Acta Math. Hungar.
103 (3) (2004), 177–196.

LEFT, RIGHT, AND INNER SOCLES OF ASSOCIATIVE SYSTEMS

J. A. ANQUELA[1], T. CORTÉS[1] (Oviedo), M. GÓMEZ-LOZANO[2]
and M. SILES-MOLINA[2] (Málaga)

Abstract. We investigate the basic properties of the different socles that can be considered in not necessarily semiprime associative systems. Among other things, we show that the socle defined as the sum of minimal (or minimal and trivial) inner ideals is always an ideal. When trivial inner ideals are included, this inner socle contains the socles defined in terms of minimal left or right ideals.

Introduction

The socle of an associative algebra is a widely present notion in the mathematical literature (see [4]; [10, §1.1], [11, §IV.3], [17, §7.1], for example). Recent papers on associative pairs and triple systems make use of pair and triple versions of the algebra socle (cf. [3, 5, 6, 8]). Surprisingly, in some of these papers, the definition of the socle [5] is given in terms of inner ideals rather than one-sided ideals. The idea comes from the theory of Jordan systems and the definition of (Jordan) socle of Fernández-López, García-Rus, and Sánchez-Campos [7], and Loos [13]. Most of the time, only semiprime associative systems are considered and, under this restriction, the equality between the different definitions of the socle is part of the mathematical folklore. However, for not necessarily semiprime systems, it is not clear whether the same coincidence holds, or if properties of a particular version of the socle apply to other versions.

Our aim is to study the different versions of associative socles and their relations for arbitrary, not necessarily semiprime, associative systems. For an associative system A, we will consider $\mathrm{Soc_l}' A$, $\mathrm{Soc_r}' A$ and $\mathrm{Soc_{in}}' A$, defined

[1]Partially supported by the Ministerio de Ciencia y Tecnología and Fondos FEDER, BFM2001-1938-C02-02.

[2]Partially supported by the Ministerio de Ciencia y Tecnología and Fondos FEDER, BFM2001-1938-C02-01, and the Junta de Andalucía, FQM 264.

Key words and phrases: associative algebra, triple system, and pair, left, right, and inner ideal, socle.

2000 Mathematics Subject Classification: 16D25, 16N60.

as the sums of all minimal left, right, and inner ideals of A, respectively, and also the left, right, and inner socles $\mathrm{Soc_l}\, A$, $\mathrm{Soc_r}\, A$ and $\mathrm{Soc_{in}}\, A$, where the sums of all trivial left, right, and inner ideals, respectively are included. In our definition we follow the idea of Loos in [13], where a satisfactory notion of socle is given for Jordan systems over an arbitrary ring of scalars.

After the preliminaries, containing basic definitions, notation and properties of associative systems, in Section 1 we study the relations between minimal left, right, and inner ideals. Section 2 is devoted to studying inner socles. Following [13], structural transformations allow us to obtain that $\mathrm{Soc_{in}}\, A$ is always an ideal of A. The special properties of one-sided multiplications as structural transformations, are also used to establish that $\mathrm{Soc_{in}}'\, A$ is also an ideal. In Section 3, we study one-sided socles. Applying elementary arguments of the theory of modules, we prove that $\mathrm{Soc_l}'\, A$, $\mathrm{Soc_r}'\, A$, $\mathrm{Soc_l}\, A$, and $\mathrm{Soc_r}\, A$ are left and right ideals of A, but not necessarily ideals in the cases of pairs and triple systems. Using the results obtained in the first section, we show that $\mathrm{Soc_l}\, A$ and $\mathrm{Soc_r}\, A$ are always contained in $\mathrm{Soc_{in}}\, A$, while the corresponding assertion for $\mathrm{Soc_l}'\, A$, $\mathrm{Soc_r}'\, A$ and $\mathrm{Soc_{in}}'\, A$ is false. When A is semiprime we prove that the six notions coincide. Finally, the fourth section deals with the interaction between socles and direct sums and the natural functors relating the categories of algebras, pairs and triple systems.

0. Preliminaries

0.1. We will deal with associative systems (algebras, pairs, and triple systems) over an arbitrary ring of scalars Φ. Recall that an *associative pair* over Φ is a pair of Φ-modules (A^+, A^-) together with a pair of trilinear maps

$$\langle\ ,\ ,\ \rangle^\sigma : A^\sigma \times A^{-\sigma} \times A^\sigma \to A^\sigma, \qquad \sigma = \pm,$$

satisfying

$$\big\langle \langle x,y,z\rangle^\sigma, u, v\big\rangle^\sigma = \big\langle x, \langle y,z,u\rangle^{-\sigma}, v\big\rangle^\sigma = \big\langle x, y, \langle z,u,v\rangle^\sigma\big\rangle^\sigma,$$

for any $x,z,v \in A^\sigma$, $y,u \in A^{-\sigma}$, $\sigma = \pm$. If A is an associative pair, it is clear that $A^{\mathrm{ex}} = (A^-, A^+)$, with obvious products, is an associative pair too. We can also consider the associative pair $A^{\mathrm{op}} = (A^+, A^-)$ obtained by reversing the products of A ($\langle x,y,z\rangle^\sigma_{\mathrm{op}} = \langle z,y,x\rangle^\sigma$).

Similarly, an *associative triple system* A over Φ is a Φ-module equipped with a trilinear map

$$\langle\ ,\ ,\ \rangle : A \times A \times A \to A,$$

satisfying

$$\langle \langle x,y,z\rangle ,u,v\rangle = \langle x,\langle y,z,u\rangle ,v\rangle = \langle x,y,\langle z,u,v\rangle \rangle ,$$

for any $x,y,z,u,v\in A$. As for pairs, one can consider the opposite triple system A^{op} of A. Due to associativity, there is no risk of ambiguity when deleting the brackets "$\langle\rangle$", thus, the products above will be usually denoted by juxtaposition, just like in the associative algebra case.

0.2. An associative algebra A gives rise to the associative triple system A_T by simply restricting to odd length products. By doubling any associative triple system A one obtains the *double associative pair* $V(A) = (A,A)$ with obvious products. From an associative pair $A=(A^+,A^-)$ one can get a (*polarized*) associative triple system $T(A)=A^+\oplus A^-$ by defining $(x^+\oplus x^-)(y^+\oplus y^-)(z^+\oplus z^-)=x^+y^-z^+\oplus x^-y^+z^-$.

0.3. Given an associative algebra, pair, or triple system A, $A^{(+)}$ denotes its *symmetrization:* Over the Φ-module A we consider the Jordan products U_xy, Q_xy or P_xy equal to the associative products xyx and, in the algebra case, the Jordan squares x^2, which are just the associative squares xx [1, 2.4, 5.1], [14, 0.7].

0.4. The wellknown notions of left and right ideals of an associative algebra have the following analogues for pairs and triple systems: Given an associative pair A, a Φ-submodule I of A^σ is called a *left* or *right ideal of* A if $A^\sigma A^{-\sigma}I\subseteqq I$ or $IA^{-\sigma}A^\sigma\subseteqq I$, respectively. An *ideal* I *of* A is a pair of Φ-submodules $I=(I^+,I^-)$, $I^\sigma\subseteqq A^\sigma$, $\sigma=\pm$, such that I^+ and I^- are both left and right ideals of A and $A^\sigma I^{-\sigma}A^\sigma\subseteqq I^\sigma$, $\sigma=\pm$. For an associative triple system A, left and right ideals of A are simply those of the pair $V(A)$, while an ideal I of A is a left and right ideal also satisfying $AIA\subseteqq I$, i.e., a Φ-submodule I of A such that (I,I) is an ideal of $V(A)$.

Notice that, if I is a left or right ideal of an associative algebra A, then it is a left or right ideal, respectively, of the associative triple system A_T. Similarly, an ideal of A is always an ideal of A_T.

0.5. For an associative algebra A, a Φ-submodule I will be called an *inner ideal of* A if it is an inner ideal of $A^{(+)}$, i.e., if it satisfies $x\widehat{A}x\subseteqq I$ for any $x\in I$, where $\widehat{A}$ denotes the usual unitization of A, equivalently, $x^2\in I$, $xAx\subseteqq I$ for any $x\in I$ (cf. [14, 0.13]).

Analogously, given an associative pair A, a Φ-submodule I of A^σ, $\sigma=\pm$, is said to be an inner ideal of A if it is an inner ideal of $A^{(+)}$, which means $xA^{-\sigma}x\subseteqq I$ for any $x\in I$ [12, 10.1]. An inner ideal of an associative triple system A is just an inner ideal of the pair $V(A)$.

0.6. An associative algebra or triple system A is said to be *nondegenerate* if, for any $x \in A$, $xAx = 0 \Longrightarrow x = 0$. An associative pair A is called nondegenerate if, for any $x \in A^{-\sigma}$, and any $\sigma \in \{+, -\}$, $xA^{\sigma}x = 0 \Longrightarrow x = 0$. Notice that nondegeneracy of an associative system A is just nondegeneracy of the Jordan system $A^{(+)}$ [12, 4.5].

0.7. An associative system is said to be *semiprime* if 0 is the only nilpotent ideal of A. Instead of going through the different notions of nilpotency of ideals for algebras, pairs, and triple systems [1, 1.6, 1.16], we recall the equivalence between semiprimeness and nondegeneracy [1, 1.18], which will be used in the sequel.

0.8. Semiprimeness (indeed, nondegeneracy) is preserved by the functors of (0.2): An associative algebra is semiprime if and only its underlying triple system is semiprime. An associative triple system A is semiprime if and only if $V(A)$ is semiprime. An associative pair A is semiprime if and only if $T(A)$ is semiprime.

0.9. Given an associative pair A, an element $x \in A^{\sigma}$ is called *von Neumann regular* if there exists $y \in A^{-\sigma}$ such that $x = xyx$. A pair of elements $(e, f) \in A^{\sigma} \times A^{-\sigma}$ is called a *pair of idempotents* if $efe = e$ and $fef = f$. Notice that any von Neumann regular element x can be completed to a pair of idempotents (e, f): $e = x$, $f = yxy$ for any y such that $x = xyx$.

Von Neumann regular elements and pairs of idempotents of an associative algebra or triple system A are simply those of the pair $V(A)$.

Notice that a usual idempotent element e of an algebra ($e^2 = e$) is von Neumann regular and (e, e) is a pair of idempotents.

0.10. Local algebras, introduced in [15], are one of the ways to connect the categories of algebras and pairs and triple systems:

Given an associative pair $A = (A^+, A^-)$ and $a \in A^{-\sigma}$, the Φ-module A^{σ} becomes an associative algebra, denoted $A^{\sigma(a)}$ and called the *a-homotope of* A, with product $x \cdot_a y := xay$, for any $x, y \in A^{\sigma}$. The set

$$\mathrm{Ker}_A\, a = \mathrm{Ker}\, a := \{x \in A^{\sigma} \mid axa = 0\}$$

is an ideal of $A^{\sigma(a)}$ and the quotient $A^{\sigma}_a := A^{\sigma(a)}/\,\mathrm{Ker}\, a$ is an associative algebra called the *local algebra of* A *at* a.

Homotopes and local algebras of an associative algebra or triple system A at an element a are simply those of the associative pair $V(A)$.

The above notions are compatible with the functors $V(\ )$ and $T(\ )$ as well as with symmetrizations (cf. [2, 0.5]).

We finally remark that, when (e, f) is a pair of idempotents of A, the local algebra of A at e is unital with unit element $\overline{f} := f + \mathrm{Ker}\, e$.

1. Minimal left, right, and inner ideals

This section is devoted to describing minimal left, right, and inner ideals of associative algebras, pairs, and triple systems. Versions of these results under more restrictive conditions or for a particular kind of system can be found in the literature (cf. [5, 4.1], [8, III.2.1.1, III.2.1.2, III.2.1.3], [9, Lemma 1.3.1], [10, Lemma 1.2.2], [17, §7.1]). However, since we are dealing with not necessarily semiprime systems, most of the known results cannot be applied. Nevertheless, going deeper into their proofs reveals that semiprimeness is not always strictly needed, which allows us to obtain more general and precise descriptions with rather elementary arguments.

As noticed by Loos in the Jordan setting, when not necessarily semiprime systems are considered, trivial inner ideals (minimal or not) play an important role in the definition of the socle (cf. [13, p. 111]). Therefore, we will consider this notion in the associative atmosphere too, and extend it to left and right ideals.

1.1. A left (right or inner) ideal I of an associative algebra A will be called *trivial* if $AI = 0$ ($IA = 0$ or $x\widehat{A}x = 0$, for all $x \in I$, respectively). A left (right or inner) ideal $I \subseteq A^\sigma$, $\sigma = \pm$, of an associative pair A will be called trivial if $A^\sigma A^{-\sigma} I = 0$ ($IA^{-\sigma}A^\sigma = 0$ or $xA^{-\sigma}x = 0$, for all $x \in I$, respectively). A trivial left (right or inner) ideal of an associative triple system A is just a trivial left (right or inner, respectively) ideal of $V(A)$.

Minimal nontrivial left, right, and inner ideals can be described as follows.

1.2. LEMMA. (i) *Let A be an associative algebra and I be a nonzero left (right or inner) ideal of A. Then, I is minimal nontrivial if and only if $Ax = I$ ($xA = I$ or $xAx = I$, respectively) for any $0 \neq x \in I$.*

(ii) *Let A be an associative pair and $I \subseteq A^\sigma$ a nonzero left (right or inner) ideal of A, $\sigma = \pm$. Then I is minimal nontrivial if and only if $A^\sigma A^{-\sigma}x = I$ ($xA^{-\sigma}A^\sigma = I$ or $xA^{-\sigma}x = I$, respectively) for any $0 \neq x \in I$.*

(iii) *Let A be an associative triple system and I a nonzero left (right or inner) ideal of A. Then I is minimal nontrivial if and only if $AAx = I$ ($xAA = I$ or $xAx = I$, respectively) for any $0 \neq x \in I$.*

PROOF. (i) Let I be a minimal nontrivial left ideal of A, and let $0 \neq x \in I$. It is obvious that Ax is a left ideal of A contained in I. By minimality of I, either $Ax = I$ or $Ax = 0$. If $Ax = 0$ for some $0 \neq x \in A$, then the linear span Φx of x is a nonzero left ideal of A contained in I, hence $I = \Phi x$ and I is trivial ($AI = Ax = 0$), which is a contradiction. The converse is straightforward. This argument holds for right ideals with obvious changes.

Consider now a minimal nontrivial inner ideal I of A. Similarly, either $xAx = I$ or $xAx = 0$ for each $0 \neq x \in I$ (notice that xAx is always an inner ideal of A, even when A is not unital). Assume $xAx = 0$ for some $0 \neq x \in A$. If $x^2 = 0$, then Φx is a nonzero inner ideal of A contained in I, so that $I = \Phi x$. Otherwise $x^2 \neq 0$ and Φx^2 is a nonzero inner ideal of A, hence $I = \Phi x^2$, similarly. In both cases, I is trivial, contradicting our assumption. The converse is clear.

(ii) and (iii) The proof of (i) applies here with straightforward changes. □

With a slightly stronger condition than nontriviality, we can even find regular elements inside left and right minimal ideals.

1.3. LEMMA. (i) *Let A be an associative algebra and I a minimal left or right ideal of A. If $II \neq 0$ (for example, when $IAI \neq 0$), then there exists a nonzero idempotent $e \in I$ such that $I = Ae$ or $I = eA$, respectively.*

(ii) *Let A be an associative pair and $I \subseteqq A^\sigma$ a minimal left or right ideal of A, $\sigma = \pm$. If $IA^{-\sigma}I \neq 0$, then there exists a pair of nonzero idempotents $e \in I$, $f \in A^{-\sigma}$ such that $I = A^\sigma A^{-\sigma}e = A^\sigma fe$ or $I = eA^{-\sigma}A^\sigma = efA^\sigma$, respectively.*

(iii) *Let A be an associative triple system and I a minimal left or right ideal of A. If $IAI \neq 0$, then there exists a pair of nonzero idempotents $e \in I$, $f \in A$ such that $I = AAe = Afe$ or $I = eAA = efA$, respectively.*

PROOF. We will deal only with pairs since the result for a triple system A follows by applying (ii) to $V(A)$, while the proof for algebras can be found in [9, Lemma 1.3.1], or obtained by simply replacing a by a formal unit 1 in the following argument.

Let A be an associative pair and $I \subseteqq A^\sigma$ a minimal left ideal of A, for some $\sigma \in \{+,-\}$. By the hypothesis, there exists $0 \neq x \in I$ and $0 \neq a \in A^{-\sigma}$ such that the left ideal Iax is nonzero. Since it is obviously contained in I, $Iax = I$ by minimality. Choose $y \in I$ such that $x = yax$ (so that $y \neq 0$) and consider the set $L := \{z - zay \mid z \in I\}$, which is obviously a left ideal of A contained in I. If $L = I$, then $Iax = Lax = 0$, since $(z - zay)ax = zax - za(yax) = zax - zax = 0$, which is not possible. Hence $L = 0$ by minimality of I and, in particular, $y = yay$ is a nonzero von Neumann regular element in I. By (0.9), $e := y$ can be completed to a pair of nonzero idempotents (e, f) of A. Moreover, since I is obviously nontrivial, we get $I = A^\sigma A^{-\sigma}e$ by (1.2). But $A^\sigma A^{-\sigma}e = A^\sigma A^{-\sigma}efe \subseteqq A^\sigma fe \subseteqq A^\sigma A^{-\sigma}e$.

A similar proof holds for right ideals. □

To obtain an analogue of (1.3) for minimal inner ideals, nontriviality is a sufficiently strong condition.

1.4. LEMMA. *Let A be an associative algebra, triple system, or pair, and I a minimal nontrivial inner ideal of A ($I \subseteqq A^\sigma$, $\sigma = \pm$, in the pair case). Then every nonzero element $e \in I$ is von Neumann regular and $I = eAe =$*

$efAfe$ ($I = eA^{-\sigma}e = efA^{\sigma}fe$ in the pair case) for any $f \in A$ ($f \in A^{-\sigma}$ in the pair case) completing e to a pair of idempotents of A.

PROOF. By (1.2), $I = eAe$ ($I = eA^{-\sigma}e$ in the pair case) for every nonzero element $e \in I$, hence e is von Neumann regular. Now, if (e,f) is a pair of idempotents of A, $eAe = efeAefe \subseteqq efAfe \subseteqq eAe$ ($eA^{-\sigma}e = efeA^{-\sigma}efe \subseteqq efA^{\sigma}fe \subseteqq eA^{-\sigma}e$ in the pair case). □

1.5. REMARK. A minimal nontrivial inner ideal I of an associative algebra A does not necessarily contain algebra idempotents: consider for example the algebra of 2×2 matrices $A = \mathcal{M}_2(\Phi)$ over a field Φ and take $I = \Phi e_{12}$.

We will finally show that, when generated by regular elements, minimal left, right, and inner ideals are deeply related. Local algebras are the suitable tool to connect these notions.

1.6. LEMMA. *Let A be an associative algebra, triple system, or pair, and $e \in A$ ($e \in A^{\sigma}$, $\sigma = \pm$, in the pair case). Then the inner ideal eAe ($eA^{-\sigma}e$ in the pair case) is minimal nontrivial if and only if the local algebra A_e ($A_e^{-\sigma}$ in the pair case) of A at e is a division algebra.*

In particular, if e is a nonzero von Neumann regular element, then eAe ($eA^{-\sigma}e$ in the pair case) is minimal if and only if A_e ($A_e^{-\sigma}$ in the pair case) is a division algebra.

PROOF. We will deal just with pairs, since the proof is valid for an algebra or a triple system just deleting the superscripts.

Suppose first that the inner ideal $eA^{-\sigma}e$ is minimal nontrivial. We claim that for any $\overline{a} := a + \operatorname{Ker} e$, $\overline{b} := b + \operatorname{Ker} e \in A_e^{-\sigma}$ with $0 \neq \overline{b}$, there are elements $\overline{c}, \overline{d} \in A_e^{-\sigma}$ such that $\overline{a} = \overline{b}\,\overline{c} = \overline{d}\,\overline{b}$, which readily implies that $A_e^{-\sigma}$ is a division algebra.

Indeed, $\overline{b} \neq 0$ means $ebe \neq 0$, so that $eA^{-\sigma}e = ebeA^{-\sigma}ebe$ by (1.2). Now $eae \in eA^{-\sigma}e$ implies the existence of some $u \in A^{-\sigma}$ such that $eae = ebeuebe$, i.e., $\overline{a} = \overline{b}\,\overline{ueb} = \overline{beu}\,\overline{b}$.

Conversely, assume that $A_e^{-\sigma}$ is a division algebra, so that $eA^{-\sigma}e$ is a nonzero inner ideal. Let $0 \neq z \in eA^{-\sigma}e$, so that $z = exe$ for some $x \in A^{-\sigma}$, and, in particular, $\overline{x}$ is a nonzero element of $A_e^{-\sigma}$. Let $\overline{y}$ be the inverse of $\overline{x}$. Hence, for any $a \in A^{-\sigma}$, we have $\overline{a} = \overline{x}\,\overline{y}\,\overline{a}\,\overline{y}\,\overline{x}$, which means $eae = exeyeaeyexe = zyeaeyz \in zA^{-\sigma}z$, so that $eA^{-\sigma}e = zA^{-\sigma}z$. This shows that $eA^{-\sigma}e$ is minimal nontrivial by (1.2). □

1.7. LEMMA. *Let A be an associative algebra, triple system, or pair, and (e,f) a pair of idempotents of A ($e \in A^{\sigma}$, $f \in A^{-\sigma}$, $\sigma = \pm$, in the pair case).*

(i) *If Afe ($A^{\sigma}fe$ in the pair case) is a minimal left ideal of A or efA (efA^{σ} in the pair case) is a minimal right ideal of A then eAe ($eA^{-\sigma}e$ in the pair case) is a minimal inner ideal of A.*

(ii) *If A is semiprime and eAe ($eA^{-\sigma}e$ in the pair case) is a minimal inner ideal of A, then Afe ($A^{\sigma}fe$ in the pair case) is a minimal left ideal of A and efA (efA^{σ} in the pair case) is a minimal right ideal of A.*

PROOF. We will consider just the case of pairs, since the proof for algebras and triple systems is obtained by deleting the superscripts.

(i) Assume, for example, that $A^{\sigma}fe$ is a minimal left ideal of A. By (1.6) we just need to show that $A_e^{-\sigma}$ is a division algebra.

Let $0 \neq \overline{x} := x + \operatorname{Ker} e \in A_e^{-\sigma}$, so that $z := exe \neq 0$. Now, $0 \neq z = exe = efexe = efz$ implies that the left ideal $A^{\sigma}fz$ is nonzero, whereas $z = exe = exefe$ forces $A^{\sigma}fz \subseteqq A^{\sigma}fe$, so that $A^{\sigma}fe = A^{\sigma}fz$ by minimality. Thus $e = efe \in A^{\sigma}fz$ and there exists $y \in A^{\sigma}$ such that $e = yfz$, and $efe = efyfz = efyfexe$, which means that $1_{A_e^{-\sigma}} = \overline{f} = \overline{fyf}\,\overline{x}$. We have shown that any nonzero element in $A_e^{-\sigma}$ is left invertible, which implies that $A_e^{-\sigma}$ is a division algebra.

(ii) Let us assume that $eA^{-\sigma}e$ is minimal and let us prove, for example, that $A^{\sigma}fe$ is minimal. For a nonzero left ideal I of A contained in $A^{\sigma}fe$, take a nonzero element $z \in I$. It is immediate that $eA^{-\sigma}z$ is an inner ideal of A and $eA^{-\sigma}z \subseteqq eA^{-\sigma}I \subseteqq eA^{-\sigma}A^{\sigma}fe \subseteqq eA^{-\sigma}e$. Moreover, since $zA^{-\sigma}z \neq 0$ by semiprimeness, and $z = yfe$ for some $y \in A^{\sigma}$, we must have $eA^{-\sigma}z \neq 0$. Therefore, $eA^{-\sigma}z = eA^{-\sigma}e$ by minimality of $eA^{-\sigma}e$. In particular, $e = efe \in eA^{-\sigma}z \in A^{\sigma}A^{-\sigma}z \in I$, so that $A^{\sigma}fe \subseteqq I$. □

1.8. REMARK. The assumptions of (1.7) can be weakened to obtain a stronger result, closer to (1.6). Indeed, it can be proved that minimality flows between one-sided ideals and inner ideals, as soon as they are generated by an element and the inner ideal is nontrivial. The result holds with similar proofs for algebras, pairs, and triple systems and for both left and right ideals, but we just state it for left ideals of an associative pair for the sake of simplicity:

Let A be an associative pair, and $e \in A^{\sigma}$, $\sigma = \pm$, be such that the inner ideal $eA^{-\sigma}e$ is nontrivial.

(i) If $A^{\sigma}A^{-\sigma}e$ is a minimal left ideal of A, then there exists a pair of idempotents $(\tilde{e}, \tilde{f})$ of A such that $A^{\sigma}A^{-\sigma}e = A^{\sigma}A^{-\sigma}\tilde{e} = A^{\sigma}\tilde{f}\tilde{e}$ and $eA^{-\sigma}e = \tilde{e}A^{-\sigma}\tilde{e}$, so that (1.7)(i) applies, and $eA^{-\sigma}e$ is minimal too.

(ii) If A is semiprime and $eA^{-\sigma}e$ is a minimal inner ideal of A, then e is von Neumann regular, so that (1.7)(ii) applies and $A^{\sigma}A^{-\sigma}e$ is a minimal left ideal of A.

Indeed, nontriviality of $eA^{-\sigma}e$ implies that $eueA^{-\sigma}eue \neq 0$ for some $u \in A^{-\sigma}$. If I denotes the left ideal $A^{\sigma}A^{-\sigma}e$, then $IA^{-\sigma}I \neq 0$ since $eue \in I$ and, in particular, I is nontrivial.

(i) Assume that I is minimal. By (1.3), there exists a pair of idempotents $e' \in I$, $f \in A^{-\sigma}$ such that $I = A^{\sigma}A^{-\sigma}e' = A^{\sigma}fe'$.

We claim that the local algebra $A_e^{-\sigma}$ is semiprime: if $0 \neq \overline{x} \in A_e^{-\sigma}$ satisfies $\overline{x} A_e^{-\sigma} \overline{x} = 0$, then $exeA^{-\sigma}exe = 0$ with $0 \neq exe \in I$. We then have $I = A^\sigma A^{-\sigma} exe$ by (1.2), and $IA^{-\sigma}I = A^\sigma A^{-\sigma} exeA^{-\sigma}A^\sigma A^{-\sigma}exe = 0$, which is a contradiction.

Let us show that $A_e^{-\sigma}$ is unital: $efe' \in efI = efA^\sigma A^{-\sigma}e \subseteqq eA^{-\sigma}e$, thus $efe' = eze$ for some $z \in A^{-\sigma}$. For any $x \in A^{-\sigma}$, $exeze = exefe' = exe$, since $exe \in I = A^\sigma fe'$ and $e'fe' = e'$. Thus $\overline{x}\,\overline{z} = \overline{x}$ in $A_e^{-\sigma}$, i.e., $\overline{z}$ is a right unit element for $A_e^{-\sigma}$. Hence $A_e^{-\sigma}$ is unital with unit element $\overline{z}$, by semiprimeness.

Now, $\overline{z}\,\overline{x}\,\overline{z} = \overline{x}$, i.e., $ezexeze = exe$ for any $x \in A^{-\sigma}$ implies that $\tilde{e} := eze = ezezeze$ is von Neumann regular and $eA^{-\sigma}e = \tilde{e}A^{-\sigma}\tilde{e}$. Moreover, $I = A^\sigma A^{-\sigma}\tilde{e}$ by (1.2) since $0 \neq \tilde{e} \in I$, and $I = A^\sigma \tilde{f}\tilde{e}$ for any $\tilde{f}$ completing $\tilde{e}$ to a pair of idempotents.

(ii) Suppose that A is semiprime and $eA^{-\sigma}e$ is minimal. By (1.4), we can take a pair of idempotents (e', f), $e' \in eA^{-\sigma}e$, such that $eA^{-\sigma}e = e'A^{-\sigma}e'$.

We claim that $e = efe'$: For any $x \in A^{-\sigma}$, $(e - efe')x(e - efe') = exe - exefe' - efe'xe + efe'xefe' = exe - exe - efe'xe + efe'xe = 0$, since exe, $efe'xe \in eA^{-\sigma}e = e'A^{-\sigma}e'$ and $e'fe' = e'$. The claim follows by semiprimeness.

Finally, $e = efe' \in efeA^{-\sigma}e \subseteqq eA^{-\sigma}e$ implies that e is von Neumann regular.

1.9. REMARK. Semiprimeness is needed in (1.7)(ii) (consequently in (1.8)(ii)). Let A denote the subalgebra of $\mathcal{M}_3(\Phi)$ spanned by $e_{11}, e_{12}, e_{31}, e_{32}$, where Φ is a field. It is immediate that $e_{11}Ae_{11}$ is a minimal inner ideal of A, while neither the left ideal $Ae_{11} = AAe_{11}$ nor the right ideal $e_{11}A = e_{11}AA$ is minimal.

2. The inner socle

2.1. Let A be an associative algebra or triple system. The sum of all minimal or trivial inner ideals of A will be called the *inner socle* of A and denoted $\mathrm{Soc_{in}}\, A$. The sum of all minimal inner ideals of A will be denoted $\mathrm{Soc_{in}}'\, A$ ($\mathrm{Soc_{in}}'\, A = 0$ if A does not have minimal inner ideals).

The inner socle $\mathrm{Soc_{in}}\, A$ of an associative pair A is the pair of Φ-submodules $(\mathrm{Soc_{in}}^+ A, \mathrm{Soc_{in}}^- A)$, where $\mathrm{Soc_{in}}^\sigma A$ is the sum of all minimal or trivial inner ideals of A contained in A^σ, $\sigma = \pm$. Similarly, we define $\mathrm{Soc_{in}}'\, A = (\mathrm{Soc_{in}}'^+ A, \mathrm{Soc_{in}}'^- A)$, where $\mathrm{Soc_{in}}'^\sigma A$ is the sum of all minimal inner ideals of A contained in A^σ, $\sigma = \pm$ ($\mathrm{Soc_{in}}'^\sigma A = 0$ if A^σ does not contain minimal inner ideals of A). For an associative triple system A, it is obvious by definition that $\mathrm{Soc_{in}}\, A = \mathrm{Soc_{in}}^{+} V(A) = \mathrm{Soc_{in}}^{-} V(A)$ and $\mathrm{Soc_{in}}' A = \mathrm{Soc_{in}}'^{+} V(A) = \mathrm{Soc_{in}}'^{-} V(A)$.

2.2. Remark. In general, the containment $\mathrm{Soc_{in}}' A \subsetneqq \mathrm{Soc_{in}}\, A$ is strict: Let $A = \mathbf{Z}x$ be a free $\mathbf{Z}$-module with basis $\{x\}$, $\Phi = \mathbf{Z}$. Let us consider the zero algebra or triple product on A ($xx = 0$ or $xxx = 0$). Thus, inner ideals (in both the algebra and the triple sense) are exactly the $\mathbf{Z}$-submodules of A (all trivial as inner ideals) and A does not have any minimal inner ideal. Thus, $\mathrm{Soc_{in}}' A = 0$ and $\mathrm{Soc_{in}}\, A = A$. Taking $V(A)$ provides a pair example of this situation.

However, the equality $\mathrm{Soc_{in}}' A = \mathrm{Soc_{in}}\, A$ holds if either A is semiprime or Φ is a field: If A is semiprime, zero is the only trivial inner ideal; if Φ is a field then any trivial inner ideal is the sum of its 1-dimensional subspaces, which are obviously minimal inner ideals.

2.3. Given A and B, two associative pairs over Φ, a *(Jordan) structural transformation from A to B* is a pair of Φ-linear maps (f,g), $f : A^{+} \to B^{+}$, $g : B^{-} \to A^{-}$, such that

$$f(x)yf(x) = f\big(xg(y)x\big), \qquad g(y)xg(y) = g\big(yf(x)y\big),$$

for any $x \in A^{+}$, $y \in B^{-}$, i.e., (f,g) is a structural transformation from $A^{(+)}$ to $B^{(+)}$ [13, p. 112].

Notice that if (f,g) is a structural transformation from A to B then (g,f) is a structural transformation from B^{ex} to A^{ex}.

2.4. We can easily find structural transformations in the multiplication algebra of an associative system:

(i) Let A be an associative pair. For $x,z \in A^{\sigma}$ and $y,u \in A^{-\sigma}$, $\sigma = \pm$, we define the *left, right,* and *middle multiplications*

$$L_{x,y} : A^{\sigma} \to A^{\sigma}, \qquad R_{x,y} : A^{-\sigma} \to A^{-\sigma}, \qquad M_{x,z} : A^{-\sigma} \to A^{\sigma},$$

by

$$L_{x,y}(a) = xya, \qquad R_{x,y}(b) = bxy, \qquad M_{x,z}(b) = xbz,$$

for any $a \in A^{\sigma}, b \in A^{-\sigma}$, respectively. Taking $\sigma = +$, it can be readily checked that:

— $(L_{x,y}, R_{x,y})$, $(R_{y,x}, L_{y,x})$, $(Id_{A^+} + L_{x,y}, Id_{A^-} + R_{x,y})$, and $(Id_{A^+} + R_{y,x}, Id_{A^-} + L_{y,x})$ are structural transformations from A to A, hence

— $(R_{x,y}, L_{x,y})$, $(L_{y,x}, R_{y,x})$, $(Id_{A^-} + R_{x,y}, Id_{A^+} + L_{x,y})$, and $(Id_{A^-} + L_{y,x}, Id_{A^+} + R_{y,x})$ are structural transformations from A^{ex} to A^{ex}.

On the other hand,

— $(M_{x,z}, M_{z,x})$ is a structural transformation from A^{ex} to A, and

— $(M_{y,u}, M_{u,y})$ is a structural transformation from A to A^{ex}.

(ii) Let A be an associative algebra and let L_x and R_x denote the left and right multiplications by an element $x \in A$, respectively. Then it is straightforward that (L_x, R_x), (R_x, L_x), $(Id_A + L_x, Id_A + R_x)$, and $(Id_A + R_x, Id_A + L_x)$ are structural transformations from $V(A)$ to $V(A)$, for any $x \in A$.

As in the Jordan setting [13, Lemma 2], structural transformations preserve minimal inner ideals in the following sense, which is a version of [5, Lemma 4.2] for not necessarily semiprime associative pairs.

2.5. LEMMA. *Let A and B be associative pairs and (f, g) a structural transformation from A to B. If $I \subseteqq A^+$ is an inner ideal of A, then $f(I) \subseteqq B^+$ is an inner ideal of B. Moreover,*

(i) *if I is trivial, then $f(I)$ is trivial;*

(ii) *if I is minimal, then $f(I)$ is either minimal or trivial;*

(iii) *if I is minimal trivial, then either $f(I)$ is minimal trivial or $f(I) = 0$.*

PROOF. For any $x \in I$, we have that $f(x)B^- f(x) = f\big(xg(B^-)x\big) \subseteqq f(xA^-x) \subseteqq f(I)$, which shows that $f(I)$ is an inner ideal of B. This containment also shows that $f(I)$ is trivial when I is trivial, i.e., (i).

(ii) We can assume $f(I) \neq 0$ and, by (i), we can restrict to the case when I is minimal nontrivial. We distinguish two cases:

— $xg(B^-)x \neq 0$ for all $0 \neq x \in I$. Notice that, for any $x \in A^+$, $xg(B^-)x \subseteqq A^+$ is always an inner ideal of A [for any $b \in B^-$, $xg(b)xA^-xg(b)x = xg\big(bf(xA^-x)b\big)x \subseteqq xg(B^-)x$]. In our case, if $0 \neq x \in I$, then $xg(B^-)x$ is a nonzero inner ideal contained in I, hence $I = xg(B^-)x$ by minimality. Let us prove that then $f(I)$ is minimal too: Let $0 \neq y \in f(I)$, $y = f(x)$ for some $0 \neq x \in I$. Thus, $I = xg(B^-)x$, and $f(I) = f\big(xg(B^-)x\big) = f(x)B^-f(x) = yB^-y$.

— $xg(B^-)x = 0$ for some $0 \neq x \in I$. By (1.2), $I = xA^-x$, hence for any $z \in I$, $z = xax$ for some $a \in A^-$, and $f(z)B^-f(z) = f\big(zg(B^-)z\big) = f\big(xaxg(B^-)xax\big) = f\big(xa\big(xg(B^-)x\big)ax\big) = 0$. We have proved that $f(I)$ is trivial.

(iii) For a minimal trivial I, we know from (i) that $f(I)$ is trivial. On the other hand, it is clear that a trivial inner ideal is minimal if and only if it is an irreducible Φ-module. Since the restriction of f from I to $f(I)$ is a surjective homomorphism of Φ-modules, that restriction is either an isomorphism or zero, by irreducibility of I. Therefore, either $f(I)$ is also an irreducible Φ-module or $f(I) = 0$. □

For the particular case of structural transformations given by left and right multiplications, something else can be said.

2.6. LEMMA. *Let A be an associative pair, and (f,g) a structural transformation from A to A such that $(Id_{A^+}+f, Id_{A^-}+g)$ is also a structural transformation from A to A. If $I \subseteqq A^+$ is a minimal nontrivial inner ideal of A, then either $f(I)$ is a minimal inner ideal of A or $(Id_{A^+}+f)(I) = \{x + f(x) \mid x \in I\}$ is a minimal inner ideal of A.*

PROOF. By (2.5), we know that $f(I)$ and $(Id_{A^+}+f)(I)$ are inner ideals of A. The result is clear if $f(I)=0$, so that we will assume $f(I) \neq 0$.

Following the proof of (2.5) for $B=A$, we have that either $xg(A^-)x \neq 0$ for all $0 \neq x \in I$, and hence $f(I)$ is minimal, or there exists $0 \neq x \in I$ such that $xg(A^-)x = 0$. In this latter case, $I = xA^-x$ by (1.2), and for any $0 \neq z \in I$, we can write $z = xax$ for some $a \in A^-$, so that, again by (1.2), $I = zA^-z = xaxA^-xax = xax(Id_{A^-}+g)(A^-)xax = z(Id_{A^-}+g)(A^-)z$. In particular $z(Id_{A^-}+g)(A^-)z \neq 0$ for any $0 \neq z \in I$, and, again by the proof of (2.5) with $B=A$ and the structural transformation $(Id_{A^+}+f, Id_{A^-}+g)$, we obtain that $(Id_{A^+}+f)(I)$ is minimal in this case. □

2.7. THEOREM. *Let A and B be associative pairs.*

(i) *If (f,g) is a structural transformation from A to B, then $f(\mathrm{Soc_{in}}^+ A) \subseteqq \mathrm{Soc_{in}}^+ B$.*

(ii) *If (f,g) is a structural transformation from A to A such that $(Id_{A^+}+f, Id_{A^-}+g)$ is also a structural transformation from A to A then $f(\mathrm{Soc_{in}}'^+ A) \subseteqq \mathrm{Soc_{in}}'^+ A$.*

PROOF. (i) By (2.5)(i, ii), if $I \subseteqq A^+$ is a minimal or trivial inner ideal of A, then $f(I)$ is a minimal or trivial inner ideal of B, so that $f(I) \subseteqq \mathrm{Soc_{in}}^+ A$, which shows the assertion.

(ii) Let $I \subseteqq A^+$ be a minimal inner ideal of A. By (2.5)(iii), if I is trivial, then $f(I)$ is either zero or minimal trivial, hence $f(I) \subseteqq \mathrm{Soc_{in}}'^+ A$. But if I is nontrivial, then either $f(I)$ or $(Id_{A^+}+f)(I)$ is minimal by (2.6). In the first case, $f(I) \subseteqq \mathrm{Soc_{in}}'^+ A$; in the second case, $(Id_{A^+}+f)(I) \subseteqq \mathrm{Soc_{in}}'^+ A$, which, together with $I \subseteqq \mathrm{Soc_{in}}'^+ A$, implies $f(I) \subseteqq \mathrm{Soc_{in}}'^+ A$. □

2.8. THEOREM. *Let A be an associative pair or triple system. Then, $\mathrm{Soc_{in}}\, A$ and $\mathrm{Soc_{in}}'\, A$ are ideals of A.*

PROOF. We will prove our assertion just for associative pairs since the result for a triple system A will follow by applying it to $V(A)$.

The fact that $\mathrm{Soc_{in}}\, A$ is an ideal of A follows from (2.7)(i) and (2.4)(i), since $\mathrm{Soc_{in}}^\sigma A = \mathrm{Soc_{in}}^{-\sigma}(A^{\mathrm{ex}})$.

In a similar way, $\mathrm{Soc_{in}}'^\sigma A$ is a left and a right ideal of A for any $\sigma \in \{+,-\}$ by (2.4)(i), (2.7)(ii), and the equality $\mathrm{Soc_{in}}'^\sigma A = \mathrm{Soc_{in}}'^{-\sigma}(A^{\mathrm{ex}})$. Hence, we just need to prove that $xIy \subseteqq \mathrm{Soc_{in}}'^{-\sigma} A$ for any minimal ideal

I of A contained in A^{σ}, and any $x, y \in A^{-\sigma}$. If such an ideal I is trivial, then $xIy = M_{x,y}(I)$ is either minimal trivial or zero by (2.5)(iii) and (2.4)(i), hence $xIy \subseteqq {\mathrm{Soc_{in}}'}^{-\sigma} A$. Otherwise, I is nontrivial, and $I = eA^{-\sigma}e$, for a nonzero pair of idempotents (e, f) by (1.4), and $e \in I$. Using (2.5)(ii) with the structural transformation $(M_{f,f}, M_{f,f})$, we get that $fIf = M_{f,f}(I)$ is either a minimal or trivial inner ideal of A. But $f = fef \in fIf$ and $0 \neq f = fef \in fA^{\sigma}f$ imply nontriviality of fIf, so that $fIf \subseteqq {\mathrm{Soc_{in}}'}^{-\sigma} A$, and $f = fef \in {\mathrm{Soc_{in}}'}^{-\sigma} A$. Finally,

$$xIy = xeA^{-\sigma}ey = xefeA^{-\sigma}ey = R_{e,y}R_{e,A^{-\sigma}}L_{x,e}(f)$$

$$\subseteqq R_{e,y}R_{e,A^{-\sigma}}L_{x,e}({\mathrm{Soc_{in}}'}^{-\sigma} A) \subseteqq {\mathrm{Soc_{in}}'}^{-\sigma} A$$

since ${\mathrm{Soc_{in}}'}^{-\sigma} A$ is a left and right ideal of A. □

2.9. Next, we will show analogous results for algebras. They cannot be directly derived from the above work for pairs, since the inner socles and ideals of the double associative pair $V(A)$ of an algebra A are not exactly those of A, unlike the triple system situation. As an example, let A be the quotient algebra $\Phi[X]/I$, where $\Phi[X]$ is the commutative associative nonunital algebra of polynomials on the variable X, I is the ideal of $\Phi[X]$ generated by X^3, and Φ is a field. Taking $a = X + I$, A is the linear span of $\{a, a^2\}$, and $a^3 = 0$. It is readily seen that the only proper inner ideal of A is Φa^2, which means that $\mathrm{Soc_{in}}\, A = \mathrm{Soc_{in}}'\, A = \Phi a^2$. However, $V(A)$ has zero product, so that any 1-dimensional subspace of A is a trivial and minimal inner ideal of $V(A)$, hence $\mathrm{Soc_{in}}\, V(A) = \mathrm{Soc_{in}}'\, V(A) = V(A)$.

To overcome this difficulty, rather than adapting the former proofs to the algebra setting, we will use unitizations since pair and algebra notions coincide for unital algebras. Namely, we will need the following assertions for an associative algebra A and its underlying triple system A_T:

(i) If A is unital, inner ideals of A are exactly inner ideals of A_T, hence minimal (respectively, trivial) inner ideals of A are exactly minimal (respectively, trivial) inner ideals of A_T.

(ii) Inner ideals of A are exactly inner ideals of $\widehat{A}$ contained in A. As a consequence, minimal (respectively, trivial) inner ideals of A are exactly minimal (respectively, trivial) inner ideals of $\widehat{A}$ contained in A.

2.10. Corollary. *Let A be an associative algebra.*

(i) $\mathrm{Soc_{in}}\, A$ *and* $\mathrm{Soc_{in}}'\, A$ *are ideals of* A.

(ii) $\mathrm{Soc_{in}}\,(A_T)$ *and* $\mathrm{Soc_{in}}'(A_T)$ *are ideals of* A.

Proof. (i) Any minimal or trivial inner ideal I of A is a minimal or trivial inner ideal of $\widehat{A}$ by (2.9)(ii), hence a minimal or trivial inner ideal of

$\widehat{A}_T$ by (2.9)(i), i.e., a minimal or trivial inner ideal of $V(\widehat{A}_T) = V(\widehat{A})$. By (2.4)(ii), (L_x, R_x) and (R_x, L_x) are structural transformations from $V(\widehat{A})$ to $V(\widehat{A})$ for any $x \in A$, so that (2.5)(i, ii) implies that xI and Ix are minimal or trivial inner ideals of $V(\widehat{A}) = V(\widehat{A}_T)$, i.e., minimal or trivial inner ideals of $\widehat{A}_T$. Therefore, they are minimal or trivial inner ideals of A, by (2.9). We have shown that $xI, Ix \subseteqq \mathrm{Soc_{in}}\, A$ for any minimal or trivial inner ideal I of A and any $x \in A$, hence $\mathrm{Soc_{in}}\, A$ is an ideal of A.

Let I be a minimal inner ideal of A. As above, I is a minimal inner ideal of $V(\widehat{A})$. For any $x \in A$, and $(f, g) = (L_x, R_x)$ or $(f, g) = (R_x, L_x)$, (2.4)(ii), (2.5)(iii), and (2.6) yield that, either $f(I)$ is a minimal inner ideal of $V(\widehat{A})$, or $f(I) = 0$, or $(Id_{\widehat{A}} + f)(I)$ is a minimal inner ideal of $V(\widehat{A})$. Again as above, this means that either $f(I)$ is a minimal inner ideal of A, or $f(I) = 0$, or $(Id_{\widehat{A}} + f)(I)$ is a minimal inner ideal of A. We obtain that, either $f(I)$ or $(Id_{\widehat{A}} + f)(I)$ is contained in $\mathrm{Soc_{in}}'\, A$. Since $I \subseteqq \mathrm{Soc_{in}}'\, A$, $f(I) \subseteqq \mathrm{Soc_{in}}'\, A$ in either case, proving that $\mathrm{Soc_{in}}'\, A$ is an ideal of A.

(ii) By definition (2.1), $\mathrm{Soc_{in}}\, V(A) = \big(\mathrm{Soc_{in}}\,(A_T), \mathrm{Soc_{in}}\,(A_T)\big)$. Using (2.7)(i) and (2.4)(ii), for any $x \in A$ we have that

$$x\big(\mathrm{Soc_{in}}\,(A_T)\big), \big(\mathrm{Soc_{in}}\,(A_T)\big)x \subseteqq \mathrm{Soc_{in}}\,(A_T),$$

proving that $\mathrm{Soc_{in}}\,(A_T)$ is an algebra ideal of A.

The former argument holds for $\mathrm{Soc_{in}}'(\ )$ after replacing (2.7)(i) by (2.7)(ii). □

3. The left and right socles

3.1. Let A be an associative algebra or triple system. The sum of all minimal or trivial left (right) ideals of A will called the *left* (*right*) *socle* of A and denoted $\mathrm{Soc_l}\, A$ ($\mathrm{Soc_r}\, A$). The sum of all minimal left (right) ideals of A will be denoted $\mathrm{Soc_l}'\, A$ ($\mathrm{Soc_r}'\, A$) ($\mathrm{Soc_l}'\, A = 0$ or $\mathrm{Soc_r}'\, A = 0$ if A does not have minimal left or right ideals, respectively).

The left socle $\mathrm{Soc_l}\, A$ of an associative pair A is the pair $(\mathrm{Soc_l}^+ A, \mathrm{Soc_l}^- A)$, where $\mathrm{Soc_l}^\sigma A$ is the sum of all minimal or trivial left ideals of A contained in A^σ, $\sigma = \pm$. Similarly, we define $\mathrm{Soc_l}'\, A = (\mathrm{Soc_l}'^+ A, \mathrm{Soc_l}'^- A)$, where $\mathrm{Soc_l}'^\sigma A$ is the sum of all minimal left ideals of A contained in A^σ, $\sigma = \pm$ ($\mathrm{Soc_l}'^\sigma A = 0$ if A^σ does not contain minimal left ideals of A). For an associative triple system A, it is obvious by definition that $\mathrm{Soc_l}\, A = \mathrm{Soc_l}^+ V(A) = \mathrm{Soc_l}^- V(A)$

and $\mathrm{Soc_l}' A = \mathrm{Soc_l}'^{+} V(A) = \mathrm{Soc_l}'^{-} V(A)$. We have analogous notions, notations, and relations for right ideals.

3.2. REMARKS. (i) The example given in (2.2) illustrates the situation when the containments $\mathrm{Soc_l}' A \subseteqq \mathrm{Soc_l} A$ and $\mathrm{Soc_r}' A \subseteqq \mathrm{Soc_r} A$ are strict. We also have the equalities $\mathrm{Soc_l}' A = \mathrm{Soc_l} A$ and $\mathrm{Soc_r}' A = \mathrm{Soc_r} A$ if either A is semiprime or Φ is a field.

(ii) In general, left and right socles do not coincide. As an example, take the subalgebra A of $\mathcal{M}_2(\Phi)$ spanned by the matrix units e_{11} and e_{21}, for a field Φ. The only proper left ideal of A is Φe_{21}, so that $\mathrm{Soc_l} A = \mathrm{Soc_l}' A = \Phi e_{21}$, whereas it is immediate that any Φ-submodule of A is a right ideal, and therefore $\mathrm{Soc_r} A = \mathrm{Soc_r}' A = A$. The situation remains the same when we consider the underlying triple system of A and the double pair $V(A)$.

3.3. THEOREM. (i) *If A is an associative pair, then* $\mathrm{Soc_l}^{\sigma} A$, $\mathrm{Soc_r}^{\sigma} A$, $\mathrm{Soc_l}'^{\sigma} A$, $\mathrm{Soc_r}'^{\sigma} A$ *are left and right ideals of A, $\sigma = \pm$.*

(ii) *If A is an associative triple system, then* $\mathrm{Soc_l} A$, $\mathrm{Soc_r} A$, $\mathrm{Soc_l}' A$, $\mathrm{Soc_r}' A$ *are left and right ideals of A.*

(iii) *If A is an associative algebra, then* $\mathrm{Soc_l} A$, $\mathrm{Soc_r} A$, $\mathrm{Soc_l}' A$, $\mathrm{Soc_r}' A$ *are ideals of A.*

PROOF. (i) Put $\sigma = +$, for example. Left ideals of A contained in A^+ are simply the submodules of A^+ over the Φ-algebra $B \subseteqq \mathrm{End}_\Phi(A^+)$ spanned by the set of all left multiplications $L_{x,y}$, for $x \in A^+$, $y \in A^-$. For fixed $x \in A^+$ and $y \in A^-$, $R_{y,x} : A^+ \to A^+$ is a B-module homomorphism. Any minimal left ideal $I \subseteqq A^+$ of A is a minimal B-submodule of A^+, hence $Iyx = R_{y,x}I$ is a minimal B-submodule or zero. Also if $I \subseteqq A^+$ is a trivial left ideal of A, Iyx is a trivial left ideal of A. The above implies that $(\mathrm{Soc_l}'^{+} A)yx \subseteqq \mathrm{Soc_l}'^{+} A$ and $(\mathrm{Soc_l}^{+} A)yx \subseteqq \mathrm{Soc_l}^{+} A$. We then have that $\mathrm{Soc_l}'^{+} A$ and $\mathrm{Soc_l}^{+} A$ are right ideals of A, but they are left ideals by definition. This proof applies to the right side with obvious changes.

(ii) follows from (i) when applied to $V(A)$.

(iii) The proof of (i) is valid for algebras deleting the superscripts and replacing y by a formal unit. □

3.4. REMARK. For an associative pair or triple system A, one-sided socles are not in general ideals of A. Consider $A = (A^+, A^-) = (\Phi e_{11}, \Phi e_{11} + \Phi e_{21})$, as a subpair of $V\big(\mathcal{M}_2(\Phi)\big)$, where Φ is a field. It can be readily seen that the only proper left ideal of A contained in A^- is Φe_{21}, thus $\mathrm{Soc_l} A = \mathrm{Soc_l}' A = (\Phi e_{11}, \Phi e_{21})$, which is not an ideal of A: $e_{11} = e_{11}e_{11}e_{11} \in A^- e_{11} A^-$, but $e_{11} \notin \Phi e_{21}$. The opposite pair A^{op} is an example where the right socle is not an ideal. The reader is referred to the last section of the paper for a triple system example of this situation.

An analogue of (2.10)(ii) for one-sided socles also holds.

3.5. Proposition. *Let A be an associative algebra. Then* $\mathrm{Soc_l}\,(A_T)$, $\mathrm{Soc_r}\,(A_T)$, $\mathrm{Soc_l}'(A_T)$, *and* $\mathrm{Soc_r}'(A_T)$ *are ideals of A.*

Proof. Let I be a left ideal of A_T, $x \in A$. We claim that

(i) if I is trivial then, then xI and Ix are trivial left ideals of A_T.

Indeed, $AAI = 0$ implies $AAxI = 0$ and $AAIx = 0$, showing that xI and Ix are both left ideals and trivial. We also claim that

(ii) if I is minimal then, $xI, Ix \subseteqq \mathrm{Soc_l}'(A_T)$.

Indeed, I is an irreducible B-module, for the Φ-algebra $B \subseteqq \mathrm{End}_\Phi(A)$ spanned by the set of all $L_{x,y} = L_xL_y$, for $x, y \in A$. Hence $Ix = R_x(I)$ is also an irreducible B-module or zero since $R_x : A \to A$ is a B-module homomorphism, i.e., Ix is either a minimal left ideal or zero, thus $Ix \subseteqq \mathrm{Soc_l}'(A)$. If I is minimal trivial, then I is an irreducible Φ-module, and $xI = L_x(I)$ is either an irreducible Φ-module or zero since $L_x : A \to A$ is a Φ-module homomorphism. If I is minimal nontrivial, then $I = AAI$ by (1.2)(iii), which implies that I is an algebra left ideal of A ($AI = AAAI \subseteqq AAI = I$), and $xI \subseteqq I$. In all cases we have shown that $xI \subseteqq \mathrm{Soc_l}'(A_T)$.

Now (ii) says that $\mathrm{Soc_l}'(A_T)$ is an ideal of A, which implies that $\mathrm{Soc_l}\,(A_T)$ is an ideal of A using (i).

The above assertion applied to the opposite algebra A^{op} yields that $\mathrm{Soc_r}\,(A_T)$ and $\mathrm{Soc_r}'(A_T)$ are also ideals of A. □

Next we study the relationship between one-sided socles and the inner socle.

3.6. Lemma. *Let A be an associative system and I a minimal left, right, or inner ideal of A. Then $I \subseteqq L$ ($I \subseteqq L^\sigma$ if $I \subseteqq A^\sigma$ in the pair case) for any ideal L of A hitting I, $I \cap L \neq 0$ ($I \cap L^\sigma \neq 0$ in the pair case).*

Proof. Just notice that $I \cap L$ ($I \cap L^\sigma$ in the pair case) is a left, right, or inner ideal of A contained in I. □

3.7. Theorem. *Let A be an associative system. Then* $\mathrm{Soc_l}\,A$, $\mathrm{Soc_r}\,A$ *are contained in* $\mathrm{Soc_{in}}\,A$. *If A is semiprime, then* $\mathrm{Soc_l}\,A = \mathrm{Soc_r}\,A = \mathrm{Soc_{in}}\,A$.

Proof. After replacing A by A^{op}, we just need to deal with $\mathrm{Soc_l}\,A$ and $\mathrm{Soc_{in}}\,A$.

Let I be a minimal or trivial left ideal of A ($I \subseteqq A^\sigma$ in the pair case). If $IAI = 0$ ($IA^{-\sigma}I = 0$ in the pair case), then I is a trivial inner ideal of A, hence $I \subseteqq \mathrm{Soc_{in}}\,A$ ($I \subseteqq \mathrm{Soc_{in}}^\sigma A$ in the pair case). Otherwise I is necessarily minimal nontrivial, and there exists a pair of nonzero idempotents (e, f) such that $I = Afe$ ($I = A^\sigma fe$ in the pair case), by (1.3) (take $f = e$ in the algebra case). By (1.7)(i), eAe ($eA^{-\sigma}e$ in the pair case) is a minimal inner ideal of A, hence

$$e = efe \in I \cap eAe \subseteqq I \cap \mathrm{Soc_{in}}\, A$$

$$(e = efe \in I \cap eA^{-\sigma}e \subseteqq I \cap \mathrm{Soc_{in}}^{\sigma} A \text{ in the pair case}),$$

and $I \subseteqq \mathrm{Soc_{in}}\, A$ ($I \subseteqq \mathrm{Soc_{in}}^{\sigma} A$ in the pair case) by (3.6) since $\mathrm{Soc_{in}}\, A$ is an ideal of A by (2.8) and (2.10). This shows $\mathrm{Soc_l}\, A \subseteqq \mathrm{Soc_{in}}\, A$.

If A is semiprime, $\mathrm{Soc_{in}}\, A$ is just the sum of minimal (necessarily nontrivial) inner ideals of A. Thus, we only need to show that any minimal nontrivial inner ideal I of A ($I \subseteqq A^{\sigma}$ in the pair case) is contained in $\mathrm{Soc_l}\, A$. By (1.4) there exists a pair of nonzero idempotents (e, f) such that $I = eAe$ ($I = eA^{-\sigma}e$ in the pair case), so that we can apply (1.7)(ii) to get that Afe ($A^{\sigma}fe$ in the pair case) is a minimal left ideal of A. Thus

$$I = eAe = eAefe \subseteqq Afe \subseteqq \mathrm{Soc_l}\, A$$

$$(I = eA^{-\sigma}e = eA^{-\sigma}efe \subseteqq A^{\sigma}fe \subseteqq \mathrm{Soc_l}^{\sigma} A \text{ in the pair case}). \quad \square$$

3.8. Remarks. (i) The examples given in (3.2)(ii) and their opposites show that the containments $\mathrm{Soc_l}\, A \subseteqq \mathrm{Soc_{in}}\, A$ and $\mathrm{Soc_r}\, A \subseteqq \mathrm{Soc_{in}}\, A$ may be strict when A is not semiprime.

(ii) An analogue of the first assertion of (3.7) for $\mathrm{Soc_l}'\, A$, $\mathrm{Soc_r}'\, A$ and $\mathrm{Soc_{in}}'\, A$ is false: Consider the quotient algebra $A = \mathbf{Q}[X]/I$, where $\mathbf{Q}[X]$ is the commutative associative unital $\mathbf{Q}$-algebra of polynomials on the variable X with rational coefficients, and I is the ideal of $\mathbf{Q}[X]$ generated by X^2. Though A is a $\mathbf{Q}$-algebra, we take $\Phi = \mathbf{Z}$. Denoting $a = X + I$, A is the linear span on $\mathbf{Q}$ of $\{1, a\}$, and $a^2 = 0$. It is easy to check that $\mathbf{Q}a$ is a minimal left ideal of A, so that $\mathrm{Soc_l}'\, A \neq 0$ and $\mathrm{Soc_r}'\, A \neq 0$ by commutativity. However, there are no minimal inner ideals of A. Indeed, if $0 \neq \alpha 1 + \beta a \in L$, an inner ideal of A, then either $\alpha \neq 0$ and $a = (\alpha 1 + \beta a)\alpha^{-2}a(\alpha 1 + \beta a) \in L$, or $\alpha = 0$, $\beta \neq 0$ and $\beta a \in L$. In either case, L contains $\mathbf{Z}\beta a$ for some $0 \neq \beta \in \mathbf{Q}$, and its infinitely many proper $\mathbf{Z}$-submodules, which are all (trivial) inner ideals of A. Notice that, since A is unital, A_T and $V(A)$ are triple system and pair examples of the same situation.

4. Further comments

In this section we study the action of the functors introduced in (0.2) on the different socles as well as the interaction of direct sums and socles.

In the following Soc stands for $\mathrm{Soc}_{\mathrm{in}}$, $\mathrm{Soc}_{\mathrm{l}}$, $\mathrm{Soc}_{\mathrm{r}}$, $\mathrm{Soc}_{\mathrm{in}}{}'$, $\mathrm{Soc}_{\mathrm{l}}{}'$, and $\mathrm{Soc}_{\mathrm{r}}{}'$.

4.1. PROPOSITION. *Let $\{A_i \mid i \in I\}$ be a family of associative systems indexed by a set I, and let $A = \sum_{i\in I} A_i$ be the direct sum of the family. Then* $\mathrm{Soc}\, A = \sum_{i\in I} \mathrm{Soc}\, A_i$.

PROOF. For any $i \in I$, let $\pi_i : A \to A_i$ denote the canonical projection and $\tau_i : A_i \to A$ the canonical injection, which are homomorphisms of associative systems. Then, our assertion can be readily obtained from the following straightforward facts for any $i \in I$:

(i) If L is a minimal left, right, or inner ideal of A, then $\pi_i(L)$ is a minimal left, right, or inner ideal of A_i, respectively, or zero.

(ii) If L is a trivial left, right, or inner ideal of A, then $\pi_i(L)$ is a trivial left, right, or inner ideal of A_i, respectively.

(iii) If L is a minimal left, right, or inner ideal of A_i, then $\tau_i(L)$ is a minimal left, right, or inner ideal of A, respectively.

(iv) If L is a trivial left, right, or inner ideal of A_i, then $\tau_i(L)$ is a trivial left, right, or inner ideal of A, respectively. □

In (2.9) we exhibited an algebra A such that $\mathrm{Soc}_{\mathrm{in}}\, V(A) \neq V(\mathrm{Soc}_{\mathrm{in}}\, A)$, equivalently, $\mathrm{Soc}_{\mathrm{in}}\, A \neq \mathrm{Soc}_{\mathrm{in}}\,(A_T)$. The same argument shows that A has an analogous behaviour with respect to $\mathrm{Soc}\, A$. However, some relations between $\mathrm{Soc}\, A$ and $\mathrm{Soc}\,(A_T)$ can be found in general.

4.2. PROPOSITION. *Let A be an associative algebra. Then,* $\mathrm{Soc}\, A \subseteqq \mathrm{Soc}\,(A_T)$. *If A is semiprime, then* $\mathrm{Soc}\, A = \mathrm{Soc}\,(A_T)$.

PROOF. The first assertion follows from:

(i) A trivial left, right, or inner ideal of A is a trivial left, right, or inner ideal of A_T, respectively (cf. (1.1)).

(ii) A minimal trivial left, right, or inner ideal of A is a minimal trivial left, right, or inner ideal of A_T, respectively. Indeed, for a trivial left, right or inner ideal being minimal is just not having proper Φ-submodules, so that (ii) is implied by (i).

(iii) A minimal nontrivial left, right, or inner ideal of A is a minimal nontrivial left, right, or inner ideal of A_T, respectively: If I is, for example, a minimal nontrivial left ideal of A, we know from (1.2) that $I = Ax$ for any $0 \neq x \in I$. It is clear that I is a left ideal of A_T. Moreover, since $I = Ax \neq 0$, there exists $y_x \in A$ such that $0 \neq y_x x \in I$. Again by (1.2), $I = Ay_x x$, hence $AAx \neq 0$ and I is a nontrivial left ideal of A_T. If L is a nonzero left ideal of A_T contained in I, we can take $0 \neq x \in L$ so that in particular, $I = Ay_x x \subseteqq L$, which shows the desired minimality.

If A is semiprime, there are no nonzero trivial left, right, or inner ideals in A, thus to get $\mathrm{Soc}\, A = \mathrm{Soc}\,(A_T)$, we just need to prove the converse of (iii). Indeed, (1.2)(iii) shows that a minimal nontrivial left, right, or inner ideal of

A_T is a left, right, or inner ideal of A, respectively, whereas its minimality in A is obvious. $\square$

We have already mentioned in the very definitions (2.1), (3.1) that $\operatorname{Soc} V(A) = V(\operatorname{Soc} A)$ for any associative triple system A. Next we show that the functor $T(\ )$ behaves equally well.

4.3. Proposition. *Let A be an associative pair. Then* $\operatorname{Soc} T(A) = T(\operatorname{Soc} A)$.

Proof. Let $\pi^\sigma : T(A) \to A^\sigma$ be the natural projection, and $\tau^\sigma : A^\sigma \to T(A)$ the natural injection, $\sigma = \pm$. The following straightforward assertions readily imply the result.

(i) If L is a minimal left, right, or inner ideal of $T(A)$, then $\pi^\sigma(L) \subseteqq A^\sigma$ is a minimal left, right, or inner ideal of A, respectively, or zero.

(ii) If L is a trivial left, right, or inner ideal of $T(A)$, then $\pi^\sigma(L) \subseteqq A^\sigma$ is a trivial left, right, or inner ideal of A, respectively.

(iii) If $L \subseteqq A^\sigma$ is a minimal left, right, or inner ideal of A, then $\tau^\sigma(L)$ is a minimal left, right, or inner ideal of $T(A)$, respectively.

(iv) If $L \subseteqq A^\sigma$ is a trivial left, right, or inner ideal of A, then $\tau^\sigma(L)$ is a trivial left, right, or inner ideal of $T(A)$, respectively. $\square$

The above results can be used to build examples. For instance, (4.3) provides a triple analogue of the situation described in (3.4). Also (4.1) can be used to obtain associative systems compiling the "evils" of several examples (e.g., for the direct sum of the algebra A of (3.2)(ii) and its opposite, left, right, and inner socles are pairwise different).

References

[1] A. D'Amour, Quadratic Jordan systems of Hermitian type, *J. Algebra,* **149** (1992), 197–233.

[2] J. A. Anquela and T. Cortés, Primitive Jordan pairs and triple systems, *J. Algebra,* **184** (1996), 632–678.

[3] A. Castellón-Serrano, A. Fernández-López, A. García-Martín and C. Martín-González, Strongly prime alternative pairs with minimal inner ideals, *Manuscripta Math.,* **90** (1996), 479–487.

[4] J. Dieudonné, Sur le socle d'un anneau et les anneaux simples infinis, *Bull. Soc. Math. France,* **70** (1942), 46–75.

[5] A. Fernández-López and E. García-Rus, Prime associative triple systems with nonzero socle, *Comm. Algebra,* **18** (1990), 1–13.

[6] A. Fernández-López, E. García-Rus, M. Gómez-Lozano and M. Siles-Molina, Goldie theorems for associative pairs, *Comm. Algebra,* **26** (1998), 2987–3020.

[7] A. Fernández-López, E. García-Rus and E. Sánchez-Campos, Von Neumann regular Jordan Banach triple systems, *J. London Math. Soc.,* (2) **42** (1990), 32–48.

[8] A. Fernández-López and M. Tocón-Barroso, *Álgebras Locales en la Teoría de Pares Asociativos,* Memoria de Licenciatura, Universidad de Málaga (Málaga, 2000).
[9] I. N. Herstein, *Noncommutative Rings,* The Carus Mathematical Monographs No. 15 The Mathematical Association of America (Menasha, Wisconsin, 1968).
[10] I. N. Herstein, *Rings with Involution,* Chicago Lectures in Mathematics, The University of Chicago Press (Chicago, 1976).
[11] N. Jacobson, *Structure of Rings,* Amer. Math. Soc. Colloquium Publications, Amer. Math. Soc. (Providence, Rhode Island, 1956).
[12] O. Loos, *Jordan Pairs,* Lecture Notes in Mathematics, Vol. 460, Springer-Verlag (New York, 1975).
[13] O. Loos, On the socle of a Jordan pair, *Collect. Math.,* **40** (1989), 109–125.
[14] K. McCrimmon and E. Zelmanov, The Structure of Strongly Prime Quadratic Jordan Algebras, *Adv. Math.,* **69** (1988), 133–222.
[15] K. Meyberg, *Lectures on Algebras and Triple Systems,* Lecture Notes, University of Virginia (Charlottesville, 1972).
[16] J. M. Osborn and M. L. Racine, Jordan rings with nonzero socle, *Trans. Amer. Math. Soc.,* **251** (1979), 375–387.
[17] L. H. Rowen, *Polynomial Identities in Ring Theory,* Academic Press (New York, 1980).

(Received November 21, 2002)

DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD DE OVIEDO
C/ CALVO SOTELO S/N
33007 OVIEDO
SPAIN
E-MAIL: ANQUE@PINON.CCU.UNIOVI.ES
CORTES@PINON.CCU.UNIOVI.ES

DEPARTAMENTO DE ÁLGEBRA, GEOMETRÍA, Y TOPOLOGÍA
FACULTAD DE CIENCIAS
UNIVERSIDAD DE MÁLAGA
CAMPUS DE TEATINOS
29071 MÁLAGA
SPAIN
E-MAIL: MAGOMEZ@AGT.CIE.UMA.ES
MERCEDES@AGT.CIE.UMA.ES