ \overline{b} \in R_a \mid abaqc \in R \right\}$.

Clearly, Λ_{aqc} is a left ideal of R_a. Now, let $\mathcal{I}$ be a nonzero left ideal of R_a. If $ayaqc = 0$ for every $\overline{y} \in \mathcal{I}$, then $\mathcal{I} \subseteq \Lambda_{aqc}$. Let $\overline{y} \in \mathcal{I}$ be such that $ayaqc \neq 0$. Since Q is semiprime, $cpayaqc \neq 0$ for some $p \in Q$. Apply that R_c is a classical left order in Q_c and take $\widetilde{u} \in \mathrm{Reg}\,(R_c)$, $\widetilde{v} \in R_c$ satisfying $\widetilde{payaq} = \widetilde{u}^{-1} \centerdot \widetilde{v}$. Then $\widetilde{0} \neq \widetilde{v} = \widetilde{u} \centerdot \widetilde{payaq} \in R_c$ implies $0 \neq cvc = cucpayaqc$.

Now we will show $aRcucpayaqc \neq 0$.

First we have $Q = \oplus Q_\alpha$ with the Q_α's as simple ideals of Q coinciding with their socles. Denote by $\pi_\alpha : Q \to Q_\alpha$ the canonical projection. For every $x \in R$, π_α induces a ring epimorphism $\mu_\alpha : Q_x \to Q_{\pi_\alpha(x)} = (Q_\alpha)_{\pi_\alpha(x)}$. Since R_x is a classical left order in Q_x, for every α such that $\pi_\alpha(x) \neq 0$, it is easy to see that $\pi_\alpha(R)_{\pi_\alpha(x)}$ is a classical left order in $(Q_\alpha)_{\pi_\alpha(x)}$. In particular $\pi_\alpha(R)_{\pi_\alpha(x)} \neq 0$, which implies that $\pi_\alpha(R)$ is a semiprime ring. Since Q_α is simple, by Proposition 2.1(iii), $(Q_\alpha)_{\pi_\alpha(x)}$ is a simple ring, and by the classical Goldie Theorem, $\pi_\alpha(R)_{\pi_\alpha(x)}$ is a prime ring, which implies (Proposition 2.1(ii)) that $\pi_\alpha(R)$ is prime. Now, let β be such that $0 \neq \pi_\beta(cvc) = \pi_\beta(cucpayaqc)$. Then $\pi_\beta(a) \neq 0$ and there exists $x \in R$ such that $0 \neq \pi_\beta(a)\pi_\beta(x)\pi_\beta(cvc) = \pi_\beta(axcvc)$. In particular $axcvc \neq 0$.

Let $\overline{r}$ and $\overline{s}$ be in $\mathrm{Reg}\,(R_a)$ and R_a, respectively, such that $\overline{xcucp} = \overline{r}^{-1} \centerdot \overline{s}$. Then $0 \neq araxcucpayaqc \in araxcRc$ and $\overline{raxcucpay} = \overline{s} \centerdot \overline{y}$ is a nonzero element in $\Lambda_{aqc} \cap \mathcal{I}$.

(2) Q is a left quotient ring of R. Let p, q be in Q with $p \neq 0$. Since Q is semiprime, $Qp \neq 0$, and since $Q = RQR$, there exists $a \in R$ such that $ap \neq 0$. Apply that $p, q \in Q = RQR$ and take $p_i, q_j \in Q$, $c_i, d_j \in R$ such that $ap = \sum_{i=1}^m ap_i c_i$ and $aq = \sum_{i=1}^n aq_j d_j$ for some $m, n \in \mathbf{N}$. By (1), $\Lambda_{ap_i c_i}$

and $\Lambda_{aq_jd_j}$ are essential left ideals of R_a and, consequently, $\Lambda = (\cap_{i=1}^m \Lambda_{ap_ic_i}) \cap (\cap_{j=1}^n \Lambda_{aq_jd_j})$ is an essential left ideal of R_a. By the classical Goldie Theorem, there exists $\overline{u} \in \mathrm{Reg}\,(R_a) \cap \Lambda$. Then $auap = \sum_{i=1}^m auap_ic_i \in R$ and $auaq = \sum_{j=1}^n auaq_jd_j \in R$. Moreover $auap \neq 0$ because $ap \neq 0$ and $\overline{u} \in \mathrm{Reg}\,(R_a)$.

(iv) $\Rightarrow$ (v). Consider $Y \subsetneqq Q$. By Proposition 3.2(iii) and (iv), R is a left local Goldie ring. Therefore Theorem 4.9 applies to obtain that there exists $s \in \mathcal{S}(R)$ such that $Y \subsetneqq sQs$. In particular for every $y \in Y$, $y = ss^\sharp ys^\sharp s \in sQs$ since by Lemma 4.1 and [3, Proposition 2.6], every element of $\mathcal{S}(R)$ has a group inverse in Q. Finally, by Proposition 3.5(ii), R_s is a classical left order in Q_s.

(v) $\Rightarrow$ (i). (1) Q is a left quotient ring of R. Indeed, let p and q be in Q with $p \neq 0$. By the hypothesis, there exist $a \in \mathcal{S}(R)$, $p_1, q_1 \in Q$ such that $p = ap_1a$, $q = aq_1a$ and R_a is a classical left order in Q_a. By the common denominator property, there exist $\overline{u} \in \mathrm{Reg}\,(R_a)$, $\overline{v}, \overline{w} \in R_a$ such that $\overline{p_1} = \overline{u}^{-1} \centerdot \overline{v}$ and $\overline{q}_1 = \overline{u}^{-1} \centerdot \overline{w}$, that is, $\overline{u} \centerdot \overline{p_1} = \overline{v}$ and $\overline{u} \centerdot \overline{q_1} = \overline{w}$. Equivalently, $\overline{uap_1} = \overline{v} \neq 0$ and $\overline{uaq_1} = \overline{w}$, which implies $aup = auap_1a \neq 0$ and $auq = auaq_1a = awa \in R$.

(2) Take $q \in Q$. By hypothesis there exists $a \in \mathcal{S}(R)$ such that $q \in aQa$ and R_a is a classical left order in the semisimple artinian ring Q_a. By (1) we can apply Lemma 4.1 and [3, Proposition 2.6] to obtain that a has a group inverse in Q. Write $q = apa$, $\overline{a^2p} = \overline{r}^{-1} \centerdot \overline{s}$ for $\overline{r}, \overline{s} \in R_a$ and $\overline{r}^{-1} = \overline{t} \in Q_a$. We claim that $a^\sharp ta^\sharp = (a^2ra^2)^\sharp$ and hence $a^3pa = atasa$ implies $q = apa = a^\sharp ta^\sharp a^2sa = (a^2ra^2)^\sharp a^2sa$, which proves our claim. $\square$

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