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Left Quotient Associative Pairs and Morita Invariant Properties[#]

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ABSTRACT

In this paper, we prove that left nonsingularity and left nonsingularity plus finite left local Goldie dimension are two Morita invariant properties for idempotent rings without total left or right zero divisors. Moreover, two Morita equivalent idempotent rings, semiprime and left local Goldie, have Fountain–Gould left quotient rings that are Morita equivalent too. These results can be obtained from others concerning associative pairs. We introduce the notion of (general) left quotient pair of an associative pair and show the existence of a maximal left quotient pair for every semiprime or left nonsingular associative pair. Moreover, we characterize those associative pairs for which their maximal left quotient pair is von Neumann regular and give a Gabriel-like characterization of associative pairs whose maximal left quotient pair is semiprime and artinian.

Key Words: Associative pair; Left quotient pair; Morita invariant property; Fountain–Gould left quotient ring.

Mathematics Subject Classification: 16S90; 16D90.

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INTRODUCTION

Along the years, the study of the Morita invariance has been a present question. In this paper, we study the Morita-invariance of properties closely related to the rings of quotients such as left nonsingularity and left nonsingularity plus finite left local Goldie dimension for idempotent rings without total left or right zero divisors (Sec. 4). We will get these results ((4.3) and (4.7)) as an application of the theory of associative pairs of quotients we develop.

Associative pairs and Morita contexts are closely related since every Morita context $(\mathcal{R}, \mathcal{S}, M, N)$ gives rise to an associative pair: (M, N) and, conversely, given an associative pair $A = (A^+, A^-)$, there exists a unital ring $\mathcal{E}$ with an idempotent e such that if $\mathcal{E} = e\mathcal{E}e \oplus (1-e)\mathcal{E}(1-e) \oplus e\mathcal{E}(1-e) \oplus (1-e)\mathcal{E}e$ is the Peirce decomposition of $\mathcal{E}$ relative to e, then A is isomorphic to $(e\mathcal{E}(1-e), (1-e)\mathcal{E}e)$ (and, of course, $(e\mathcal{E}e, (1-e)\mathcal{E}(1-e), e\mathcal{E}(1-e), (1-e)\mathcal{E}e)$ is a Morita context). This ring $\mathcal{E}$ is called the standard imbedding of A. Associative pairs play also a fundamental role in the new approach to Zelmanov's classification of strongly prime Jordan pairs (see D'Amour, 1992), and have been already used by Loos in the classification of the nondegenerate Jordan pairs of finite capacity (see Loos, 1975).

After a preliminary paragraph on associative pairs, in the first section we study the left singular ideal of an associative pair: we prove that, in general, it is a pair of two-sided ideals, and an ideal when the associative pair has no total right zero divisors (this is the case, for example, if the pair is semiprime). Nonsingularity of an associative pair A (i.e., $Z_l(A) = 0$, where $Z_l(A)$ denotes the left singular ideal of A) is equivalent to that of its standard imbedding.

In Sec. 2, we introduce the notion of left quotient pair of a pair, extending the well-known definition of left quotient ring given by Utumi (1956). We connect properties of a pair with those of its left quotient pairs, and find the maximal left quotient pair of every associative pair which has neither total left nor total right zero divisors, or it is left nonsingular.

Section 3 is devoted to Johnson and Gabriel's Theorems for associative pairs (see Johnson's Theorem for pairs and (3.3)). As a corollary we obtain the Gabriel's Theorem for associative pairs.

0. PRELIMINARIES

An *associative pair* over a unital commutative associative ring of scalars Φ is a couple (A^+, A^-) of Φ-modules together with Φ-trilinear maps

$$A^\sigma \times A^{-\sigma} \times A^\sigma \to A^\sigma$$
$$(x, y, z) \mapsto xyz$$

satisfying the following identities:

$$uv(xyz) = u(vxy)z = (uvx)yz \tag{0.1}$$

for all $u, x, z \in A^\sigma$, $y, v \in A^{-\sigma}$ and $\sigma = \pm$. (See Fernández López et al., 1998 and Loos, 1995 for definitions and results on associative pairs.)

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By Loos (1995, (2.3)) given an associative pair $A = (A^+, A^-)$, there exists a unital associative algebra $\mathscr{E}$ with an idempotent e such that if $\mathscr{E}_{11} \oplus \mathscr{E}_{12} \oplus \mathscr{E}_{21} \oplus \mathscr{E}_{22}$ is the Peirce decomposition of $\mathscr{E}$ relative to e, that is, $\mathscr{E}_{ij} = u_i \mathscr{E} u_j$, with $u_1 = e$ and $u_2 = 1 - e$, then A is isomorphic to the associative pair $(\mathscr{E}_{12}, \mathscr{E}_{21})$, where $\mathscr{E}_{11}$ (resp. $\mathscr{E}_{22}$) is spanned by e and all products $x_{12}y_{21}$ (resp. $1 - e$ and all products $y_{21}x_{12}$) for $x_{12} \in \mathscr{E}_{12}$, $y_{21} \in \mathscr{E}_{21}$, and has the following property:

$$x_{11}\mathscr{E}_{12} = \mathscr{E}_{21}x_{11} = 0 \text{ implies } x_{11} = 0, \quad \text{and}$$
$$x_{22}\mathscr{E}_{21} = \mathscr{E}_{12}x_{22} = 0 \text{ implies } x_{22} = 0. \tag{0.2}$$

The pair $(\mathscr{E}, e)$ is called the *standard imbedding* of A.

Now let $\mathscr{A}$ be the subalgebra of $\mathscr{E}$ generated by the elements x_{12} and x_{21}. It is immediate that $\mathscr{A}$ is an ideal of $\mathscr{E}$. We will call $\mathscr{A}$ the *envelope* of the associative pair A. Notice that $\mathscr{A}_{12} := e\mathscr{A}(1 - e) = \mathscr{E}_{12}$ and $\mathscr{A}_{21} := (1 - e)\mathscr{A}e = \mathscr{E}_{21}$, hence the associative pair A is isomorphic to the associative pair $(\mathscr{A}_{12}, \mathscr{A}_{21})$. It is not difficult to see, by using (0.2), that $\mathscr{A}$ is an essential ideal of $\mathscr{E}$.

In what follows, an expression of the type xy, with $x \in A^\sigma$, $y \in A^{-\sigma}$, must be understood by considering the associative pair A inside its envelope $\mathscr{A}$. And $xy = 0$ means $\lambda_A(x, y) = 0$ and $\rho_A(x, y) = 0$, where $\lambda(x, y)z = \rho(y, z)x = xyz$.

1. THE LEFT SINGULAR IDEAL OF AN ASSOCIATIVE PAIR

The notion of left singular ideal for semiprime associative pairs was introduced by the authors jointly with Fernández López and García Rus in Fernández López et al. (1998), where it was used as a tool to study Fountain–Gould orders in associative pairs. Here we analyze some properties of this left singular ideal without the semiprimeness hypothesis.

We recall that a nonzero element a in a ring $\mathscr{R}$ is said to be a *total right zero divisor* if $\mathscr{R}a = 0$.

1.1. Definitions. Let A be an associative pair. We will say that an element $a \in A^\sigma$ is a *total right zero divisor* if a is nonzero and $A^\sigma A^{-\sigma} a = 0$. When A^+ and A^- have no total right zero divisors we will say that the associative pair *A has no total right zero divisors.*

These definitions are consistent with the classical ones because a right zero divisor a in a ring $\mathscr{R}$ is a right zero divisor in the associative pair $(\mathscr{R}, \mathscr{R})$. Moreover $\mathscr{R}$ has no total right zero divisors if and only if $(\mathscr{R}, \mathscr{R})$ has no total right zero divisors.

If A is an associative pair which has no total right zero divisors then, for every subset X of A^σ,

$$lan_A(X) = \{b \in A^{-\sigma} : A^\sigma b X = 0\}. \tag{1.2}$$

Indeed, if $b \in A^{-\sigma}$ satisfies $A^\sigma bx = 0$ for every $x \in X$, then $bxA^{-\sigma}$ must be zero. Otherwise, suppose $bxd \neq 0$ for some $d \in A^{-\sigma}$. Since A has no total right zero divisors, then there exist $u \in A^\sigma$, $v \in A^{-\sigma}$ satisfying $0 \neq vubxd \in vA^\sigma bxA^{-\sigma} = 0$, a contradiction.

If moreover A has no total left zero divisors (for example, if it is a semiprime pair),

$$lan_A(X) = \{b \in A^{-\sigma} : bXA^{-\sigma} = 0\}. \quad (1.3)$$

1.4. Lemma. *If an associative pair A has no total right zero divisors, then* (0.2) *is equivalent to*:

$$\mathscr{A}_{21}x_{11} = 0 \text{ implies } x_{11} = 0, \quad \text{and}$$
$$\mathscr{A}_{12}x_{22} = 0 \text{ implies } x_{22} = 0.$$

Proof. Clearly, this conditions imply (0.2). Conversely, $\mathscr{A}_{21}x_{11} = 0$ and $x_{11} \neq 0$ implies, by (0.2), $x_{11}a_{12} \neq 0$ for some $a_{12} \in \mathscr{A}_{12}$. Since A has no total right zero divisors, then $0 \neq \mathscr{A}_{12}\mathscr{A}_{21}x_{11}a_{12}$, which is a contradiction. □

A left ideal $\mathscr{L}$ of a ring $\mathscr{R}$ is *dense* in $\mathscr{R}$ if for every $x, y \in \mathscr{R}$, with $x \neq 0$, there exists an element $a \in \mathscr{R}$ such that $ay \in \mathscr{L}$ and $ax \neq 0$. By Utumi (1956, (1.6)), this condition is equivalent to say that $\mathscr{R}$ is a left quotient ring of $\mathscr{L}$.

1.5. Lemma. *Let A be an associative pair and write $\mathscr{A}$ and $\mathscr{E}$ to denote the envelope and the standard imbedding, respectively, of A. Then, the following conditions are equivalent*:

(i) *A has no total right zero divisors.*
(ii) *$\mathscr{A}$ has no total right zero divisors.*
(iii) *$\mathscr{A}$ is a dense left ideal of $\mathscr{E}$.*

Proof. (i) ⇒ (iii) Let x and y be elements in $\mathscr{E}$ with $x \neq 0$. Suppose first that $x_{12} \neq 0$. By the hypothesis there exists $(a_{12}, a_{21}) \in A$ such that $a_{12}a_{21}x_{12} \neq 0$. Then $a_{21}x \neq 0$ (otherwise $0 = a_{21}x = a_{21}x_{11} + a_{21}x_{12}$ implies $a_{21}x_{11} = a_{21}x_{12} = 0$, a contradiction) and $a_{21}y \in \mathscr{A}$ since $\mathscr{A}$ is an ideal of $\mathscr{E}$. The case $x_{21} \neq 0$ is analogue. Now, suppose $x_{12} = x_{21} = 0$. In this case x_{11} (or x_{22}) must be nonzero and by (1.4), $b_{21}x_{11} \neq 0$ for some $b_{21} \in \mathscr{A}_{21}$. Then, as it is easy to show, $b_{21}x \neq 0$ and $b_{21}y \in \mathscr{A}$.

(iii) ⇒ (ii) follows by Utumi (1956, (1.6)).

(ii) ⇒ (i) Take $0 \neq a_{12} \in \mathscr{A}_{12}$. By the hypothesis, there exist $b, c \in \mathscr{A}$ such that $bca_{12} \neq 0$. This implies $\mathscr{A}_{12}\mathscr{A}_{21}a_{12} \neq 0$. □

For an associative pair A, it is defined

$$Z_l(A)^\sigma = \{z \in A^\sigma : lan(z) \subseteq A^{-\sigma} \text{ is an essential left ideal of} A\}.$$

The "moreover" part of the following lemma can be obtained as a corollary of (1.9). However we include here a direct proof of the result.

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1.6. Lemma. *For an associative pair A we have that* $Z_l(A) = (Z_l(A)^+, Z_l(A)^-)$ *is a pair of two-sided ideals of A. Moreover, if A has no total right zero divisors, then* $Z_l(A)$ *is an ideal of A. In particular,* $Z_l(A)$ *is an ideal of A if A is a semiprime associative pair.*

Proof. Being $Z_l(A)$ a pair of two-sided ideals of A follows the ideas of the proof of Theorem 3.1 in Fernández López et al. (1998). Now, suppose that A has no total right zero divisors. Let $x, y \in A^\sigma$ and $z \in Z_l(A)^{-\sigma}$, and take a nonzero element l in a nonzero left ideal L of A contained in $A^{-\sigma}$. If $A^\sigma lx = 0$, then $A^{-\sigma}A^\sigma lxzy = 0$ and since A has no total right zero divisors, this implies $0 \neq A^{-\sigma}A^\sigma l \subseteq L \cap lan(xzy)$. If $A^\sigma lx \neq 0$, since $lan(z)$ is an essential left ideal of A, there exists $0 \neq alx \in A^\sigma lx \cap lan(z)$. Apply that A has no total right zero divisors to find $b \in A^\sigma$, $c \in A^{-\sigma}$ such that $bcalx \neq 0$. Then $0 \neq cal \in L \cap lan(xzy)$. In any case $L \cap lan(xzy) \neq 0$, so $lan(xzy)$ is essential, which completes the proof. □

1.7. Definitions. Given an associative pair A, the pair $Z_l(A) = (Z_l(A)^+, Z_l(A)^-)$ of two-sided ideals of A will be called the *left singular* (*two − sided*) *ideal* of A. An associative pair $A = (A^+, A^-)$ will be called *left nonsingular* if its left singular ideal $Z_l(A)$ is zero. *Right nonsingular* associative pairs are defined similarly, while *nonsingular* means that A is both left and right nonsingular.

1.8. Lemma. *Let A be a left nonsingular associative pair. Then A has no total right zero divisors.*

Proof. We prove first the following property:

(1) For every nonzero $x_{ii} \in \mathscr{A}_{ii}$ we have $\mathscr{A}_{ji}x_{ii} \neq 0$ (for $i \neq j$).

Suppose $\mathscr{A}_{ji}x_{ii} = 0$ for some nonzero $x_{ii} \in \mathscr{A}_{ii}$. Then $\mathscr{A}_{ji} = lan_A(x_{ii}\mathscr{A}_{ij})$. Since $x_{ii} \neq 0$, by (0.2), $x_{ii}\mathscr{A}_{ij} \neq 0$ and we have just proved that $x_{ii}\mathscr{A}_{ij} \subseteq Z_l(A)^\sigma$, for $\sigma = +$ or $\sigma = -$, contrary to the hypothesis.

Now, let a_{12} be an element of $\mathscr{A}_{12}$ such that $\mathscr{A}_{12}\mathscr{A}_{21}a_{12} = 0$; then, by (1), $\mathscr{A}_{21} = lan_A(a_{12})$, which implies $a_{12} \in Z_l(A)^+ = 0$ and proves that A has no total right zero divisors. □

1.9. Proposition. *Let A be an associative pair without total right zero divisors and denote by* $\mathscr{A}$ *and* $\mathscr{E}$ *its envelope and standard imbedding, respectively. Then*

$$Z_l(A)^\sigma = Z_l(\mathscr{E}) \cap A^\sigma = Z_l(\mathscr{A}) \cap A^\sigma,$$

and the following are equivalent conditions.

(i) *A is left nonsingular.*
(ii) $\mathscr{A}$ *is left nonsingular.*
(iii) $\mathscr{E}$ *is left nonsingular.*

Proof. (1) $Z_l(\mathscr{E}) \cap A^\sigma \subseteq Z_l(A)^\sigma$.

Suppose $x \in Z_l(\mathscr{E}) \cap \mathscr{A}_{12}$ and take a nonzero left ideal L of A contained in $\mathscr{A}_{21}$. If $Lx = 0$ then $L \subseteq lan_A(x)$. If $l_{21}x \neq 0$ for some $l_{21} \in L$, since $\mathscr{E}l_{21}$ is a nonzero left ideal of $\mathscr{E}$ and $lan_{\mathscr{E}}(x)$ is an essential left ideal of $\mathscr{E}$, there exists $0 \neq ul_{21} \in \mathscr{E}l_{21} \cap lan_{\mathscr{E}}(x)$. Write $ul_{21} = u_{12}l_{21} + u_{22}l_{21}$. If $u_{22}l_{21} \neq 0$, then we have a nonzero element in $L \cap lan_A(x)$. If $u_{12}l_{21} \neq 0$, by (1.4), $0 \neq \mathscr{A}_{21}u_{12}l_{21} \subseteq L \cap lan_A(x)$. Anyway, $lan_A(x)$ is an essential left ideal of A contained in A^-, which implies $x \in Z_l(A)^+$.

(2) $Z_l(A)^\sigma \subseteq Z_l(\mathscr{A}) \cap A^\sigma$.

Let x be in $Z_l(A)^+$. Take a nonzero left ideal $\mathscr{L}$ of $\mathscr{A}$. If $\mathscr{L}x = 0$ then $\mathscr{L} \subseteq lan_{\mathscr{A}}(x)$. Suppose $\mathscr{L}x \neq 0$ and take $l \in \mathscr{L}$ such that $lx \neq 0$. If $l_{21} \neq 0$, then $\mathscr{E}_{22}l_{21}$ is a nonzero left ideal of A contained in A^-; applying $x \in Z_l(A)^+$ we find $u_{22} \in \mathscr{E}_{22}$ such that $0 \neq u_{22}l_{21} \in lan_A(x)$ and this implies $u_{22}lx = u_{22}l_{21}x = 0$, so $0 \neq u_{22}l \in \mathscr{L} \cap lan_{\mathscr{A}}(x)$. If $l_{11} \neq 0$, by (1.5), $\mathscr{A}l_{11} \neq 0$, so there exists $a_{21} \in \mathscr{A}_{21}$ such that $0 \neq a_{21}l_{11}$, and the element $a_{21}l$ satisfies the conditions of the previous case.

(3) $Z_l(\mathscr{A}) = Z_l(\mathscr{E}) \cap \mathscr{A}$.

This follows by Proposition 3.2(ii) of Gómez Lozano and Siles Molina (2002) since by (1.5), $\mathscr{A}$ is a dense ideal of $\mathscr{E}$.

By (1), (2) and (3), $Z_l(A)^\sigma \subseteq Z_l(\mathscr{A}) \cap A^\sigma = Z_l(\mathscr{E}) \cap A^\sigma \subseteq Z_l(A)^\sigma$ and the first statement has been proved.

(i) $\Rightarrow$ (iii) Consider $x \in Z_l(\mathscr{E})$. Then $x_{12} \in Z_l(\mathscr{E}) \cap A^+ = Z_l(A)^+ = 0$ and analogously $x_{21} = 0$. If $x_{ii} \neq 0$, by (1.8) and (1.4), $0 \neq \mathscr{A}_{ji}x_{ii} \subseteq Z_l(\mathscr{E}) \cap \mathscr{A}_{ji} = Z_l(A)^\sigma = 0$, which is a contradiction.

(ii) $\Leftrightarrow$ (iii) follows by Proposition 3.2 (iii) of Gómez Lozano and Siles Molina (2002), taking into account (1.5).

(ii) $\Rightarrow$ (i) follows since $Z_l(A)^\sigma = Z_l(\mathscr{A}) \cap A^\sigma$. □

1.10. Proposition. *Let A be an associative pair without total right zero divisors and denote by $\mathscr{A}$ and $\mathscr{E}$ its envelope and standard imbedding, respectively. Then*

$$Z_l(\mathscr{A}_{ii}) = Z_l(\mathscr{E}) \cap \mathscr{A}_{ii} = Z_l(\mathscr{A}) \cap \mathscr{A}_{ii},$$

and the following are equivalent conditions.

(i) *$\mathscr{A}_{11}$ and $\mathscr{A}_{22}$ are left nonsingular.*
(ii) *$\mathscr{E}$ is left nonsingular.*

Moreover, if A has no total left zero divisors, then

(iii) *$\mathscr{A}_{11}$ is left nonsingular if and only if $\mathscr{A}_{22}$ is left nonsingular.*

Proof. $Z_l(\mathscr{E}) \cap \mathscr{A}_{11} \subseteq Z_l(\mathscr{A}_{11})$: Take $x_{11} \in Z_l(\mathscr{E}) \cap \mathscr{A}_{11}$. We will see that $lan_{\mathscr{A}_{11}}(x_{11})$ is an essential left ideal of $\mathscr{A}_{11}$. Consider a nonzero element y in an ideal I of $\mathscr{A}_{11}$. Then $\mathscr{E}y$ is a nonzero ideal of $\mathscr{E}$ and there exists $0 \neq uy \in lan_{\mathscr{E}}(x_{11}) \cap \mathscr{E}y$. If $0 \neq euy \in lan_{\mathscr{A}_{11}}(x_{11}) \cap I$ we have finished. Otherwise, $0 \neq uy = (1-e)uy$ and since A has no total right zero divisors, $0 \neq \mathscr{A}_{12}uy \subseteq lan_{\mathscr{A}_{11}}(x_{11}) \cap I$.

$Z_l(\mathscr{A}_{11}) \subseteq Z_l(\mathscr{A}) \cap \mathscr{A}_{11}$: Consider $x \in Z_l(\mathscr{A}_{11})$. We want to prove that $lan_{\mathscr{A}}(x_{11})$ is an essential left ideal of $\mathscr{A}$. Take a nonzero left ideal I of $\mathscr{A}$. If $Ie = 0$ then $I \subseteq lan_{\mathscr{A}}(x_{11})$. If $Ie \neq 0$ we have $eIe \neq 0$ (suppose $(1-e)Ie \neq 0$; apply that A has no total right zero divisors to obtain $0 \neq \mathscr{A}_{12}Ie \cap lan_{\mathscr{A}_{11}}(x_{11})$, which implies $0 \neq \mathscr{A}_{12}I \cap lan_{\mathscr{A}_{11}}(x_{11}) \subseteq I \cap lan_{\mathscr{A}}(x_{11})$) and hence $0 \neq eIe \cap lan_{\mathscr{A}_{11}}(x_{11})$, which implies $0 \neq eI \cap lan_{\mathscr{A}}(x_{11}) \subseteq I \cap lan_{\mathscr{A}}(x_{11})$.

Now, notice that $Z_l(\mathscr{A}_{ii}) \subseteq Z_l(\mathscr{A}) \cap \mathscr{A}_{ii} =$ (by Gómez Lozano and Siles Molina (2002, 3.2(ii)) in the proof of (1.9)) $Z_l(\mathscr{E}) \cap \mathscr{A}_{ii} \subseteq Z_l(\mathscr{A}_{ii})$, which proves the first statement.

(i) $\Leftrightarrow$ (ii) If $\mathscr{E}$ is left nonsingular, by (1.9), A is left nonsingular. Then $Z_l(\mathscr{A}_{11}) = Z_l(\mathscr{A}_{22}) = 0$. Conversely, suppose $Z_l(\mathscr{A}_{ii}) = 0$ for $i, j = 1, 2, i \neq j$. Since $Z_l(\mathscr{E})$ is an ideal, $x_{ij} \in Z_l(\mathscr{E})$, and by the first statement, $x_{ii} \in Z_l(\mathscr{A}_{ii})$, for $i = 1, 2$. Now, since A has no total right zero divisors, if $x_{ij} \neq 0$, $\mathscr{A}_{ji}x_{ij} \neq 0$. Hence $0 \neq \mathscr{A}_{ji}x_{ij} \neq Z_l(\mathscr{E}) \cap \mathscr{A}_{jj} = Z_l(\mathscr{A}_{jj}) = 0$, a contradiction.

(iii) Suppose $\mathscr{A}_{11}$ left nonsingular, and consider $0 \neq x_{22} \in Z_l(\mathscr{A}_{22})$. By (1.4) $\mathscr{A}_{12}x_{22} \neq 0$ and since A has no total left zero divisors, $0 \neq \mathscr{A}_{12}x_{22}\mathscr{A}_{21}\mathscr{A}_{12}$. This means $0 \neq \mathscr{A}_{12}x_{22}\mathscr{A}_{21} \subseteq \mathscr{A}_{12}Z_l(\mathscr{A}_{22})\mathscr{A}_{21} \subseteq$ (by the previous statement) $Z_l(\mathscr{E}) \cap \mathscr{A}_{11} = Z_l(\mathscr{A}_{11}) = 0$, which is not possible. Hence $Z_l(\mathscr{A}_{22}) = 0$. □

2. LEFT QUOTIENT PAIRS

The notion of left quotient ring was introduced by Utumi (1956) and has proved to be very useful in order to study Fountain–Gould left orders in rings (see Gómez Lozano and Siles Molina, 2002 and the related references therein). Let $\mathscr{R}$ be a subring of a ring $\mathscr{Q}$. We say that $\mathscr{Q}$ is a (general) *left quotient ring* of $\mathscr{R}$ if for every $x, y \in \mathscr{Q}$, with $x \neq 0$, there is an $a \in \mathscr{R}$ such that $ax \neq 0$ and $ay \in \mathscr{R}$. Notice that a ring is a left quotient ring of itself if and only if it has no total right zero divisors. If $\mathscr{R}$ has no total right zero divisors, then by Utumi (1956) it has a unique maximal left quotient ring, which is unital, called the *Utumi left quotient ring* of $\mathscr{R}$.

2.1. Definition. Let $A = (A^+, A^-)$ be a subpair of an associative pair $Q = (Q^+, Q^-)$. We say that Q is a *left quotient pair* of A if given $p, q \in Q^\sigma$ with $p \neq 0$ (and $\sigma = +$ or $\sigma = -$) there exist $a \in A^\sigma$, $b \in A^{-\sigma}$ such that $abp \neq 0$ and $abq \in A^\sigma$.

For example, the associative pair $(\mathscr{M}_{1\times 2}(\mathbb{Q}), \mathscr{M}_{2\times 1}(\mathbb{Q}))$ is a left quotient pair of the associative pair $(\mathscr{M}_{1\times 2}(4\mathbb{Z}), \mathscr{M}_{2\times 1}(8\mathbb{Z}))$. Moreover, it is maximal among the left quotient pairs of $(\mathscr{M}_{1\times 2}(4\mathbb{Z}), \mathscr{M}_{2\times 1}(8\mathbb{Z}))$. Every associative pair without total right zero divisors is a left quotient pair of itself.

The notion of left quotient pair extends that of Utumi of left quotient ring since given a subring $\mathscr{R}$ of an associative ring $\mathscr{Q}$, $\mathscr{Q}$ is a left quotient ring of $\mathscr{R}$ if and only if $Q = (\mathscr{Q}, \mathscr{Q})$ is a left quotient pair of the associative pair $R = (\mathscr{R}, \mathscr{R})$.

The following lemma will be used in the sequel although without mentioning it.

2.2. Lemma. *Let $Q = (Q^+, Q^-)$ be a left quotient pair of an associative pair $A = (A^+, A^-)$. Then, given $q_1, \ldots, q_n \in Q^\sigma$ with $q_1 \neq 0$ ($\sigma = +$ or $\sigma = -$), there exist $a \in A^\sigma$, $b \in A^{-\sigma}$ such that $abq_1 \neq 0$ and $abq_i \in A^\sigma$ for all $i \in \{1, \ldots, n\}$.*

Proof. The case $n = 1$ follows from the definition. Suppose the result is true for $n-1$. By the induction assumption there exist $x \in A^\sigma$, $y \in A^{-\sigma}$ such that $xyq_1 \neq 0$ and $xyq_i \in A^\sigma$ for $i \in \{1, \ldots, n-1\}$. Given $xyq_1 \neq 0$ and xyq_n, there exist $z \in A^\sigma$, $t \in A^{-\sigma}$ such that $ztxyq_1 \neq 0$ and $ztxyq_n \in A^\sigma$. Take $a = ztx \in A^\sigma$ and $b = y \in A^{-\sigma}$ to complete the proof. □

2.3. Let Q be an associative pair which is a left quotient pair of a subpair A. Then, it is not difficult to see that for any finite family $\{(p_i, q_i)\}_{i=1}^n$ in $(Q^\sigma, Q^{-\sigma})$, we have that $(A^\sigma : \sum_{i=1}^n q_i p_i) = \{x \in A^\sigma \mid x \sum_{i=1}^n q_i p_i \in A^\sigma\}$ is an essential left ideal of A.

2.4. Lemma. *Let Q be an associative pair which is a left quotient pair of a subpair A. If A has no total left zero divisors (for example, if it is semiprime) or it is left nonsingular, then for any finite family $\{(p_i, q_i)\}_{i=1}^n$ in $(Q^\sigma, Q^{-\sigma})$ and every nonzero $a \in A^{-\sigma}$ we have:*

$$\left(A^\sigma : \sum_{i=1}^n q_i p_i\right) a \neq 0.$$

Proof. Suppose first A without total left zero divisors. Then, given the nonzero element a, and applying that A has no total right zero divisors, there exist $b, c \in A^\sigma$ satisfying $cab \neq 0$. Apply that Q is a left quotient pair of A to find $x \in A^\sigma$, $y \in A^{-\sigma}$ such that $xycab \neq 0$ and $xyc \sum_{i=1}^n q_i p_i \in A^\sigma$. Since the element xyc is in $(A^\sigma : \sum_{i=1}^n q_i p_i)$ and $xyca \neq 0$, we have finished.

Now, suppose A left nonsingular. If $(A^\sigma : \sum_{i=1}^n q_i p_i)a = 0$, then $(A^\sigma : \sum_{i=1}^n q_i p_i) \subseteq lan_A(a)$, which implies, by (2.3), $0 \neq a \in Z_l(A)^{-\sigma}$, a contradiction. □

2.5. Proposition. *Let A be a subpair of an associative pair Q, and write $\mathscr{A}$ and $\mathscr{Q}$ to denote the envelopes of A and Q, respectively.*

(i) *If Q is a left quotient pair of A, and $(\mathscr{E}, e)$ and $(\mathscr{E}', e')$ are the standard imbeddings of A and Q, then $\mathscr{E} \subseteq \mathscr{E}'$, $e = e'$ and $1_{\mathscr{E}} = 1_{\mathscr{E}'}$.*
(ii) *If $\mathscr{Q}$ is a left quotient ring of $\mathscr{A}$, then Q is a left quotient pair of A.*

Suppose that A has no total left zero divisors or it is left nonsingular.

(iii) *If Q is a left quotient pair of A, then $\mathscr{Q}$ is a left quotient ring of $\mathscr{A}$.*

Proof. (i) By the construction of $\mathscr{E}$, $\mathscr{E}_{11}$ is the subalgebra of $End_\phi(A) \times End_\phi(A^-)^{op}$ generated by $\{(\lambda_A(x, y), \rho_A(x, y)) \mid x \in A^+, y \in A^-\}$ and (Id_{A^+}, Id_{A^-}), where the index A, A^+ or A^- under each operator means where it acts. Analogously we have that $\mathscr{E}'_{11}$ is the subalgebra of $End_\phi(Q^+)\times, End_\phi(Q^-)^{op}$ generated by $\{(\lambda_Q(x, y), \rho_Q(x, y)) \mid x \in Q^+, y \in Q^-\}$ and (Id_{Q^+}, Id_{Q^-}). Hence, to prove $\mathscr{E} \subseteq \mathscr{E}'$, and since $A^+ = \mathscr{E}_{12} \subseteq \mathscr{E}'_{12} = Q^+$, it is enough to show that for every $n \in \mathbb{N} \cup \{0\}$, $x_i \in A^+$, $y_i \in A^-$,

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the map from $\mathscr{E}_{11}$ into $\mathscr{E}'_{11}$ which sends $n(Id_{A^+}, Id_{A^-}) + (\sum \lambda_A(x_i, y_i), \sum \rho_A(x_i, y_i))$ to $n(Id_{Q^+}, Id_{Q^-}) + (\sum \lambda_Q(x_i, y_i), \sum \rho_Q(x_i, y_i))$ is well-defined. Or, equivalently, that

$$n(Id_{A^+}, Id_{A^-}) + \left(\sum \lambda_A(x_i, y_i), \sum \rho_A(x_i, y_i)\right) = 0 \text{ implies}$$
$$n(Id_{Q^+}, Id_{Q^-}) + \left(\sum \lambda_Q(x_i, y_i), \sum \rho_Q(x_i, y_i)\right) = 0. \qquad (*)$$

Suppose $n(Id_{Q^+}, Id_{Q^-}) + (\sum \lambda_Q(x_i, y_i), \sum \rho_Q(x_i, y_i)) \neq 0$. Then, for some element $(p, q) \in Q$, $(np + \sum x_i y_j p, nq + \sum q x_i y_i) \neq 0$, so either $np + \sum x_i y_j p \neq 0$ or $nq + \sum q x_i y_i \neq 0$. In the first case, since Q is a left quotient pair of A, there exists $(a, b) \in A$ satisfying $0 \neq nabp + \sum abx_i y_i p = a\{[nId_{A^-} + \sum \rho_A(x_i, y_i)]b\}p$. This implies $nId_{A^-} + \sum \rho_A(x_i, y_i) \neq 0$. If $nq + \sum q x_i y_i \neq 0$, apply that Q is a left quotient pair of A to find $(v, u) \in (A^-, A^+)$ such that $vuq \in A^-$ and $0 \neq nvuq + \sum vuqx_i y_i = (nId_{A^-} + \sum \rho_A(x_i, y_i))vuq$. Hence $nId_{A^-} + \sum \rho_A(x_i, y_i) \neq 0$, and $(*)$ has been proved. For $\mathscr{E}_{22}$ we can reasoning analogously. □

Notice that with this reasoning we have:

2.6. If p_{ii} is a nonzero element of $\mathscr{Q}_{ii}$, then $\mathscr{A}_{ji} p_{ii} \neq 0$, for $i \neq j$, $i, j \in \{1, 2\}$.

Now we will see $e = e'$. Since $\mathscr{A}_{21}(e - e') = 0$, by (2.6), $e = e'$. Analogously we can prove $1_{\mathscr{E}} - e = 1_{\mathscr{E}'} - e'$, which leads to $1_{\mathscr{E}} = 1_{\mathscr{E}'}$.

(ii) Take $p_{12}, q_{12} \in \mathscr{Q}_{12}$ with $p_{12} \neq 0$. By the hypothesis there exists $b \in \mathscr{A}$ such that $bp_{12} \neq 0$ and $bp_{12}, bq_{12} \in \mathscr{A}$. This implies $b_{11}p_{12} \neq 0$ or $b_{21}p_{12} \neq 0$. Suppose first $0 \neq b_{11}p_{12} \in \mathscr{A}_{12}$. By (1.5), A has no total right zero divisors, hence there exists $(c_{12}, c_{21}) \in A$ satisfying $c_{12}c_{21}b_{11}p_{12} \neq 0$. If we denote $d_{21} = c_{21}b_{11}$, then $c_{12}d_{21}p_{12} \neq 0$ and $c_{12}d_{21}q_{12} \in \mathscr{A}_{12}$.

Now, consider $0 \neq b_{21}p_{12} \in \mathscr{A}_{22}$. By (1.5) and (1.4), there exists $c_{12} \in \mathscr{A}_{12}$ such that $c_{12}b_{21}p_{12} \neq 0$. Moreover $c_{12}b_{21}q_{12} = c_{12}bq_{12} \in \mathscr{A}_{12}$.

(iii) Let p and q be in $\mathscr{Q}$ with $p \neq 0$. We distinguish two cases. Suppose first $p_{11} \neq 0$ (the case $p_{22} \neq 0$ is analogue). By (2.6) there exists $a_{21} \in \mathscr{A}_{21}$ such that $a_{21}p_{11} \neq 0$. Apply the hypothesis to find $b_{12} \in \mathscr{A}_{12}$, $b_{21} \in \mathscr{A}_{21}$ satisfying $b_{21}b_{12}a_{21}p_{11} \neq 0$ and $b_{21}b_{12}a_{21}p_{11}$, $b_{21}b_{12}a_{21}q_{11} \in \mathscr{A}_{21}$. Write $c_{21} = b_{21}b_{12}a_{21}$. By (2.4), $(\mathscr{A}_{12} : c_{21}q_{12})c_{21}p_{11} \neq 0$. Let c_{12} be in $(\mathscr{A}_{12} : c_{21}q_{12})$ satisfying $c_{12}c_{21}p_{11} \neq 0$. Then the element $d = c_{12}c_{21} \in \mathscr{A}$ verifies: $dp = dp_{11} + dp_{12} \neq 0$ (since $dp_{11} \neq 0$) and $dq \in \mathscr{A}$.

Now, suppose $p_{11} = p_{22} = 0$. In this case p_{12} or p_{21} must be nonzero. Consider, for example, $p_{12} \neq 0$. Apply that Q is a left quotient pair of A to find $a_{21} \in \mathscr{A}_{21}$ such that $a_{21}p_{12} \neq 0$. Then, given $a_{21}p_{12}$ and $a_{21}q$, by the previous case there exists $b \in \mathscr{A}$ such that $ba_{21}p = ba_{21}p_{12} \neq 0$ and $ba_{21}q \in \mathscr{A}$, which concludes the proof. □

2.7. Example. There exist two associative pairs A and Q such that $A \subseteq Q$ but there is no ring monomorphisms $f : \mathscr{A} \to \mathscr{Q}$ such that $f(a) = a$ for every $a \in A^\sigma$, where $\mathscr{A}$ and $\mathscr{Q}$ denote the envelopes of A and Q, respectively.

Proof. Consider a ring $\mathscr{R}$ such that $\mathscr{R}^5 \neq 0$ but $\mathscr{R}^6 = 0$ and define $A = (\mathscr{R}^2, \mathscr{R}^2)$, $Q = (\mathscr{R}, \mathscr{R})$. It is clear that $A \subseteq Q$. If $f : \mathscr{A} \to \mathscr{Q}$ is a ring monomorphism such that $f(a) = a$ for every $a \in A^\sigma$, then $\lambda_A(a, b) = 0$ implies $\lambda_Q(a, b) = 0$, with $(a, b) \in A$.

But this is not the case: Choose $a, x, y, z, t \in \mathscr{R}$ such that $axyzt \neq 0$ and define $b = xyz$. Then $\lambda_A(a,b)A^\sigma = ab\mathscr{R}^2 \subseteq \mathscr{R}^6 = 0$, and $0 \neq abt \in \lambda_Q(a,b)Q^\sigma$. □

2.8. Lemma. *If Q is a left quotient pair of a subpair A, and $(\mathscr{E}', e')$ and $(\mathscr{E}, e')$ are the standard imbeddings of Q and A, respectively, then $\mathscr{E}'_{11} := e'\mathscr{E}'e'$ is a left quotient ring of $\mathscr{A}_{11} := e'\mathscr{A}e'$, where $\mathscr{A}$ denotes the envelope of A.*

Proof. By (2.5) (i), $\mathscr{E} \subseteq \mathscr{E}'$, so $\mathscr{A}_{11} \subseteq \mathscr{E}'_{11}$. Consider $p, q \in \mathscr{E}'_{11}$, with $p \neq 0$. By (2.6), $a_{21}p \neq 0$ for some $a_{21} \in \mathscr{A}_{21}$. By the hypothesis there exist $b_{12} \in \mathscr{A}_{12}$, $c_{21} \in \mathscr{A}_{21}$ such that $c_{21}b_{12}a_{21}p \neq 0$ and $c_{21}b_{12}a_{21}p, c_{21}b_{12}a_{21}q \in \mathscr{A}_{21}$. Since A has no total right zero divisors, there exists $d_{12} \in \mathscr{A}_{12}$ such that $d_{12}c_{21}b_{12}a_{21}p \neq 0$. If we denote $x = d_{12}c_{21}b_{12}a_{21} \in \mathscr{A}_{11}$, then $xp \neq 0$ and $xq \in \mathscr{A}_{11}$. □

2.9. Theorem. *Let A be an associative pair, and denote by $\mathscr{A}$ and $(\mathscr{E}, e)$ its envelope and standard imbedding, respectively.*

(i) *If A has no total right zero divisors, then $\mathscr{E}$ is a left quotient ring of $\mathscr{A}$ and, consequently, $\mathscr{A}$ and $\mathscr{E}$ have the same Utumi left quotient ring. Denote it by $\mathscr{Q}$.*

(ii) *If A has neither total left zero divisors nor total right zero divisors, or it is left nonsingular, then $Q := (e\mathscr{Q}(1-e), (1-e)\mathscr{Q}e)$ is a left quotient pair of A and given a left quotient pair T of A there exists a monomorphism of associative pairs $f : T \to Q$ which is the identity on A.*

Proof. (i) By (1.5), $\mathscr{A}$ is a dense ideal of $\mathscr{E}$ and by Exercise 13.21 of Lam (1999), $Q^l_{max}(\mathscr{A}) = Q^l_{max}(\mathscr{E})$. (ii) First of all we are going to see:

2.10. For every $0 \neq x \in \mathscr{A}, e\mathscr{E}x \neq 0$ and $(1-e)\mathscr{E}x \neq 0$.

Indeed, let x be in $\mathscr{A}$ such that $e\mathscr{E}x = 0$. Then $0 = ex = exe + ex(1-e)$, which implies $x_{11} = x_{12} = 0$. Now, $\mathscr{A}_{12}x \subseteq e\mathscr{E}x = 0$ implies $\mathscr{A}_{12}x_{21} = \mathscr{A}_{12}x_{22} = 0$; apply that A is a left quotient pair of A and (2.6) to obtain $x_{21} = x_{22} = 0$ and, hence $x = 0$.

Now, let p and q be in $e\mathscr{Q}(1-e)$ with $p \neq 0$. Apply that $\mathscr{Q}$ is a left quotient ring of $\mathscr{A}$ to take $a \in \mathscr{A}$ such that $0 \neq ap \in \mathscr{A}$ and $aq \in \mathscr{A}$. By (2.10) applied twice, there exist $x, y \in \mathscr{E}$ such that $ey(1-e)xaep = ey(1-e)xap \neq 0$. Moreover the elements $ey(1-e) \in \mathscr{A}_{12}$ and $(1-e)xae \in \mathscr{A}_{21}$ satisfy $ey(1-e)xaeq = ey(1-e)xaq \in \mathscr{A}_{12}$.

Finally, suppose that T is a left quotient pair of A and write $(\mathscr{E}_T, e)$ to denote the standard imbedding of T. By (2.5) (iii), $\mathscr{E}_T$ is a left quotient ring of $\mathscr{A}$ and therefore $\mathscr{A} \subseteq \mathscr{T} \subseteq \mathscr{Q}$. So, $(e\mathscr{T}(1-e), (1-e)\mathscr{T}e) \subseteq (e\mathscr{Q}(1-e), (1-e)\mathscr{Q}e)$. □

2.11. Definition. If A is an associative pair without total left zero divisors and without total right zero divisors, or it is left nonsingular, then by (2.9)(ii), it has a unique (up to isomorphisms) maximal left quotient associative pair which will be called the *maximal left quotient pair* of A and if we denote it by $Q^l_{max}(A)$, then (by 2.9(ii)),

$$Q^l_{max}(A) = (e\mathscr{Q}(1-e), (1-e)\mathscr{Q}e),$$

where $\mathscr{Q} = Q^l_{max}(\mathscr{A}) = Q^l_{max}(\mathscr{E})$, for $\mathscr{A}$ and $(\mathscr{E}, e)$ the envelope and the standard imbedding of A, respectively.

A natural question which arises is if the envelope of the maximal left quotient pair of an associative pair A and the maximal left quotient ring of the envelope coincide. The answer is negative, as it is shown with the following example.

2.12. Example. Let V be a left vector space over a field K of infinite dimension, $\mathcal{Q} = End_K(V)$ and $\mathcal{A} = Soc(\mathcal{Q})$. Consider two idempotents $e, f \in \mathcal{Q}$ such that $e + f = 1$, $e \in \mathcal{A}$, $f \notin \mathcal{A}$. Then the associative pair $A = (e\mathcal{A}f, f\mathcal{A}e)$ has the ring $\mathcal{A}$ as an envelope, $Q^l_{max}(A) = (e\mathcal{Q}f, f\mathcal{Q}e)$, $Q^l_{max}(\mathcal{A}) = \mathcal{Q}$ and the envelope of $Q^l_{max}(A)$ is $e\mathcal{Q}f \oplus f\mathcal{Q}e \oplus e\mathcal{Q}f\mathcal{Q}e \oplus f\mathcal{Q}e\mathcal{Q}f = \mathcal{A} \neq \mathcal{Q}$.

2.13. Lemma. *If Q is a left quotient pair of an associative pair A, then*:

(i) $L \cap A^\sigma \neq 0$ *for any nonzero left ideal L of Q contained in Q^σ*, $\sigma = \pm$.
(ii) *A semiprime (prime) implies Q semiprime (prime).*

Proof. (i) follows from the definition.

(ii) Suppose A prime. If I and J are two nonzero left ideals of Q, by (i), $I^\sigma \cap A^\sigma$ and $J^\sigma \cap A^\sigma$ are nonzero, hence $(I^\sigma \cap A^\sigma)A^{-\sigma}(J^\sigma \cap A^\sigma) \neq 0$, which implies $I^\sigma Q^{-\sigma} J^\sigma \neq 0$. The case A semiprime follows analogously by considering $J = I$. □

With the following proposition we show the relationship between some properties of an associative pair and the analogues of its left quotient pairs (see Fernández López et al., 1998 for the definitions).

We recall the notion of local ring at an element of an associative pair (see Fernández López et al., 1998): Let $A = (A^+, A^-)$ be an associative pair and $a \in A^\sigma$. Then the submodule $aA^{-\sigma}a$ equipped with the multiplication defined by $(axa) \cdot (aya) = axaya$ is a ring called the *local ring* of A *at* a and denoted by A_a. Note that if a is *von Neumann regular*, i.e., $a \in aA^{-\sigma}a$, then A_a is unital with a as the unity.

A family of left ideals $\{L_i\}_{i\in\Gamma}$ of an associative pair is said to be *independent* if the sum of its ideals is direct.

2.14. Proposition. *Let Q be a left quotient pair of an associative pair A. Then*:

(i) $lan_Q(X) \cap A^\sigma = lan_A(X)$ *for any subset X of* A^σ.
(ii) $lan_Q(X) \subseteq lan_Q(Y)$ *implies* $lan_A(X) \subseteq lan_A(Y)$ *for any* $X, Y \subseteq A^\sigma$.
(iii) $Z_l(A) = Z_l(Q) \cap A$.
(iv) *A is left nonsingular if and only if Q is so.*
(v) *If $\{L_i\}_{i\in\Gamma}$ is a family of independent nonzero left ideals of A contained in A^σ, then for every $i \in \Gamma$ there exist $l_i \in L_i$ and $b_i \in A^{-\sigma}$ such that $\{Q^\sigma b_i l_i\}_{i\in\Gamma}$ is a family of independent nonzero left ideals of Q contained in Q^σ.*
(vi) *If $\{\widetilde{L}_i\}_{i\in\Gamma}$ is a family of independent nonzero left ideals of Q contained in Q^σ, then $\{\widetilde{L}_i \cap A^\sigma\}_{i\in\Gamma}$ is a family of independent nonzero left ideals of A contained in A^σ.*

(vii) *Q has finite left Goldie dimension if and only if A has finite left Goldie dimension. In fact, u-dim A = u-dim Q.*

(viii) *For any $a \in A^\sigma$, $u\text{-}dim_A(a) \leq u\text{-}dim_Q(a)$. Hence, if Q has finite left local Goldie dimension then A has finite left local Goldie dimension too.*

If A is semiprime, then:

(ix) *For every element $a \in A^\sigma$, the local ring of Q at a, Q_a, is a left quotient ring of A_a.*
If A has no total left zero divisors or it is left nonsingular, then:

(x) *$lan_Q(X) \subseteq lan_Q(Y)$ if and only if $lan_A(X) \subseteq lan_A(Y)$ for any $X, Y \subseteq A^\sigma$.*

(xi) *For any $a \in A^\sigma$, $u\text{-}dim_A(a) = u\text{-}dim_Q(a)$.*

Proof.

(i) It is clear that for any $X \subseteq A^+$, $lan_Q(X) \cap A^- \subseteq lan_A(X)$. Conversely, let z be in $lan_A(X)$. Then, for every $x \in X$, $0 = zx \in \mathscr{A}_{22} \subseteq \mathscr{Q}_{22}$ (by (2.5) (i)). Therefore, $z \in lan_Q(X)$.

(ii) follows by (i).

(iii) Let z be in $Z_l(A)^+$. We have to see that $lan_Q(z)$ is an essential left ideal of Q. Let L be a nonzero left ideal of Q contained in Q^-. By (2.13)(i), $L \cap A^-$ is a nonzero left ideal of A. Since $lan_A(z)$ is an essential left ideal of A, $lan_A(z) \cap L \cap A^- \neq 0$, and by (i) $lan_Q(z) \cap L \neq 0$.

Conversely, suppose $z \in Z_l(Q) \cap A$ and let prove that $lan_A(z)$ is an essential left ideal of A. Given a nonzero left ideal L of A contained in A^-, since A has no total right zero divisors we have $A^-A^+L \neq 0$, which implies $Q^-al \neq 0$ for some $a \in A^+$, $l \in L$. Apply that $lan_Q(z)$ is an essential left ideal of Q to find $0 \neq pal \in Q^-al \cap lan_Q(z)$. Since Q is a left quotient pair of A, given $0 \neq pal$ and p, there exist $u \in A^+$, $v \in A^-$ such that $vupal \neq 0$ and $vup \in A^+$. Then $0 \neq vupal \in A^+A^-L \cap lan_Q(z) \subseteq L \cap lan_A(z)$ (by (i)).

(iv) By (iii), $Z_l(Q) = 0$ implies $Z_l(A) = 0$. Conversely, if $Z_l(A) = 0$ then $Z_l(Q)$ must be zero by (2.13)(i).

(v) Let $\{L_i\}_{i\in\Gamma}$ be as in the statement. For every $i \in \Gamma$, choose a nonzero element $l_i \in L_i$. Since A has no total right zero divisors, there exist $a_i \in A^\sigma$, $b_i \in A^{-\sigma}$ such that $0 \neq a_ib_il_i$. Then $\widetilde{L}_i = Q^\sigma b_il_i$ is a nonzero left ideal of Q contained in Q^σ. Now we see that the sum of the $\widetilde{L}_i$'s is direct.

Suppose $q_1b_1l_1 = q_2b_2l_2 + \cdots + q_nb_nl_n$ for $q_jb_jl_j \in \widetilde{L}_{i_j}$, with $q_1b_1l_1 \neq 0$. Apply that Q is a left quotient pair of A to find $u \in A^\sigma$, $v \in A^{-\sigma}$ such that $uvq_1b_1l_1 \neq 0$ and $uvq_i \in A^\sigma$. Then $0 \neq uvq_1b_1l_1 = \sum_{j=2}^n uvq_jb_jl_j \in L_{i_1} \cap \left(\sum_{j\neq 1} L_{i_j}\right)$, a contradiction.

(vi) follows immediately by taking into account (2.13)(i).

(vii) is a direct consequence of (v) and (vi).

(viii) follows by (v).

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(ix) Take $apa, aqa \in Q_a$, with $apa \neq 0$. Apply that Q is a left quotient pair of A to find $x \in A^\sigma$, $y \in A^{-\sigma}$ satisfying $0 \neq xyapa \in A^\sigma$. By (2.4), given $xya \in Q^\sigma$, $q \in Q^{-\sigma}$ and $0 \neq xyapa \in A^\sigma$, we have $(A^{-\sigma} : xyaq)xyapa \neq 0$. Let z be in $A^{-\sigma}$ with $zxyaq \in A^{-\sigma}$ and $zxyapa \neq 0$. By (1.4) and the semiprimeness of A, $0 \neq atuzxyapa$ for some $t \in A^{-\sigma}$, $u \in A^\sigma$. Then the element $atuzxya$, which is in A_a, satisfies: $atuzxya.apa = atuzxyapa \neq 0$ and $atuzxya.aqa = atuzxyaqa \in A_a$.

(x) By (ii), $lan_Q(X) \subseteq lan_Q(Y)$ implies $lan_A(X) \subseteq lan_A(Y)$. Now we prove the converse. Suppose $lan_A(X) \subseteq lan_A(Y)$ but $lan_Q(X) \nsubseteq lan_Q(Y)$ and let q be in $lan_Q(X)$ such that $pqy \neq 0$ for some $y \in Y$, $p \in Q^\sigma$ (this is possible by virtue of (1.4)). Since Q is a left quotient pair of A, there exist $u \in A^\sigma$, $v \in A^{-\sigma}$ such that $uvpqy \neq 0$ and $uvpqy \in A^\sigma$. By (2.4), $(A^{-\sigma} : uvpq)\ uvpqy \neq 0$, so there exists $b \in A^{-\sigma}$ such that $buvpq \in A^{-\sigma}$ and $0 \neq buvpqy$. Then $buvpq \in lan_A(X)$ (because $q \in lan_Q(X)$) but $buvpq \notin lan_A(Y)$, which contradicts the initial hypothesis.

(xi) By (viii), u-dim$_A(a) \leq$ u-dim$_Q(a)$. Now, let $\{\widetilde{L}_i\}_{i\in\Gamma}$ be a family of nonzero left ideals of Q contained in $Q^\sigma Q^{-\sigma}a$. We can take $0 \neq \sum p_{k_i} q_{k_i} a \in \widetilde{L}_i$, with $p_{k_i}, q_{k_i} \in Q^{-\sigma}$. By (2.4), $(A^{-\sigma} : p_{k_i} q_{k_i})a \neq 0$, hence there exists y_i in $A^{-\sigma}$ such that $\sum y_i p_{k_i} q_{k_i} \in A^{-\sigma}$ and $\sum y_i p_{k_i} q_{k_i} a \neq 0$. By (1.4) we can find $x \in A^\sigma$ such that $l_i := xy_i \sum p_{k_i} q_{k_i} a \neq 0$. Since $l_i \in A^\sigma A^{-\sigma} a \cap \widetilde{L}_i$, $\{A^\sigma A^{-\sigma} l_i\}_{i\in\Gamma}$ is a family of nonzero left ideals of A contained in $A^\sigma A^{-\sigma} a$. Moreover its sum is direct (because $A^\sigma A^{-\sigma} l_i \subseteq \widetilde{L}_i$ and the sum of the $\widetilde{L}_i$'s was direct), which proves our claim. □

3. JOHNSON AND GABRIEL'S THEOREMS FOR ASSOCIATIVE PAIRS

While Johnson's Theorem characterizes those rings $\mathscr{R}$ for which $Q^l_{max}(\mathscr{R})$ is von Neumann regular (Lam, 1999, 13.36), Gabriel's Theorem (Lam, 1999, 13.40) specializes it further by asking for characterizations for those rings $\mathscr{R}$ for which $Q^l_{max}(\mathscr{R})$ is semisimple, i.e., isomorphic to a finite direct product of rings of the form $End_\Delta(V)$ for a suitable finite left vector space V over a division ring Δ. In this section we prove that every associative pair A for which $Q^l_{max}(A)$ is von Neumann regular is left nonsingular (and conversely), and characterize those associative pairs whose maximal left quotient pair is isomorphic to $\Pi_{\alpha\in\Gamma}(Hom_{\Delta_\alpha}(V_\alpha, W_\alpha), Hom_{\Delta_\alpha}(W_\alpha, V_\alpha))$, where for each $\alpha \in \Gamma$, V_α and W_α are left vector spaces over the same division ring Δ_α. In particular we get a characterization of those associative pairs whose maximal left quotient pair is semisimple and artinian.

3.1. Johnson's Theorem for Associative Pairs. Let A be an associative pair. Then A is left nonsingular if and only if $Q^l_{max}(A)$ is a von Neumann regular associative pair.

Proof. Suppose A is left nonsingular. By (1.9), $\mathscr{E}$ is left nonsingular. By Johnson's Theorem for rings (Lam, 1999, 13.36), the maximal left quotient ring of $\mathscr{E}$, name it $\mathscr{Q}$, is von Neumann regular. By 2.9(ii), $Q^l_{max}(A) = (e\mathscr{Q}(1-e), (1-e)\mathscr{Q}e)$, and it is easy to prove that this is a von Neumann regular associative pair. Conversely, as every von Neumann regular associative pair is left nonsingular (by Proposition 3.4 of Fernández López et al., 1998), 2.14(iv) completes the proof. □

Let $\mathcal{R}$ be an element in an arbitrary ring $\mathcal{R}$. Recall that the *local ring* of $\mathcal{R}$ *at a* is defined as the ring obtained from the abelian group $(a\mathcal{R}a, +)$ by considering the product given by $axa \cdot aya = axaya$. Denote it by $\mathcal{R}_a$. The reader is referred to Gómez Lozano and Siles Molina (2002) to see the relation among some properties of a ring and the corresponding ones of its local rings at elements.

Now, we will introduce some notation. Given a ring $\mathcal{R}$ and an element $x \in \mathcal{R}$, the left Goldie (or uniform) dimension of x will be denoted by u-dim(x) (u-dim$_{\mathcal{R}}(x)$ to specify the ring). By u-dim(R) we understand the uniform dimension of ${}_R R$. We put $I(\mathcal{R}) = \{x \in \mathcal{R} \mid \mathrm{u} - \dim(x) < \infty\}$. Condition (iii) in the next Proposition was proved by Ánh and Márki (1996, Proposition 1). Here we give a different proof by using Johnson's Theorem for rings.

3.2. Proposition. *Let $\mathcal{R}$ be a ring and denote by* Soc$(\mathcal{R})$ *the socle of the ring $\mathcal{R}$.*

(i) *If $\mathcal{R}$ is semiprime, then*: $I(\mathcal{R}) = \{a \in \mathcal{R} \mid \mathrm{u} - \dim(R_a) < \infty\} \supseteq \mathrm{Soc}(\mathcal{R})$.
(ii) *If $\mathcal{R}$ is von Neumann regular, then* $I(\mathcal{R}) = \mathrm{Soc}(\mathcal{R})$.
(iii) *If $\mathcal{R}$ is left nonsingular, then $I(\mathcal{R})$ is an ideal of $\mathcal{R}$.*

Proof. (i) The equality holds by [6, Proposition 2.1 (iv)] of Gómez Lozano and Siles Molina (2002). Now, let x be in Soc$(\mathcal{R})$. Since Soc$(\mathcal{R})$ is a von Neumann regular ideal of $\mathcal{R}$, $x\mathcal{R}x = x\mathrm{Soc}(\mathcal{R})x$, which obviously implies u-dim$(R_x) =$ u-dim$(\mathrm{Soc}(\mathcal{R})_x)$. By Proposition 2.1 (vi) of Gómez Lozano and Siles Molina (2002), u-dim$(\mathrm{Soc}(\mathcal{R})_x) < \infty$ and hence $x \in I(\mathcal{R})$.

(ii) By (i) we only need to prove $I(\mathcal{R}) \subseteq \mathrm{Soc}(\mathcal{R})$. Take a nonzero $x \in I(\mathcal{R})$. By Proposition 2.1 (i), (iv) and (ix) of Gómez Lozano and Siles Molina (2002), $\mathcal{R}_x$ is a semiprime left Goldie ring. By the classical Goldie's Theorem, $\mathcal{R}_x$ is a classical left order in a semisimple artinian ring $\mathcal{T}$. Since $\mathcal{R}$ is von Neumann regular, the ring $\mathcal{R}_x$ is unital and von Neumann regular. Therefore, $\mathrm{Reg}(\mathcal{R}_x) = \mathrm{Inv}(\mathcal{R}_x)$, where "Reg" and "Inv" denote the set of regular and invertible elements, respectively. Since $\mathcal{T}$ is generated by $\mathcal{R}_x$ and the inverses of the elements of $\mathrm{Reg}(\mathcal{R}_x)$, we have $\mathcal{T} = \mathcal{R}_x$ and $\mathcal{T}(= \mathcal{R}_x)$ artinian implies, by Proposition 2.1 (v) of Gómez Lozano and Siles Molina (2002), $x \in \mathrm{Soc}(\mathcal{R})$.

(iii) By Johnson's Theorem (Lam, 1999, 13.36), $\mathcal{Q} := Q^l_{max}(\mathcal{R})$ is a von Neumann regular ring. Since, by (ii), $I(\mathcal{Q}) = \mathrm{Soc}(\mathcal{Q})$ is an ideal of $\mathcal{Q}$, clearly $I(\mathcal{Q}) \cap \mathcal{R}$ is an ideal of $\mathcal{R}$. We conclude the proof by applying Proposition 3.2 (iv) of Gómez Lozano and Siles Molina (2002), which says $I(\mathcal{R}) = I(\mathcal{Q}) \cap \mathcal{R}$. □

Given an associative pair A, denote by $I(A)^\sigma$ the set of the elements of A^σ having finite left Goldie dimension, and set $I(A) = (I(A)^+, I(A)^-)$. We denote by Soc(A) the socle of A (the reader is referred to Loos, 1991 for the study of the socle of an associative pair).

3.3. Proposition. *Let A be an associative pair and denote by* $\mathscr{A}$ *and* $(\mathscr{E}, e)$ *its envelope and standard imbedding, respectively.*

(i) *If A is semiprime, then* $I(A)^\sigma = \{a \in A^\sigma | \text{u} - \dim(A_a) < \infty\} = I(\mathscr{A}) \cap A^\sigma = I(\mathscr{E}) \cap A^\sigma \supseteq \mathrm{Soc}(A)$.

(ii) *If A is left nonsingular, then* $I(A)^\sigma = I(\mathscr{A}) \cap A^\sigma = I(\mathscr{E}) \cap A^\sigma$.

(iii) *If A is von Neumann regular, then* $I(A) = \mathrm{Soc}(A)$.

(iv) *If A is left nonsingular, then* $I(A)$ *is an ideal of A.*

Proof. (i) By Proposition 5.2 (iv) of Fernández López et al. (1998), $a \in I(A)^\sigma$ if and only if u-dim$(A_a) < \infty$, and so the first equality holds.

Since A semiprime implies $\mathscr{E}$ and $\mathscr{A}$ semiprime (by Proposition 4.2 of Fernández López et al., 1998, $\mathscr{E}$ is semiprime, and $\mathscr{A}$ is semiprime because it is an ideal of $\mathscr{E}$), by condition (i) of (3.2) and taking into account that $aA^{-\sigma}a = a\mathscr{A}a = a\mathscr{E}a$, for every $a \in A^\sigma$, we have $I(A)^\sigma = I(\mathscr{A}) \cap A^\sigma = I(\mathscr{E}) \cap A^\sigma$.

Now, take an element $a \in \mathrm{Soc}(A)^\sigma$ and let b be in $A^{-\sigma}$ satisfying $aba = a$ (which is possible by virtue of Loos, 1989, Theorem 1). Then $aA^{-\sigma}a = abaA^{-\sigma}aba \subseteq a\,\mathrm{Soc}(A)^\sigma a$ (because $\mathrm{Soc}(A)$ is an ideal of A). Since, obviously, $a\,\mathrm{Soc}(A)^\sigma a \subseteq a\,A^{-\sigma}a$, we have $\mathrm{Soc}(A)_a = A_a$. Finally, apply Proposition 5.2(v) of Fernández López et al. (1998) to infer that u-dim$(\mathrm{Soc}(A)) < \infty$. By Proposition 5.2(iv) of Fernández López et al. (1998), u-dim$(\mathrm{Soc}(A)_a) < \infty$ hence $\mathrm{Soc}(A) \subseteq I(A)$.

(ii) By Theorem 2.9, there exists $\mathscr{Q} := Q^l_{max}(\mathscr{A}) = Q^l_{max}(\mathscr{E})$ and $Q := (\mathscr{Q}_{12}, \mathscr{Q}_{21})$ is a left quotient pair of A. Moreover, since $\mathscr{A}$ is left nonsingular (by Proposition 1.9), $\mathscr{Q}$ is von Neumann regular (Johnson's Theorem). Take an element $a_{12} \in A^+$. Then

$$\begin{aligned}\text{u-dim}_A(a_{12}) &\overset{(a)}{=} \text{u-dim}_Q(a_{12}) \overset{(b)}{=} \text{u-dim}(Q_{a_{12}}) = \text{u-dim}(\mathscr{Q}_{a_{12}}) \\ &\overset{(c)}{=} \text{u-dim}_{\mathscr{Q}}(a_{12}) \overset{(d)}{=} \text{u-dim}_{\mathscr{A}}(a_{12}) \overset{(e)}{=} \text{u-dim}_{\mathscr{E}}(a_{12}).\end{aligned}$$

(a) Because Q is a left quotient pair of A and by 2.14(xi).

(b) By (i) and nondegeneracy of Q (which is von Neumann regular since $\mathscr{Q}$ is).

(c) By condition (i) in Proposition 3.2, which can be applied since $\mathscr{Q}$ is nondegenerate.

(d) Because $\mathscr{Q}$ is a left quotient ring of $\mathscr{A}$ and by Proposition 3.2 (iv) of Gómez Lozano and Siles Molina (2002).

(e) Since $\mathscr{Q}$ is a left quotient ring of $\mathscr{A}$ and by Proposition 3.2 (iv) of Gómez Lozano and Siles Molina (2002), u-dim$_{\mathscr{Q}}(a_{12}) =$ u-dim$_{\mathscr{E}}(a_{12})$.

(iii) Let a be in $I(A)$. By Proposition 5.2(i), (iv) and Proposition 5.5 of Fernández López et al. (1998), A_a is a semiprime left Goldie ring. Now we follow the same reasoning as in the proof of condition (ii) in (3.2) (notice that A_a is von Neumann regular since A is so) to prove A_a artinian. By (Fernández López et al. (1998), Proposition 5.2 (v)) this implies $a \in \mathrm{Soc}(A)$.

(iv) By (2.9) (ii) and (2.9), $Q := Q^l_{max}(A) = (e\mathscr{Q}(1-e), (1-e)\mathscr{Q}e)$ is a von Neumann regular associative pair, so we can apply (iii) to obtain $I(Q)^\sigma = \mathrm{Soc}(Q)^\sigma$, which is an ideal of A. Since $I(A)^\sigma = I(Q)^\sigma \cap A^\sigma$, by (2.14) (xi), we have proved the required statement. □

3.4. Theorem. *For an associative pair A the following conditions are equivalent*:

(i) *A is left nonsingular and* $I(A)^\pm$ *is an essential left ideal of A contained in* $A^\pm$.

(ii) $Q := Q^l_{max}(A) \cong \Pi_{\alpha\in\Gamma}(Hom_{\Delta_\alpha}(V_\alpha, W_\alpha), Hom_{\Delta_\alpha}(W_\alpha, V_\alpha))$, *where for each* $\alpha \in \Gamma$, V_α *and* W_α *are left vector spaces over the same division ring* Δ_α.

Proof. (i) ⇒ (ii) By Johnson's Theorem for associative pairs, $Q = Q^l_{max}(A)$ is a von Neumann regular associative pair and by 2.9 (ii), $Q = (e\mathcal{Q}(1-e), (1-e)\mathcal{Q}e)$, where $\mathcal{Q} = Q^l_{max}(\mathcal{A})$ and $\mathcal{A}$ is the envelope of A. Denote by $\mathcal{S}$ the subalgebra of $\mathcal{Q}$ generated by Q, that is, $\mathcal{S}$ is the envelope of Q. It is not difficult to see that $\mathcal{S}$ is an ideal of $\mathcal{Q}$. Moreover,

(1) $\mathcal{Q}$ is a von Neumann regular ring.

By 1.9, A left nonsingular implies $\mathcal{A}$ left nonsingular, and by Johnson's Theorem for rings, $\mathcal{Q}$ is a von Neumann regular ring.

(2) $I(\mathcal{A})$ is dense in $\mathcal{A}$.

For every $a \in A^\sigma$, $u-\dim_A(a) = u-\dim_Q(a)$ (by 2.14 (xi)). This implies $I(A)^\sigma \subseteq I(Q)^\sigma$. Since $I(A)^\sigma$ is an essential left ideal of A, $I(Q)^\sigma$ is an essential ideal of Q. By 3.2 (ii), $I(Q)^\sigma = I(\mathcal{Q}) \cap Q^\sigma$. We claim that $I(\mathcal{Q})$ is an essential ideal of $\mathcal{Q}$: Consider a nonzero ideal J of Q^σ. By Proposition 4.1 (i) of Fernández López et al. (1998) $J \cap Q^\sigma$ is a nonzero ideal of Q^σ. Hence $0 \neq I(Q)^\sigma \cap J \cap Q^\sigma = I(\mathcal{Q}) \cap J \cap Q^\sigma$. This shows our claim.

Finally, being $\mathcal{Q}$ a left quotient ring of $\mathcal{A}$ implies $I(\mathcal{A}) = I(\mathcal{Q}) \cap \mathcal{A}$ is an essential ideal of $\mathcal{A}$ and consequently it is a dense ideal of $\mathcal{A}$ (apply that $\mathcal{A}$ is left nonsingular).

(3) The conclusion.

By Theorem 3.24 of Lam (1999), which can be applied taking into account (2) and that $\mathcal{A}$ is left nonsingular, $\mathcal{Q} := Q^l_{\max}(\mathcal{A}) \cong \Pi\mathcal{Q}_\alpha$, where each $\mathcal{Q}_\alpha$ is an ideal of $\mathcal{Q}$ isomorphic (as a ring) to $End_{\Delta_\alpha}(U_\alpha)$ *for* a suitable left vector space U_α over some division ring Δ_α.

Define $V_\alpha := U_\alpha e_\alpha$, $W_\alpha = U_\alpha(1-e)_\alpha$, $e_\alpha = \pi_\alpha(e)$, $(1-e)_\alpha = \pi_\alpha(1-e)$. Then we have $(e\mathcal{Q}(1-e), (1-e)\mathcal{Q}e) \cong \Pi(Hom_{\Delta_\alpha}(V_\alpha, W_\alpha), Hom_{\Delta_\alpha}(W_\alpha, V_\alpha))$.

(ii) ⇒ (i) Define $U_\alpha = V_\alpha \oplus W_\alpha$, $\mathcal{Q} = \Pi_\alpha End_{\Delta_\alpha}(U_\alpha)$, $e = (e_\alpha)$ and $f = (f_\alpha)$, where

$$e_\alpha : U_\alpha \to U_\alpha \quad f_\alpha : U_\alpha \to U_\alpha$$
$$v_\alpha + w_\alpha \mapsto v_\alpha \quad v_\alpha + w_\alpha \mapsto w_\alpha$$

Then $e + f = 1_\mathcal{Q}$ and $Q = (e\mathcal{Q}(1-e), (1-e)\mathcal{Q}e) \cong \Pi_\alpha(Hom_{\Delta_\alpha}(V_\alpha, W_\alpha), Hom_{\Delta_\alpha}(W_\alpha, V_\alpha))$. This implies Q von Neumann regular and so by Fernández López et al. (1998, Proposition 3.4), Q is left nonsingular. By 2.14(iv), A is left nonsingular. Finally, $I(A)^\sigma = I(Q) \cap A^\sigma$ (by 2.14 (xi)) $= I(\mathcal{Q}) \cap A^\sigma$ (by 3.3(i)) implies that $I(A)^\sigma$ must be an essential left ideal of A. □

Notice that finiteness of left Goldie dimension of A implies that the direct product in the previous theorem must be finite as well as the dimensions of the vector spaces involved.

3.5. Gabriel's Theorem for Associative Pairs. For an associative pair A the following conditions are equivalent:

(1) A is left nonsingular and has finite left Goldie dimension.
(2) $Q^l_{max}(A) = \Pi_{i=1}^n (Hom_{\Delta_i}(V_i, W_i), Hom_{\Delta_i}(W_i, V_i))$, where for each $i \in \{1, \cdots, n\}$, V_i and W_i are finite left vector spaces over the same division ring Δ_i.

4. APPLICATIONS TO MORITA CONTEXTS

Let R and S be two rings, ${}_RN_S$ and ${}_SM_R$ two bimodules and $(-,-) : N \times M \to R$, $[-,-] : M \times N \to S$ two maps. Then the following conditions are equivalent:

(i) $\begin{pmatrix} R & N \\ M & S \end{pmatrix}$ is a ring with componentwise sum and product given by:

$$\begin{pmatrix} r_1 & n_1 \\ m_1 & s_1 \end{pmatrix}\begin{pmatrix} r_2 & n_2 \\ m_2 & s_2 \end{pmatrix} = \begin{pmatrix} r_1r_2 + (n_1, m_2) & r_1n_2 + n_1s_2 \\ m_1r_2 + s_1m_2 & [m_1, n_2] + s_1s_2 \end{pmatrix},$$

(ii) $[-,-]$ is S-bilinear and R-balanced, $(-,-)$ is R-bilinear and S-balanced and the following associativity conditions holds:

$$(n,m)n' = n[m,n'] \quad \text{and} \quad [m,n]m' = m(n,m').$$

$[-,-]$ being S-bilinear and R-balanced and $(-,-)$ being R-bilinear and S-balanced is equivalent to having bimodule maps $\varphi : N \otimes_S M \to R$ and $\psi : M \otimes_R N \to S$, given by

$$\varphi(n \otimes m) = (n,m) \quad \text{and} \quad \psi(m \otimes n) = [m,n],$$

so that the associativity condition above reads

$$\varphi(n \otimes m)n' = n\psi(m \otimes n') \quad \text{and} \quad \psi(m \otimes n)m' = m\varphi(n,m').$$

A *Morita context* is a sextuple $(R, S, N, M, \varphi, \psi)$ satisfying the conditions given above. The associated ring is called the *Morita ring* of the context.

In classical Morita theory it is shown that two rings with identity R and S are Morita equivalent (i.e., R- and S-mod are equivalent categories) if and only if there exists a Morita context $(R, S, N, M, \varphi, \psi)$, with φ and ψ surjective. The approach to Morita theory for rings without identity by means of Morita contexts appears in a number of papers (see Marín, 1998 and the references therein) in which many consequences are obtained from the existence of a Morita context for two rings R and S. In particular it is shown in Theorem of Kyuno (1974) that, if R and S are arbitrary rings such that there is a surjective Morita context for these rings, then the categories R-Mod and S-Mod are equivalent (and the rings R and S are said to be *Morita-equivalent*). It is proved in Proposition 2.3 of García and Simón (1991) that the converse implication holds for idempotent rings.

Recall that an *idempotent ring* is a ring R such that $R^2 = R$. For an idempotent ring R we denote by R-Mod the full subcategory of the category of all left R-modules whose objects are the "unital" nondegenerate modules. Here a left R-module is said to be *unital* if $M = RM$, and is said to be *nondegenerate* if, for $m \in M$, $Rm = 0$ implies $m = 0$. Note that if R has an identity, then R-Mod is the usual category of left R-modules.

The following result can be found in García and Simón (1991) (see Proposition 2.5 and Theorem 2.7).

4.1. Theorem. *Let R and S be two idempotent rings. Then R-Mod and S-Mod are equivalent categories if and only if there exists a Morita context $(R, S, M, N, \varphi, \psi)$, with $M \in R$-Mod $\cap$ Mod-S, $N \in S$-Mod $\cap$ Mod-R, and φ and ψ surjective.*

4.2. Remark. If $(R, S, M, N, \varphi, \psi)$ is a Morita context for two idempotent rings R and S, with $M \in R$-Mod $\cap$ Mod-S and $N \in S$-Mod $\cap$ Mod-R, and T is the Morita ring of the context, then (M, N) is an associative pair and if R has no total left or right zero divisors and S has no total left or right zero divisors, then T is the envelope of the associative pair.

4.3. Theorem. *Let R and S be two Morita-equivalent idempotent rings such that R has no total left or right zero divisors and S has no total right zero divisors, and let $T = (R, S, M, N)$ be the Morita ring of the context. Then the following are equivalent conditions*:

(i) *R is left nonsingular.*
(ii) *S is left nonsingular.*
(iii) *$A = (M, N)$ is a left nonsingular associative pair.*
(iv) *T is left nonsingular.*

Proof. Notice that by 4.2, T is the envelope of the associative pair A. Since the modules are left and right nondegenerate, and the rings are idempotent, we have that A has neither total left right zero divisors nor total right zero divisors. Apply (1.10) to obtain (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

By 1.9, (iii) $\Leftrightarrow$ (iv). □

The conditions over R and S in the previous theorem cannot be dropped, since there exist two Morita equivalent idempotent rings R and S such that R is left nonsingular while $Z_l(S) \neq 0$ (consider R and $S = R/J$, with R and J as in the following lemma).

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4.4. Lemma. *Let R be a commutative idempotent ring, and consider an ideal J of R such that $JR = 0$ and R/J is semiprime. Then $\begin{pmatrix} R & R/J \\ R/J & R/J \end{pmatrix}$, with product given by*

$$\begin{pmatrix} r & \bar{m} \\ \bar{x} & \bar{s} \end{pmatrix}\begin{pmatrix} a & \bar{n} \\ \bar{y} & \bar{b} \end{pmatrix} = \begin{pmatrix} ra + my & \overline{rn + mb} \\ \overline{xa + sy} & \overline{xn + sb} \end{pmatrix}$$

defines a surjective Morita context for the idempotent rings R and R/J. Hence the rings R and R/J are Morita equivalent.

Proof. It is not difficult to prove that the product is well defined. By the idempotency of R, $(R/J)^2 = R/J$. Moreover, given $r \in R = R^2$, $r = \sum_{i=1}^{\alpha} m_i y_i$, with $\alpha \in \mathbb{N}$ and $m_i, y_i \in R$. Hence, $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \sum_{i=1}^{\alpha} \begin{pmatrix} 0 & \overline{m_i} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \overline{y_i} & 0 \end{pmatrix}$. This proves the surjectivity. The modules of the context are unital by the idempotency of R. Finally, we will prove that the modules are nondegenerate. Indeed $r\bar{x} = \bar{0}$ for every $r \in R$ implies $\overline{R}\bar{x} = \bar{0}$ and by the semiprimeness of $\overline{R}$, $\bar{x} = \bar{0}$. The semiprimeness of $\overline{R}$ implies too that R/J is a nondegenerate R/J-module. □

We recall that a ring R is said to have *finite left local Goldie dimension* if any element of R has finite left Goldie (or uniform) dimension. The left Goldie dimension of an element $a \in R$ will be denoted by u-dim(a).

4.5. Theorem. *Let R and S be two Morita-equivalent idempotent rings such that R has no total left or right zero divisors and S has no total left or right zero divisors, and suppose R left nonsingular (equivalently S left nonsingular). Let $T = (R, S, M, N)$ be the Morita ring of the context, and define $A := (M, N)$. Then the following are equivalent conditions:*

(i) *R has finite left local Goldie dimension.*
(ii) *S has finite left local Goldie dimension.*
(iii) *Every element of M has finite left Goldie dimension in A.*
(iv) *Every element of N has finite left Goldie dimension in A.*
(v) *T has finite left local Goldie dimension.*

Proof. Fix the following notation: $(\mathscr{E}, e)$ is the standard imbedding of the associative pair A, $\mathscr{Q} := Q^l_{max}(\mathscr{A})$, which exists by (1.8) and (2.9), $\mathscr{A}_{11} = R$, $\mathscr{A}_{22} = S$ and $\mathscr{A} = T$.

(1) $\mathscr{Q}_{ii}$ is a left quotient ring of $\mathscr{A}_{ii}$.

Take, for example, $p_{11}, q_{11} \in \mathscr{Q}_{11}$, with $p_{11} \neq 0$. Since $\mathscr{Q}$ is a left quotient ring of $\mathscr{A}$, we can choose $a \in \mathscr{A}$ such that $ap_{11} \neq 0$ and $aq_{11} \in \mathscr{A}$. If $a_{11}p_{11} \neq 0$, then we have finished. Suppose $a_{21}p_{11} \neq 0$. The absence of total right zero divisors in A implies $b_{12}a_{21}p_{11} \neq 0$ for some $b_{12} \in \mathscr{A}_{12}$. Then the element $c_{11} := b_{12}a_{21} \in \mathscr{A}_{11}$ satisfies: $c_{11}p_{11} \neq 0$ and $c_{11}q_{11} \in \mathscr{A}$.

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(2) $I(\mathscr{A}_{ii}) = I(\mathscr{A}) \cap \mathscr{A}_{ii} = I(\mathscr{E}) \cap \mathscr{A}_{ii}$.
For an element $a_{11} \in \mathscr{A}_{11}$ we have:

$$\text{u-dim}_{\mathscr{A}_{11}}(a_{11}) \overset{(a)}{=} \text{u-dim}_{\mathscr{Q}_{11}}(a_{11}) \overset{(b)}{=} \text{u-dim}(\mathscr{Q}_{11 a_{11}}) = \text{u-dim}(\mathscr{Q}_{a_{11}})$$
$$\overset{(c)}{=} \text{u-dim}_{\mathscr{Q}}(a_{11}) \overset{(d)}{=} \text{u-dim}_{\mathscr{A}}(a_{11}) \overset{(e)}{=} \text{u-dim}_{\mathscr{E}}(a_{11}).$$

(a) Because by (1), $\mathscr{Q}_{11}$ is a left quotient ring of $\mathscr{A}_{11}$ and by Proposition 3.2(iv) of Gómez Lozano and Siles Molina (2002).
(b) By Proposition 2.1(iv) of Gómez Lozano and Siles Molina (2002), which can be applied since $\mathscr{Q}$ von Neumann regular implies $\mathscr{Q}_{11}$ nondegenerate.
(c) Because $\mathscr{Q}$ is nondegenerate and by Proposition 2.1(iv) of Gómez Lozano and Siles Molina (2002).
(d) It is a consequence of Proposition 3.2(iv) of Gómez Lozano and Siles Molina (2002).
(e) Since $\mathscr{Q}$ is a left quotient ring of $\mathscr{A}$ and by Proposition 3.2(iv) of Gómez Lozano and Siles Molina (2002), $\text{u-dim}_{\mathscr{Q}}(a_{11}) = \text{u-dim}_{\mathscr{E}}(a_{11})$.

(v) ⇒ (i), (ii), (iii), (iv) follows by 3.3(ii) and by (2).

(i), (ii), (iii) or (iv) ⇒ (v) by 3.3(ii) and (2), $I(\mathscr{A}_{ij}) = I(\mathscr{A}) \cap \mathscr{A}_{ij}$, for $i, j = 1, 2$, $i \neq j$. Taking into account that $I(\mathscr{A})$ is an ideal of $\mathscr{A}$ (by 3.2) and that $\mathscr{A}$ is generated, as an ideal of $\mathscr{E}$, by $\mathscr{A}_{ij}$, the result follows. □

Let $\mathscr{A}_{11}$ and $\mathscr{A}_{22}$ be two Morita-equivalent idempotent rings, denote the Morita ring of the context by $\mathscr{A} = (\mathscr{A}_{11}, \mathscr{A}_{22}, \mathscr{A}_{12}, \mathscr{A}_{21})$, and suppose that there exists $Q^l_{max}(\mathscr{A}_{11})$ and $Q^l_{max}(\mathscr{A}_{22})$ (as under the hypothesis of 4.3 and 4.5). The natural questions that arise are the following: are these two rings Morita-equivalent too?, and, if $\mathscr{Q} := Q^l_{max}(\mathscr{A})$, do $\mathscr{Q}_{11}$ and $\mathscr{Q}_{22}$ coincide with $Q^l_{max}(\mathscr{A}_{11})$ and $Q^l_{max}(\mathscr{A}_{22})$, respectively? The answer is negative in both cases.

4.6. Example. Consider a simple and non unital ring $\mathscr{R}$ which coincides with its socle, and take a minimal idempotent $e \in \mathscr{R}$. Then $\begin{pmatrix} e\mathscr{R}e & e\mathscr{R} \\ \mathscr{R}e & \mathscr{R} \end{pmatrix}$ is a Morita context for the idempotent rings $e\mathscr{R}e$ and $\mathscr{R}$ which have no total right zero divisors. On the one hand, by Proposition 4.3.7 of Lambek (1976), $Q^l_{max}(\mathscr{R}) = End_\Delta(V)$, with V a left vector space over a division ring Δ (which is isomorphic to $e\mathscr{R}e$). On the other hand, $Q^l_{max}(e\mathscr{R}e) = e\mathscr{R}e$ (because $e\mathscr{R}e$ is a division ring). But $End_\Delta(V)$ and Δ are not Morita equivalent rings because A is left nonsingular and has finite left local Goldie dimension, while $End_\Delta(V)$, which is left nonsingular, has not finite left local Goldie dimension, and this property is Morita invariant, by virtue of 4.5.

Finally we prove that for semiprime left local Goldie rings, the Fountain–Gould left orders of two idempotent Morita equivalent rings are Morita equivalent too. This contrasts with the previous example, which shows that under the same conditions (semiprime and left local Goldie), the maximal left quotient rings of two Morita equivalent rings are not Morita equivalent.

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4.7. Theorem. *Let R and S be two Morita equivalent semiprime idempotent rings, with R left local Goldie. Then*:

(i) *S is a left local Goldie ring.*
(ii) *If T_1 and T_2 denote the Fountain–Gould left quotient rings of R and S, respectively, then T_1 and T_2 are Morita equivalent rings.*

Proof. (i) follows by (4.3) and (4.5).

(ii) Consider a surjective Morita context (R, S, M, N) for the rings R and S, and let $A = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ be the Morita ring of the context. Denote by Q_1 and Q_2 the maximal left quotient rings of R and S, respectively.

Consider the unital ring $B = \begin{pmatrix} R^1 & M \\ N & S^1 \end{pmatrix}$, where R^1 and S^1 denote the unitizations of R and S, respectively. This ring has two orthogonal idempotents $e = \begin{pmatrix} 1_{R^1} & 0 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 0 & 1_{S^1} \end{pmatrix}$ such that $e + f = 1_B$ and $Ae + eA \subseteq A$. By (2.7) of Aranda Pino et al. (To appear), there exist two orthogonal idempotents $u, v \in Q := Q^l_{max}(A)$ such that $u + v = 1_Q$ and $R = uAu$, $S = vAv$, $M = uAv$, and $N = vAu$ are contained in Q. Moreover, $Q_1 = Q^l_{max}(R) = Q^l_{max}(uAu) \cong$ (by Lemma 1.8 of Aranda Pino et al. (To appear), which can be used because $Au + uA \subseteq A$ and $lan_A(Au) = ran_A(uA){=}0$) $uQ^l_{max}(A)u$. And analogously $Q_2 = Q^l_{max}(S) = Q^l_{max}(vAv) \cong vQ^l_{max}(A)v$. This means that M, N, Q_1 and Q_2 can be considered inside Q as uQv, vQu, uQu and vQv, respectively. By 4.9 of Gómez Lozano and Siles Molina (2002), $T_1 = RQ_1$ and $T_2 = SQ_2$. We claim that $T = \begin{pmatrix} RQ_1 & RQ_1MQ_2 \\ SQ_2NQ_1 & SQ_2 \end{pmatrix}$ is a surjective Morita context for the idempotent rings RQ_1 and SQ_2.

$RQ_1RQ_1 = RQ_1$ since every element $q \in T = RQ_1$ can be written as $q = aa^{\#^2}ab$, with $a, b \in R$. The same argument for SQ_2 shows that it is an idempotent ring. Moreover, this implies $RQ_1RQ_1MQ_2 = RQ_1MQ_2$ and $SQ_2SQ_2NQ_1 = SQ_2NQ_1$.

Now, $RQ_1MQ_2 = RQ_1MS^2Q_2 \subseteq RQ_1MQ_2SQ_2 \subseteq RQ_1MQ_2$. Hence $RQ_1MQ_2SQ_2 = RQ_1MQ_2$ and analogously $SQ_2NQ_1RQ_1 = SQ_2NQ_1$, which shows that the modules are unital.

In what follows, we will show the surjectivity of the Morita context.

$RQ_1MQ_2SQ_2NQ_1 \subseteq RQ_1 = RQ_1RQ_1RQ_1RQ_1 = RQ_1MNQ_1MNMNQ_1MNQ_1 \subseteq RQ_1MQ_2SQ_2NQ_1$. Hence $RQ_1MQ_2SQ_2NQ_1{=}RQ_1$. Analogously, $SQ_2NQ_1RQ_1MQ_2 = SQ_2$.

Finally, we have that the modules are nondegenerate:

Indeed, suppose $0 \neq t = \sum_{i=1}^n r^i q_1^i m^i q_2^i \in RQ_1MQ_2$. $(M, N) \subseteq (RQ_1MQ_2, SQ_2NQ_1) \subseteq (uQv, vQu){=}Q^l_{max}((M, N))$ by (2.9). This implies that (RQ_1MQ_2, SQ_2NQ_1) is a left quotient pair of (M, N). Hence, if $t \neq 0$, for some $(m, n) \in (M, N)$, $0 \neq mnt \in M$. Since M is a nondegenerate right S-module and S is idempotent, $0 \neq mntS^2 \subseteq MNtSQ_2$. This implies RQ_1MQ_2 nondegenerate as a right SQ_2-module and as a left RQ_1-module.

Now, changing the roles of R and S, the proof is complete. □

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