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3-Graded Lie Algebras with Jordan Finiteness Conditions#

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ABSTRACT

A notion of socle is introduced for 3-graded Lie algebras (over a ring of scalars Φ containing $\frac{1}{6}$) whose associated Jordan pairs are non-degenerate. The socle turns out to be a 3-graded ideal and is the sum of minimal 3-graded inner ideals each of which is a central extension of the TKK-algebra of a division Jordan pair. Non-degenerate 3-graded Lie algebras having a large socle are essentially determined by TKK-algebras of simple Jordan pairs with minimal inner ideals and their derivation algebras, which are also 3-graded. Classical Banach Lie algebras of compact operators on an infinite dimensional Hilbert space provide a source of examples of infinite dimensional strongly prime 3-graded Lie algebras with non-zero socle. Other examples can be found within the class of finitary simple Lie algebras

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1. INTRODUCTION

Let $L = L_1 \oplus L_0 \oplus L_{-1}$ be a 3-graded Lie algebra over a ring of scalars Φ. If $\frac{1}{6} \in \Phi$, then $V := (L_1, L_{-1})$ becomes a Jordan pair for the triple product given by $\{x, y, z\} = [[x, y], z]$ for $x, z \in L_\sigma$, $y \in L_{-\sigma}$, $\sigma = \pm 1$. Our purpose is to use the information on V to study the structure of the whole L. Of course, some restrictions should be imposed on the grading if one wants L to be essentially determined by V.

An example of this approach can be found in Neher's description of Lie algebras graded by 3-graded root systems (Neher, 1996, 1.2): A Lie algebra L is graded by a 3-graded root system R if and only if it is a central extension of the Tits–Kantor–Koecher algebra of a Jordan pair V (TKK(V) for short) covered by a grid whose associated 3-graded root system is isomorphic to R. He gives the classification of Jordan pairs covered by a grid and describes their Tits–Kantor–Koecher algebras.

In recent years, a very rich socle theory has been developed for non-degenerate Jordan pairs (see Loos, 1989) and, following the pattern of the structure of prime rings with minimal one-sided ideals (cf. Jacobson, 1968), strongly prime Jordan pairs with non-zero socle have been classified (Fernández Lopéz and Tocón, 2003). We note that any simple Jordan pair covered by a grid with division coordinate algebra coincides with its socle, so in this case the socle theory and the grid theory agree.

The aim of this paper is to develop a similar socle theory for 3-graded Lie algebras making use of their close relationship with Jordan pairs, and to describe non-degenerate 3-graded Lie algebras with *large* socles and their central extensions (see 5.4 and 5.7). Let L be a 3-graded Lie algebra such that its associated Jordan pair V is non-degenerate. If V has socle $\mathrm{Soc}(V) = (\mathrm{Soc}(V)^+, \mathrm{Soc}(V)^-)$, then $\mathrm{Soc}(V)^- \oplus [\mathrm{Soc}(V)^+, \mathrm{Soc}(V)^-] \oplus \mathrm{Soc}(V)^+$ turns out to be an ideal of L (4.3) that we call the socle of L and denote by $\mathrm{Soc}(L)$. In fact, the socle can be computed in terms of a certain class of minimal 3-graded inner ideals (in Zelmanov's sense) of the Lie algebra L, each of which is a central extension of the Tits–Kantor–Koecher algebra of a division Jordan pair (4.7). Moreover, if L itself is non-degenerate, then so is V, $\mathrm{Soc}(L)$ is then isomorphic to $\mathrm{TKK}(\mathrm{Soc}(L))$.

In general, the derivation algebra $\mathrm{Der}\, L$ of a 3-graded Lie algebra L is 5-graded. We prove (3.3) that $\mathrm{Der}\, L$ is actually 3-graded when L's associated Jordan pair is von Neumann regular (the socle of any non-degenerate Jordan pair is regular). In (5.4) we show that any strongly prime 3-graded Lie algebra with non-zero socle can be trapped between

$$\mathrm{ad}(\mathrm{TKK}(V)) \triangleleft L \leq \mathrm{Der}(\mathrm{TKK}(V)),$$

where V is a simple Jordan pair coinciding with its socle and $\mathrm{Der}(\mathrm{TKK}(V))$ is itself a strongly prime 3-graded Lie algebra with the same socle, $\mathrm{ad}(\mathrm{TKK}(\mathrm{Soc}(V)))$, as L and without outer derivations. Non-degenerate 3-graded Lie algebras with essential socle can be described this way too since they are essential subdirect products of strongly prime ones with non-zero socle (5.4). Furthermore, we also characterize 3-graded Lie algebras which are central extensions of non-degenerate 3-graded Lie algebras with essential socle (5.7).

Any simple finite-dimensional Lie algebra over an algebraically closed field of characteristic 0 which is not of type E_8, F_4 or G_2 has a (non-trivial) 3-grading and, relative to any of these gradings, coincides with its socle. Examples of infinite dimensional strongly prime 3-graded Lie algebras with non-zero socle are the classical Banach Lie algebras of compact operators on an infinite dimensional complex Hilbert space (see De La Harpe, 1972 or Strasek and Zalar, 2002). Other examples can be found within the class of *finitary* simple Lie algebras (see Baranov, 1999).

2. 3-GRADED LIE ALGEBRAS AND JORDAN PAIRS

2.1. Throughout this paper, we will be dealing with Lie algebras L and Jordan pairs $V = (V^+, V^-)$ over a ring of scalars Φ containing $\frac{1}{6}$. As usual, $[x, y]$ will denote the Lie product and $\operatorname{ad} x$ the adjoint mapping determined by x. Jordan products will be denoted by $Q_x y$, for any $x \in V^\sigma, y \in V^{-\sigma}, \sigma = \pm$, with linearizations $Q_{x,z}y = \{x, y, z\} = D_{x,y}z$. The reader is referred to (Jacobson, 1962; Loos, 1975; Neher, 1996) for basic results, notation and terminology. Nevertheless, we will stress some notions and basic properties for both Jordan pairs and Lie algebras.

2.2. Recall that $(\delta^+, \delta^-) \in \operatorname{End}_\Phi(V^+) \times \operatorname{End}_\Phi(V^-)$ is a *derivation* of V if

$$\delta^\sigma(\{x, y, z\}) = \{\delta^\sigma(x), y, z\} + \{x, \delta^{-\sigma}(y), z\} + \{x, y, \delta^\sigma(z)\}$$

for any $x, z \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$ (Loos, 1975, 1.4). The set $\operatorname{Der} V$ of all derivations of V is a Lie subalgebra of $(\operatorname{End}_\Phi(V^+) \times \operatorname{End}_\Phi(V^-))^{(-)}$.

For $x \in V^+, y \in V^-, \delta(x, y) := (D_{x,y}, -D_{y,x})$, is a derivation of V (Loos, 1975, JP12), called an *inner derivation*. It deserves to be mentioned that this fact, together with the symmetry of the triple Jordan product, $\{x, y, z\} = \{z, y, x\}$, are the defining axioms of a Jordan pair, whenever $\frac{1}{6} \in \Phi$ (see Loos, 1975, p. 55). We denote by $\operatorname{IDer} V$ the Φ-module spanned by all inner derivations of V. In fact, $\operatorname{IDer} V$ is an ideal of $\operatorname{Der} V$.

2.3. An element $x \in V^\sigma$ is called an *absolute zero divisor* if $Q_x = 0$. Then V is said to be *non-degenerate* if it has no non-zero absolute zero divisors, *semiprime* if $Q_{B^\pm} B^\mp = 0$ implies $B = 0$, and *prime* if $Q_{B^\pm} C^\mp = 0$ implies $B = 0$ or $C = 0$, for $B = (B^+, B^-), C = (C^+, C^-)$ ideals of V. Similarly, $x \in L$ is an *absolute zero divisor* of L if $(\operatorname{ad} x)^2 = 0$, and L is *non-degenerate* if it has no non-zero absolute zero divisors, *semiprime* if $[I, I] = 0$ implies $I = 0$, and *prime* if $[I, J] = 0$ implies $I = 0$ or $J = 0$, for I, J ideals of L. A Jordan pair or Lie algebra is *strongly prime* if it is prime and non-degenerate.

2.4. Non-zero ideals of non-degenerate (strongly prime) Jordan pairs inherit non-degeneracy (strong primeness) (Loos, 1975, JP3; McCrimmon, 1984). The same is true for Lie algebras: every non-zero ideal of a non-degenerate (strongly prime) Lie algebra is non-degenerate (strongly prime) (Zelmanov, 1984, Lemma 4; García, 2003a, 0.4, 1.5).

2.5. Given a subset S of L, the *annihilator* of S in L consists of the elements $x \in L$ such that $[x, S] = 0$, and it is an ideal as soon as S is. We denote by $\operatorname{Ann}(I)$ the annihilator of an ideal I. Clearly, $\mathrm{Ann}(L) = Z(L)$, the center of L. If L is semiprime, $I \cap \mathrm{Ann}(I) = 0$ for any ideal I of L, and then an ideal E is *essential* ($E \cap I \neq 0$ for every non-zero ideal I of L) if and only if $\mathrm{Ann}(E) = 0$. Notice also that L is prime if and only if the annihilator of every non-zero ideal of L is zero.

2.6. We will denote by $\mathrm{Der}\, L$ the set of derivations of L. The Lie bracket of two derivations $[\delta, \mu] = \delta\mu - \mu\delta$ is again a derivation, hence $\mathrm{Der}\, L \leq (\mathrm{End}_\Phi L)^{(-)}$.

If M is an ideal of L with $\mathrm{Ann}(M) = 0$, then L can be embedded in $\mathrm{Der}\, M$ via the adjoint mapping: $L \cong \mathrm{ad}_M L \leq \mathrm{Der}\, M$.

2.7. If $L = \bigoplus_\alpha I_\alpha$ is a direct sum of simple ideals, then $\mathrm{Der}\, L \cong \prod_\alpha \mathrm{Der}\, I_\alpha$. Indeed, any derivation δ of L stabilizes $I_\alpha : \delta(I_\alpha) = \delta([I_\alpha, I_\alpha]) \subset [\delta(I_\alpha), I_\alpha] \subset I_\alpha$.

2.8. A $(2n+1)$-*grading* of a Lie algebra L is a decomposition

$$L = L_n \oplus \cdots \oplus L_1 \oplus L_0 \oplus L_{-1} \oplus \cdots \oplus L_{-n},$$

where each L_i is a submodule of L satisfying $[L_i, L_j] \subset L_{i+j}$, and where $L_{i+j} = 0$ if $i + j \neq 0, \pm 1, \ldots, \pm n$. A Lie algebra is $(2n+1)$-*graded* if it has a $(2n+1)$-grading.

2.9. If $L = L_n \oplus \cdots \oplus L_1 \oplus L_0 \oplus L_{-1} \oplus \cdots \oplus L_{-n}$ is $(2n+1)$-graded, then $V := (L_n, L_{-n})$ is a Jordan pair for the triple products defined by $\{x, y, z\} := [[x, y], z]$ for all $x, z \in L_\sigma, y \in L_{-\sigma}, \sigma = \pm n$, and it is called the *associated Jordan pair of* L. Moreover, if L is non-degenerate, so is V (Zelmanov, 1985, Lemma 1.8).

In this paper we are mainly interested in 3-graded Lie algebras. A standard example of a 3-graded Lie algebra is that given by the TKK-algebra of a Jordan pair.

For $x \in L_n, y \in L_{-n}$ and $z_0 = [x, y]$, we have that $(D_{x,y}, -D_{y,x}) = (\mathrm{ad}\, z_0, \mathrm{ad}\, z_0)$ is a derivation on $V = (L_n, L_{-n})$ (2.2). See Zelmanov, 1985, p. 351:

2.10. For any Jordan pair V there exists a 3-graded Lie algebra $\mathrm{TKK}(V) = L_1 \oplus L_0 \oplus L_{-1}$, the *Tits–Kantor–Koecher algebra of* V (Kantor, 1964, 1967, 1972; Koecher, 1967, 1968; Tits, 1962), uniquely determined by the following conditions (cf. Neher, 1996, 1.5(6)):

(TKK1) The associated Jordan pair (L_1, L_{-1}) of L is isomorphic to V.
(TKK2) $[L_1, L_{-1}] = L_0$.
(TKK3) $[x_0, L_1 \oplus L_{-1}] = 0$ implies $x_0 = 0$, for any $x_0 \in L_0$.

In general, by TKK-*algebra* we mean a Lie algebra of the form $\mathrm{TKK}(V)$ for some Jordan pair V. In the literature, 3-graded Lie algebras satisfying (TKK2) have been called *Jordan 3-graded Lie algebras*. Notice that Jordan 3-graded Lie algebras are not far from the ones directly built out of Jordan pairs by the TKK construction. Indeed, as soon as they satisfy (TKK3), for example when they are centerfree, they are isomorphic to the TKK-algebras of their associated Jordan pairs.

The transfer of regularity conditions between TKK-algebras and their associated Jordan pairs, which has been studied in García and Neher (2003, 1.6) and García (2003b, 1.6, 1.7), can be extended to 3-graded Lie algebras satisfying (TKK3) in most cases.

2.11. Proposition. *Let L be a 3-graded Lie algebra and denote by V its associated Jordan pair.*

(i) *If L is non-degenerate so is its associated Jordan pair $V = (L_1, L_{-1})$. Moreover, $L_1 \oplus [L_1, L_{-1}] \oplus L_{-1} \cong \mathrm{TKK}(V)$.*
Moreover, if L satisfies (TKK3), then
(ii) *V is non-degenerate if and only if L is non-degenerate. In this case, $\mathrm{TKK}(V)$ is an essential ideal of L.*
(iii) *V is strongly prime if and only if L is strongly prime.*

Proof. (i) Non-degeneracy of V follows from (2.9). Moreover, since the ideal $L_1 \oplus [L_1, L_{-1}] \oplus L_{-1}$ is non-degenerate (2.4), it is centerfree and hence isomorphic to $\mathrm{TKK}(V)$ (2.10).

(ii) By (i) we only need to prove that non-degeneracy of V implies non-degeneracy of L. Let $x = x_1 + x_0 + x_{-1}$ be an absolute zero divisor in L, i.e., $[x, [x, L]] = 0$. Then $[x, [x, L_{-1}]] = 0$ and, by grading decomposition properties, $[x_1, [x_1, L_{-1}]] = 0$, i.e., $\{x_1, L_{-1}, x_1\} = 0$, implying $x_1 = 0$ by non-degeneracy of V. Similarly, $x_{-1} = 0$, and thus $x = x_0 \in L_0$. Set $\delta_0 := \mathrm{ad}\, x_0$. Then $\delta_0^2 = 0$ and hence $[\delta_0, [\delta_0, \mathrm{ad}\, L]] = \mathrm{ad}\, \delta_0^2(L) = 0$. Expanding this expression we get that $0 = \delta_0(\delta_0 \mathrm{ad}\, L - \mathrm{ad}\, L\delta_0) - (\delta_0 \mathrm{ad}\, L - \mathrm{ad}\, L\delta_0)\delta_0 = -2\delta_0 \mathrm{ad}\, L\delta_0$. Since L is 2-torsion free by our initial assumption, $\mathrm{ad}\, x_0 \mathrm{ad}\, y \mathrm{ad}\, x_0 = 0$ for all $y \in L$. Therefore, $[\delta_0(y), \delta_0(z)] = \delta_0[y, \delta_0(z)] - [y, \delta_0^2(z)] = \delta_0 \mathrm{ad}\, y\, \delta_0(z) = 0$ for all $y, z \in L$, and $[\delta_0(L), \delta_0(L)] = 0$.

Now, let us show that $\delta_0(L_1) = 0$. Indeed, for any $y_1 \in L_1, y_{-1} \in L_{-1}$, we have

$$
\begin{aligned}
2Q_{\delta_0(y_1)} y_{-1} &= \{\delta_0(y_1), y_{-1}, \delta_0(y_1)\} = [[\delta_0(y_1), y_{-1}], \delta_0(y_1)] \\
&= [\delta_0[y_1, y_{-1}], \delta_0(y_1)] - [[y_1, \delta_0(y_{-1})], \delta_0(y_1)] \\
&= -\{y_1, \delta_0(y_{-1}), \delta_0(y_1)\} = [[\delta_0(y_{-1}), \delta_0(y_1)], y_1] = 0
\end{aligned}
$$

since $[\delta_0(L), \delta_0(L)] = 0$. Hence, $\delta_0(L_1) = 0$ by non-degeneracy of V, and similarly $\delta_0(L_{-1}) = 0$. Then $\delta_0(L_1 \oplus L_{-1}) = [x_0, L_1 \oplus L_{-1}] = 0$ implies $x_0 = 0$ because L satisfies (TKK3).

As we have seen, $\mathrm{TKK}(V)$ is isomorphic to the ideal $L_1 \oplus [L_1, L_{-1}] \oplus L_{-1}$, but this ideal has zero annihilator: $x = x_1 + x_0 + x_{-1} \in \mathrm{Ann}_L(L_1 \oplus [L_1, L_{-1}] \oplus L_{-1})$ implies $[x_0, L_1] = [x_{-1}, L_1] = 0$ because of the grading. Similarly, $[x_0, L_{-1}] = [x_1, L_{-1}] = 0$. But $[x_0, L_1 \oplus L_1] = 0$ implies $x_0 = 0$ by (TKK3), and $[x_\sigma, L_{-\sigma}] = 0$ for $\sigma = \pm 1$ gives $x_\sigma = 0$ by non-degenracy of V. Then $\mathrm{TKK}(V)$ is an essential ideal of L by (2.5).

(iii) That V strongly prime implies L strongly prime follows as in García and Neher (2003, 1.6) for TKK Lie (super)algebras and (ii). The converse can be obtained taking into account that $\mathrm{TKK}(V)$ is strongly prime by (2.4) and hence V is non-degenerate by (i) and prime by García and Neher (2003, 1.6).

3. DERIVATIONS OF 3-GRADED LIE ALGEBRAS

Graded derivations, which appear naturally when dealing with graded structures, generalize the adjoint mapping and are characterized by the way they act on the homogeneous parts (see Martinez, 2001, p. 805).

3.1. Given a 3-graded Lie algebra $L = L_1 \oplus L_0 \oplus L_{-1}$, we say that a derivation $\delta : L \to L$ is *μ-graded* ($\mu = 0, \pm 1, \pm 2$) if $\delta(L_\sigma) \subset L_{\sigma+\mu}$ for $\sigma = 0, \pm 1$. In this case we write $\delta \in \mathscr{D}_\mu$. Moreover, the set of all derivations of a 3-graded Lie algebra L coincides with $\operatorname{Der} L = \mathscr{D}_2 \oplus \mathscr{D}_1 \oplus \mathscr{D}_0 \oplus \mathscr{D}_{-1} \oplus \mathscr{D}_{-2}$, which is naturally 5-graded, since any derivation δ on L can be decomposed as the sum

$$\begin{aligned}\delta &= E_1\delta E_{-1} + (E_1\delta E_0 + E_0\delta E_{-1}) \\ &\quad + (E_1\delta E_1 + E_0\delta E_0 + E_{-1}\delta E_{-1}) \\ &\quad + (E_0\delta E_1 + E_{-1}\delta E_0) + E_{-1}\delta E_1 \\ &\quad \in \mathscr{D}_2 \oplus \mathscr{D}_1 \oplus \mathscr{D}_0 \oplus \mathscr{D}_{-1} \oplus \mathscr{D}_{-2},\end{aligned}$$

where each E_i denotes the projection on L_i.

We must accept the possibility that there may exist *unpleasant* ± 2-graded derivations on a 3-graded Lie algebra. For example, take an abelian 3-graded algebra $L = L_1 \oplus L_0 \oplus L_{-1}$. Then any linear mapping $f : L_{-1} \to L_1$ can be extended to a 2-graded derivation on L. However, under certain conditions, the algebra of derivations of a 3-graded Lie algebra is itself 3-graded. We begin with a lemma proving that ± 2-graded derivations vanish on von Neumann regular elements. Recall that $x \in V^\sigma$ is called (*von Neumann*) *regular* if $x = Q_x y$ for some $y \in V^{-\sigma}$. A Jordan pair is called (*von Neumann*) *regular* if all its elements are regular.

3.2. Lemma. *Any ± 2-graded derivation of a 3-graded Lie algebra L annihilates the regular elements of the associated Jordan pair of L.*

Proof. Let δ be a derivation in $\mathscr{D}_{-2}$ (a similar argument works for 2-graded derivations), i.e., $\delta(L_1) \subset L_{-1}$ and $\delta(L_{-1}) = \delta(L_0) = 0$. Given a regular element x in V^+, take $y \in V^-$ such that $Q_x y = x$. We clain that $\delta(x) = 0$. Indeed,

$$\begin{aligned}2\delta(x) &= \delta(\{x, y, x\}) = \delta([[x, y], x]) = [\delta([x, y]), x] + [[x, y], \delta(x)] \\ &= [[x, y], \delta(x)] = -\{y, x, \delta(x)\} \quad (\text{because } [x, y] \in L_0).\end{aligned}$$

Therefore,

$$\begin{aligned}\delta(x) &= -\frac{1}{2}\{y, x, \delta(x)\} = \frac{1}{4}\{y, x, \{y, x, \delta(x)\}\} \\ &= \frac{1}{2}Q_y Q_x \delta(x) + \frac{1}{4}\{y, Q_x y, \delta(x)\} \quad (\text{using Loos (1975, JP9)}) \\ &= \frac{1}{2}Q_y Q_x \delta(x) - \frac{1}{2}\delta(x) \quad (\text{since } Q_x y = x),\end{aligned}$$

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whence

$$\frac{3}{2}\delta(x) = \frac{1}{2}Q_yQ_x\delta(x) \quad \text{and}$$

$$\delta(x) = \frac{1}{3}Q_yQ_x\delta(x) = \frac{1}{9}Q_yQ_xQ_yQ_x\delta(x)$$

$$= \frac{1}{9}Q_yQ_{Q_xy}\delta(x) \quad \text{(by Loss (1975, JP3))}$$

$$= \frac{1}{9}Q_yQ_x\delta(x) \text{ (since } Q_xy = x) = \frac{1}{3}\delta(x),$$

implying $\delta(x) = 0$, since $\frac{1}{6} \in \Phi$.

3.3. Corollary. *Let L be a 3-graded Lie algebra whose associated Jordan pair is regular. Then* $\operatorname{Der} L = \mathscr{D}_1 \oplus \mathscr{D}_0 \oplus \mathscr{D}_{-1}$ *is also 3-graded.*

3.4. Corollary. *Let $L = L_1 \oplus L_0 \oplus L_{-1}$ be a simple 3-graded Lie algebra whose associated Jordan pair $V = (L_1, L_{-1})$ has a non-zero regular element. Then there are no ± 2-graded derivations in* $\operatorname{Der} L$.

Proof. Let $x^\sigma \in V^\sigma$ be regular. By Loos (1975, 5.2) there exists $x^{-\sigma} \in V^{-\sigma}$ such that (x^+, x^-) is a *Jordan pair idempotent* of V, i.e., $x^\sigma = Q_{x^\sigma}x^{-\sigma}, \sigma = \pm$. We know from (3.2) that any $\sigma 2$-graded derivation δ has $\delta(x^{-\sigma}) = 0, \sigma = \pm$. It is easy then to check by using the Jacobi identity that δ vanishes on the ideal of L generated by $x^{-\sigma}$, and L being simple implies $\delta(L) = 0$, i.e., $\delta = 0$.

4. THE SOCLE OF A 3-GRADED LIE ALGEBRA

The notion of socle for non-degenerate Jordan pairs (see Loos, 1989) is extended here to 3-graded Lie algebras with non-degenerate associated Jordan pair. But before dealing with Lie algebras, it will be useful to recall the form of the socle for some standard examples of Jordan pairs.

4.1. Examples. (1) Let X_1, X_2 be vector spaces over a division Φ-algebra Δ. Denote by $\mathscr{L}(X_i, X_j)$ the Φ-module of all Δ-linear mapping from X_i to X_j, and put $\mathscr{L}(X) := \mathscr{L}(X, X)$ for any vector space X. Then $(\mathscr{L}(X_1, X_2), \mathscr{L}(X_2, X_1))$ is a Jordan pair with $Q_ab = aba$. It will be called the *rectangular Jordan pair* defined by (X_1, X_2). Any rectangular Jordan pair $(\mathscr{L}(X_1, X_2), \mathscr{L}(X_2, X_1))$ is strongly prime with socle equal to $(\mathscr{F}(X_1, X_2), \mathscr{F}(X_2, X_1))$, where $\mathscr{F}(X_i, X_j)$ is the set of all $a \in \mathscr{L}(X_i, X_j)$ having finite rank (see Fernández Lopéz and Tocón, 2003).

(2) Let X be a vector space over a field F, and let $q : X \to F$ be a quadratic form on X with associated bilinear form $q(x, y) := q(x + y) - q(x) - q(y)$. Then (X, X) becomes a Jordan pair for the product given by $Q_xy = q(x, y)x - q(x)y$. It will be called the *Clifford pair* defined by q. If q is non-degenerate, then the Clifford pair

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(X, X) is non-degenerate and coincides with its socle (see Loos, 1975, 12.8). Moreover, it is simple if $\dim X \neq 2$ (Jacobson, 1981, p. 14, Example 4).

(3) Let $(A, *)$ be an associative algebra with involution. Denote by $H(A, *) := \{a \in A | a = a^*\}$ the set of all hermitian elements. Then $(H(A, *), H(A, *))$ is a subpair of $(A, A)^{(+)}$. It will be called the *hermitian pair* defined by $(A, *)$. For instance, we have the hermitian Jordan pair $(H(\mathscr{L}(X), *), H(\mathscr{L}(X), *))$, where X is a vector space over a field F endowed with a non-degenerate symmetric bilinear form $g : X \times X \to F$, and where $a \mapsto a^*$ is the *adjoint involution*: $g(xa, y) = g(x, ya^*)$. Any hermitian pair $V = (H(\mathscr{L}(X), *),\ H(\mathscr{L}(X), *))$ is strongly prime with $\mathrm{Soc}(V) = (H(\mathscr{F}(X), *), H(\mathscr{F}(X), *))$, where $\mathscr{F}(X)$ is the set of all $a \in \mathscr{L}(X)$ having finite rank (see Fernández Lopéz and Tocón, 2003).

Now we return to Lie algebras by proving the following lemma.

4.2. Lemma. *Let $L = L_1 \oplus L_0 \oplus L_{-1}$ be a 3-graded Lie algebra, and let (M_1, M_{-1}) be an ideal of the associated Jordan pair (L_1, L_{-1}). If (M_1, M_{-1}) is* perfect, *i.e.*, $M_\sigma = \{M_\sigma, M_{-\sigma}, M_\sigma\} = [[M_\sigma, M_{-\sigma}], M_\sigma], \sigma = \pm 1,$ *then* $M := M_1 \oplus [M_1, M_{-1}] \oplus M_{-1}$ *is invariant under derivations of L, hence it is an ideal of L.*

Proof. By perfection, the ideal (M_1, M_{-1}) is invariant under derivations of the Jordan pair (L_1, L_{-1}) and, taking into account that 0-graded derivations of L define Jordan pair derivations, we get

$$\delta_0(M_i) = \delta_0(\{M_i, M_{-i}, M_i\}) \subset M_i, \quad i = \pm 1, \tag{1}$$

for any 0-graded derivation $\delta_0 \in \mathrm{Der}\, L$. From (1),

$$\delta_0([M_1, M_{-1}]) \subset [\delta_0(M_1), M_{-1}] + [M_1, \delta_0(M_{-1})] \subset [M_1, M_{-1}]. \tag{2}$$

Now let us check that $\delta_i(M) \subset M$ for every i-graded derivation $\delta_i \in \mathrm{Der}\, L, i = \pm 1$:

$$\begin{aligned} &\delta_i([M_1, M_{-1}]) \subset [M_i, \delta_i(M_{-i})] \subset [M_i, L_0] \subset M_i, \\ &\delta_i(M_i) = 0, \quad \text{and} \\ &\delta_i(M_{-i}) = \delta_i[[M_{-i}, M_i], M_{-i}] \subset [M_1, M_{-1}]. \end{aligned} \tag{3}$$

Finally, for $2i$-graded derivations $\delta_{2i} \in \mathrm{Der}\, L, i = \pm 1$, we have

$$\begin{aligned} \delta_{2i}(M_i \oplus [M_1, M_{-1}]) &= 0, \\ \delta_{2i}(M_{-i}) &= \delta_{2i}(\{M_{-i}, M_i, M_{-i}\}) \\ &\subset [\delta_{2i}[M_{-i}, M_i], M_{-i}] + [[M_{-i}, M_i], \delta_{2i}(M_{-i})] \\ &\subset [[M_1, M_{-1}], L_i] \subset M_i. \end{aligned} \tag{4}$$

The proof is now complete altogether.

4.3. Let $L = L_1 \oplus L_0 \oplus L_{-1}$ be a 3-graded Lie algebra whose associated Jordan pair $V = (L_1, L_{-1})$ is non-degenerate. From Loos, (1989, Theorem 1), $\mathrm{Soc}(V)$ is regular and hence a perfect ideal of V. Then, by (4.2), the ideal of L generated by $\mathrm{Soc}(V)$ coincides with $\mathrm{Soc}(V)^+ \oplus [\mathrm{Soc}(V)^+, \mathrm{Soc}(V)^-] \oplus \mathrm{Soc}(V)^-$. This ideal will be called

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the *socle* of L and denoted by $\mathrm{Soc}(L)$. We remark that this definition of socle does not depend only on the Lie algebra L, but also on the chosen 3-grading on L.

4.4. According to Benkart's definition (see Benkart, 1977), a submodule K of a Lie algebra L is an *inner ideal* if $[K,[K,L]] \subset K$. Because of the grading, inner ideals (minimal inner ideals) $K \subset L_\sigma$ $(\sigma = \pm 1)$ of the associated Jordan pair $V = (L_1, L_{-1})$ of a 3-graded Lie algebra L are inner ideals (minimal inner ideals) of L. In particular, any $x \in L_\sigma$ $(\sigma = \pm 1)$ generates the *principal inner ideal* $[x,[x,L]]$ (notice that$(\mathrm{ad}\, x)^3 = 0$). If V is non-degenerate, then $\mathrm{Soc}(L_\sigma), \sigma = \pm 1$, is the sum of all inner ideals of V contained in L_σ (see Loos, 1989), so $\mathrm{Soc}(L)$ is generated as a Lie algebra by these minimal inner ideals.

4.5. Zelmanov (1985, Sec. 5) introduces a notion of inner ideals for **Z**-graded Lie algebras which in the particular case of a 3-graded Lie algebra reads as follows (cf. García, to appear, 1.1).

Let L be a 3-graded Lie algebra. We say that a graded Φ-submodule $B = B_1 \oplus B_0 \oplus B_{-1}$ of L is a 3-*graded inner ideal* if

(i) B is a subalgebra of L.
(ii) B_1, B_{-1} are inner ideals of $V = (L_1, L_{-1})$.

It is clear that if $(B_1, B_{-1}) \subset (L_1, L_{-1})$ is a pair of inner ideals of V, then $B_1 \oplus [B_1, B_{-1}] \oplus B_{-1}$ is a 3-graded inner ideal of L. In particular, a Jordan pair idempotent $e = (e^+, e^-)$ of V determines the 3-graded inner ideal $L(e) := V_2(e)^+ \oplus [V_2(e)^+, V_2(e)^-] \oplus V_2(e)^-$ (see Loos, 1975, 5.5).

4.6. Let $K^\sigma \subset V^\sigma, \sigma = \pm$, be two minimal inner ideals of a Jordan pair V. For the Jordan subpair $K := (K^+, K^-)$ the following conditions are equivalent:

(i) K is a division Jordan pair,
(ii) K is non-degenerate, and
(iii) $Q_{K^+}K^- \neq 0$.

Moreover, in this case $K = V_2(e)$ is the Peirce 2-space of a division Jordan pair idempotent e of V. Indeed, (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are clear. To prove (iii) $\Rightarrow$ (i), let $x^+ \in K^+$ be such that $Q_{x^+}K^- \neq 0$. Then $Q_{x^+}K^- = K^+$ by minimality of K^+ and hence there exists $x^- \in K^-$ such that $Q_{x^+}x^- = x^+$. Then $e := (x^+, Q_{x^-}x^+)$ is a division Jordan pair idempotent (see Loos, 1975, 5.5) and $K = V_2(e)$.

4.7. Proposition. *Let L be a 3-graded Lie algebra with non-degenerate associated Jordan pair V. Then $\mathrm{Soc}(L) = \sum_e L(e)$, where the sum is taken over all division Jordan pair idempotents of V. Furthermore, $L(e)$ is a central extension of the TKK-algebra of the division Jordan pair $V_2(e)$.*

Proof. The containments $\sum_e L(e) \subset \mathrm{Soc}(L)$ and $\mathrm{Soc}(V)^+ \oplus \mathrm{Soc}(V)^- \subset \sum_e L(e)$ are clear. Thus, it is enough to show that $[\mathrm{Soc}(V)^+, \mathrm{Soc}(V)^-]$ is also contained in the sum of all $L(e)$.

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We know that $[\mathrm{Soc}(V)^+, \mathrm{Soc}(V)^-]$ is generated as a Φ-module by elements of the form $[a^+, b^-]$, for division Jordan pair idempotents $a = (a^+, b^-)$ and $b = (b^+, b^-)$ of V. Now

$$\begin{aligned} 2[a^+, b^-] &= [[[a^+, a^-], a^+], b^-] = [[[a^+, a^-], b^-], a^+] + [[a^+, a^-], [a^+, b^-]] \\ &= [a^+, \{b^-, a^+, a^-\}] + [[a^+, [a^+, b^-]], a^-] + [a^+, [a^-, [a^+, b^-]]] \\ &= 2[a^+, \{b^-, a^+, a^-\}] - [Q_{a^+} b^-, a^-] \\ &= 2[a^+, Q_{(b^- + a^-)} a^+] - 2[a^+, Q_{b^-} a^+] - 2[a^+, a^-] - [Q_{a^+}, b^-, a^-]. \end{aligned}$$

Clearly, both $[a^+, a^-]$ and $[Q_{a^+} b^-, a^-]$ belong to $L(a)$, while $[a^+, Q_{b^-} a^+] \in L(e)$ for some division Jordan pair idempotent e because $(Q_{a^+} V^-, Q_{Q_{b^- a^+}} V^+)$ is a pair of minimal inner ideals of V that satisfies (4.6)(iii) when $[a^+ Q_{b^-} a^+] \neq 0$. Thus, we only need to consider the first summand $[a^+, Q_{(b^- + a^-)} a^+]$.

Without loss of generality we can assume that $Q_{a^+}(b^- + a^-) = Q_{a^+} b^- + a^+ \neq 0$ (in case $Q_{a^+}(b^- + a^-) = 0$ replace b^- by $-b^-$ in the above formula and work with $2[a^+, Q_{(-b^- + a^-)} a^+] + 2[a^+, Q_{b^-} a^+] - 2[a^+, a^-] + 2[Q_{a^+} b^-, a^-]$, where now $Q_{a^+}(-b^- + a^-) = -Q_{a^+} b^- + a^+ = 2a^+ \neq 0$). In this case, the subpair $K := (Q_{a^+} V^-, Q_{(b^- + a^-)} Q_{a^+} V^-)$ satisfies (4.6)(iii) and hence $[a^+, Q_{(b^- + a^-)} a^+] \in L(e)$ for some division Jordan pair idempotent e, which completes the proof.

Recall that V is non-degenerate if L is so, by (2.11)(i). In this case $\mathrm{Soc}(L)$ is non-degenerate by (2.4) and hence it is isomorphic to $\mathrm{TKK}(\mathrm{Soc}(V))$ by (2.10).

4.8. Proposition. *Let $L = L_1 \oplus L_0 \oplus L_{-1}$ be a non-degenerate 3-graded Lie algebra, with associated Jordan pair V. Then*

(1) $\mathrm{Soc}(L) = \bigoplus_\alpha M_\alpha$, *where each M_α is a 3-graded ideal of L. In fact, $M_\alpha = V_\alpha^+ \oplus [V_\alpha^+, V_\alpha^-] \oplus V_\alpha^-$, with the $V_\alpha = (V_\alpha^+, V_\alpha^-)$ being the simple ideals of* $\mathrm{Soc}(V)$. *Moreover, each $M_\alpha \cong \mathrm{TKK}(V_\alpha)$ is a simple ideal of L.*
(2) *For any 3-graded ideal I of L,* $\mathrm{Soc}(I) = I \cap \mathrm{Soc}(L)$.

Proof. By Loos (1989, Theorem, 2), $\mathrm{Soc}(V) = \bigoplus V_\alpha$ where the V_α are the simple ideals of $\mathrm{Soc}(V)$. Moreover, using that $V_\alpha^+ = \{V_\alpha^+, V_\alpha^-, V_\alpha^+\}$, we get that $[V_\alpha^+, V_\beta^+] = 0$ when $\alpha \neq \beta$. Therefore

$$\begin{aligned} \mathrm{Soc}(L) &= \mathrm{Soc}(V)^+ \oplus [\mathrm{Soc}(V)^+, \mathrm{Soc}(V)^-] \oplus \mathrm{Soc}(V)^- \\ &= \bigoplus_\alpha (V_\alpha^+ \oplus [V_\alpha^+, V_\alpha^-] \oplus V_\alpha^-) = \bigoplus_\alpha M_\alpha \end{aligned}$$

and the M_α are ideals of L by (4.2). Since L is non-degenerate, each M_α is also non-degenerate by (2.4), hence $M_\alpha = \mathrm{TKK}(V_\alpha)$ is simple by García and Neher (2003, 1.6).

For (2), denote by $U = (I_1, I_{-1})$ the Jordan pair associated to the 3-graded ideal I. First notice that U is non-degenerate so it makes sense to consider $\mathrm{Soc}(I)$.

By Jordan socle theory (cf. Loos, 1989, Proposition 3), $\mathrm{Soc}(U) = \mathrm{Soc}(V) \cap U$ and the simple ideals of $\mathrm{Soc}(U)$ are precisely the simple ideals of V contained in U. Now (2) follows from (1).

5. 3-GRADED LIE ALGEBRAS WITH ESSENTIAL SOCLE

A structure theorem is given for non-degenerate 3-graded Lie algebras with essential socle. We begin by transferring regularity conditions from *large* 3-graded ideals of a 3-graded Lie algebra to the whole algebra. Our approach will consist in reducing the question to a Jordan pair one.

5.1. Let V be a Jordan pair. Given $X \subset V^{\sigma}$, denote by $\mathrm{Ann}(X) \subset V^{-\sigma}$ the *annihilator* of X in V (cf. Loos, 1975, 10.3), and set $\mathrm{Ann}(I) := (\mathrm{Ann}(I^-), \mathrm{Ann}(I^+))$ to denote annihilator (ideal) of an ideal I of V. It is not difficult to see (Loos, 1975, JP21) that if I is non-degenerate, then $a \in \mathrm{Ann}(I^{\sigma})$ if and only if $\{a, I^{\sigma}, V^{-\sigma}\} = 0$ and $\{I^{\sigma}, a, V^{-\sigma}\} = 0$.

5.2. Lemma. *Let $I \lhd V$ be a nondegenerate ideal of a Jordan pair V. If* $\mathrm{Ann}(I) = 0$ *then V is non-degenerate. Moreover, V is strongly prime if and only if I is strongly prime.*

Proof. Let $v \in V^{-\sigma}$ be an absolute zero divisor of V, and let $y \in I^{-\sigma}$, and $y \in I^{\sigma}$. By Loos (1975, JP21), for any $a \in I^{-\sigma}$ we have

$$\begin{aligned} Q_{\{v,y,z\}}a &= Q_vQ_yQ_za + Q_zQ_yQ_va + D_{z,y}Q_vD_{y,z}a - Q_{v,Q_zQ_yv}a \\ &= -\{v, a, Q_zQ_yv\} \end{aligned}$$

since v is an absolute zero divisor. Let $b \in I^{-\sigma}$. By the above equality,

$$\begin{aligned} Q_{Q_{\{v,y,z\}}a}b = Q_{\{v,a,Q_zQ_yv\}}b &= -\{v, b, Q_{Q_zQ_yv}Q_av\} \\ &= -\{v, b, Q_zQ_yQ_vQ_yQ_zQ_av\} = 0, \end{aligned}$$

using again that v is an absolute zero divisor. Then, by non-degeneracy of I, $Q_{\{v,y,z\}}a = 0$ for every $a \in I^{-\sigma}$, and

$$\{v, I^{-\sigma}, V^{\sigma}\} = 0. \qquad (5)$$

Now let $x \in I^{-\sigma}$ and $z \in V^{-\sigma}$. By Loos (1975, JP20), for any $a \in I^{\sigma}$ we have

$$\begin{aligned} Q_{\{x,v,z\}}a &= Q_xQ_vQ_za + Q_zQ_vQ_xa + Q_{x,z}Q_vQ_{x,z}a - \{Q_xv, a, Q_zv\} \\ &= -\{Q_xv, a, Q_zv\}, \end{aligned}$$

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since v is an absolute zero divisor. Hence, by Loos (1975, JP21), for $b \in I^\sigma$ we have

$$Q_{Q_{\{x,v,z\}}a}b = Q_{\{Q_xv,a,Q_zv\}}b = Q_{Q_xv}Q_aQ_{Q_zv}b + Q_{Q_zv}Q_aQ_{Q_xv}b \\ - \{Q_zv, b, Q_{Q_xv}Q_aQ_zv\} + \{Q_xv, a, Q_{Q_zv}\{a, Q_xv, b\}\} = 0,$$

since v is an absolute zero divisor and hence so is any Q_xv (Loos, 1975, JP3). Again, by non-degeneracy of I, we get

$$\{I^{-\sigma}, v, V^{-\sigma}\} = 0. \tag{6}$$

Therefore, from (5) and (6) we get that $v \in \mathrm{Ann}(I) = 0$.

Finally, it is straightforward to see that every non-zero ideal of V has nonzero intersection with I. Thus, V is prime if I is so, which together with (2.4) completes the proof.

5.3. Theorem. *Let $M = M_1 \oplus M_0 \oplus M_{-1}$ be a graded ideal of a 3-graded Lie algebra L. Suppose that $M_0 = [M_1, M_{-1}]$ and $\mathrm{Ann}_L(M) = 0$, and denote by $U = (M_1, M_{-1}) \lhd V = (L_1, L_{-1})$ the associated Jordan pairs of M and L, respectively. Then L is non-degenerate (strongly prime) if and only if U is so. Moreover, if U is non-degenerate, then $\mathrm{Soc}(L) = \mathrm{Soc}(M)$.*

Proof. Clearly, $\mathrm{Ann}_L(M) = 0$ implies both M and L satisfy (TKK3). Moreover, $\mathrm{Ann}_V(U) = 0$: if $a_\sigma \in \mathrm{Ann}_V(M_{-\sigma})$ $(\sigma = \pm 1)$, then

$$[[a_\sigma, M_{-\sigma}], L_\sigma] = \{a_\sigma, M_{-\sigma}, L_\sigma\} = 0$$

and similarly,

$$[[M_{-\sigma}, a_\sigma], L_{-\sigma}] = \{M_{-\sigma}, a_\sigma, L_{-\sigma}\} = 0.$$

Hence, $[a_\sigma, M_{-\sigma}] = 0$ by (TKK3), and $a_\sigma \in \mathrm{Ann}_L(M) = 0$.

In general, if L is non-degenerate (strongly prime) then so is U (2.4) and (2.11)(i) and (iii). Suppose then that U is non-degenerate (strongly prime). Then V is non-degenerate (strongly prime) by (5.2), and hence L is non-degenerate (strongly prime) by (2.11)(iii). The equality $\mathrm{Soc}(L) = \mathrm{Soc}(M)$ follows from (4.8) since $\mathrm{Ann}_L(M) = 0$.

Now everything is ready to prove the main result of this paper.

5.4. Theorem. *For a 3-graded Lie algebra L, the following statements are equivalent:*

(i) *L satisfies (TKK3) and its associated Jordan pair is non-degenerate with essential socle.*
(ii) *L is non-degenerate with essential socle.*
(iii) $\mathrm{ad}(\mathrm{TKK}(V)) \lhd L \leq \mathrm{Der}(\mathrm{TKK}(V))$, *where V is a non-degenerate Jordan pair coinciding with its socle.*
(iv) $\bigoplus \mathrm{ad}(\mathrm{TKK}(V_\alpha)) \lhd L \leq \prod \mathrm{Der}(\mathrm{TKK}(V_\alpha))$, *where each V_α is a simple Jordan pair with minimal inner ideals.*

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If L is as in (iii), $\mathrm{Soc}(L) = \mathrm{ad}(\mathrm{TKK}(V))$ *and L is strongly prime if and only if* $\mathrm{Soc}(L)$ *is simple, if and only if V is simple. Moreover, in this case,* $\mathrm{Der}(\mathrm{TKK}(V))$ *is the largest strongly prime* 3*-graded Lie algebra having socle equal to* $\mathrm{ad}(\mathrm{TKK}(V))$, *and every derivation of* $\mathrm{Der}(\mathrm{TKK}(V))$ *is inner.*

Proof. (i) $\Rightarrow$ (ii). Set $V = (L_1, L_{-1})$ and $M = \mathrm{Soc}(L)$. Then $(M_1, M_{-1}) = \mathrm{Soc}(V)$. By (5.3), we only need to prove that $\mathrm{Ann}_L(M) = 0$. Since $\mathrm{Ann}_L(M)$ is a graded ideal because so is M, it suffices to consider homogeneous elements. For $a_\sigma \in \mathrm{Ann}_L(M)$ $(\sigma = \pm 1)$, we have by (5.1) that $a_\sigma \in \mathrm{Ann}_V(M_{-\sigma}) = 0$, since V is non-degenerate. For $a_0 \in \mathrm{Ann}_L(M)$ and $\sigma = \pm 1$, $[a_0, L_\sigma] \subset \mathrm{Ann}_L(M) \cap L_\sigma = 0$, and hence $a_0 = 0$ by (TKK3).

(ii) $\Rightarrow$ (iii). Set $\mathrm{Soc}(L) = \mathrm{TKK}(V)$, where V now denotes the socle of the Jordan pair (L_1, L_{-1}). The adjoint representation defines then a 3-graded Lie algebra isomorphism of L into $\mathrm{Der}(\mathrm{TKK}(V))$ by (2.6):

$$\mathrm{ad}(\mathrm{TKK}(V)) \triangleleft \mathrm{ad}\, L \leq \mathrm{Der}(\mathrm{TKK}(V)).$$

(iii) $\Rightarrow$ (iv). Let V be as in (iii). By Loos (1989, Theorem 2) $V = \bigoplus V_\alpha$, with each V_α being a simple Jordan pair containing minimal inner ideals. Now the implication follows since $\mathrm{TKK}(\bigoplus V_\alpha) \cong \bigoplus \mathrm{TKK}(V_\alpha)$, and $\mathrm{Der}(\bigoplus \mathrm{TKK}(V_\alpha)) \cong \prod \mathrm{Der}(\mathrm{TKK}(V_\alpha))$ by (2.7) because $\mathrm{TKK}(V)$ is a direct sum of the ideals $\mathrm{TKK}(V_\alpha)$, which are simple by García and Neher (2003, 1.6).

(iv) $\Rightarrow$ (i). Let $V = \bigoplus V_\alpha$ be a non-degenerate Jordan pair coinciding with its socle, and set $M := \bigoplus \mathrm{ad}(\mathrm{TKK}(V_\alpha)) = \mathrm{ad}(\bigoplus \mathrm{TKK}(V_\alpha)) \cong \mathrm{ad}(\mathrm{TKK}(\bigoplus V_\alpha))$. Since $M \triangleleft L \leq \prod \mathrm{Der}(\mathrm{TKK}(V_\alpha))$, $\mathrm{Ann}_L(M) = 0$ and hence L clearly satisfies (TKK3). Moreover, $\mathrm{Soc}(L) = M$ is an essential ideal of L.

Finally, suppose that L is as in (iii). By (5.3) and (4.8), L is strongly prime if and only if $\mathrm{Soc}(L)$ is simple, equivalently, V is simple. In this case, putting $M = \mathrm{TKK}(V)$, we have that $\mathrm{Der}(M)$ is the largest strongly prime 3-graded Lie algebra having socle equal to ad M. Now let L be a strongly prime 3-graded Lie algebra with non-zero socle M. Then the mapping $\delta \mapsto \delta_{|M}$, associating to any deviation δ on $\mathrm{Der}(L)$ its restriction to M, is an isomorphism of $\mathrm{Der}(L)$ into $\mathrm{Der}(M)$, and therefore $\mathrm{Der}(L)$ is strongly prime with the same socle as L. Hence, taking $L = \mathrm{Der}(M)$, we have that every derivation on $\mathrm{Der}(M)$ is inner.

5.5. Remarks. (1) Since, by (4.4), principal inner ideals of the Jordan pair of a 3-graded Lie algebra L are principal inner ideals of L, it follows from Loos (1989, Corollary 1) that non-degenerate 3-graded Lie algebras satisfying the descending chain condition on principal inner ideals have essential socle. Note also that, by the von Neumann regularity of the Jordan socle, 3-graded Lie algebras with non-zero socle are $*$-Lie algebras in the sense of Benkart (1977).

(2) A Lie algebra is called *Artinian* if it satisfies the descending chain condition on all inner ideals. Since the associated Jordan pair of an Artinian 3-graded Lie is also Artinian, we have by Loos (1975, 12.12) and (5.4) that non-degenerate Artinian

3-graded Lie algebras L are of the form

$$\bigoplus_{1}^{n} \mathrm{ad}(\mathrm{TKK}(V_i)) \triangleleft L \leq \prod_{1}^{n} \mathrm{Der}(\mathrm{TKK}(V_i)),$$

where the V_i are simple Artinian Jordan pairs.

If L is actually a semisimple 3-graded Lie algebra which is finite-dimensional over a field F of characteristic zero, then L is non-degenerate (Benkart, 1977, p. 64) and clearly Artinian. Moreover,

$$L = \bigoplus_{1}^{n} \mathrm{ad}(\mathrm{TKK}(V_i)) \cong \bigoplus_{1}^{n} \mathrm{TKK}(V_i),$$

since every derivation in a simple finite-dimensional Lie algebra over a field of characteristic 0 is inner (Jacobson, 1962, Sec. 3 (6.6)).

5.6. By using the equivalence (i) $\Leftrightarrow$ (ii) of Theorem 5.4, central extensions of non-degenerate 3-graded Lie algebras with essential socle can also be determined. We will say that a 3-graded Lie algebra L satisfies *weak*-(TKK3) if

$$[x_0, L_1 \oplus L_{-1}] = 0 \Rightarrow x_0 \in Z(L)$$

for $x_0 \in L_0$. Notice that any 3-graded Lie algebra L with (TKK2) or (TKK3) verifies weak-(TKK3).

5.7. Corollary. *Let L be a 3-graded Lie algebra with non-degenerate associated Jordan pair V. The following conditions are equivalent*:

(i) *$L/Z(L)$ is non-degenerate with essential socle.*
(ii) *V has essential socle and L satisfies weak-*(TKK3).

Proof. First we claim that non-degeneracy of V implies that $Z(L) \subset L_0$. Indeed, if $x = x_1 + x_0 + x_{-1} \in Z(L)$ then $[x, L_{-\sigma}] = 0$, $\sigma = \pm 1$, so $[x_\sigma, L_{-\sigma}] = 0$, which gives $x_\sigma = 0$ by nondegeneracy of V. Hence V is also the associated Jordan pair of $L/Z(L)$.

(i) $\Rightarrow$ (ii) Since $L/Z(L)$ satisifies (TKK3) by (5.4)(ii) $\Rightarrow$ (i), L has weak-(TKK3). Moreover, being V the associated Jordan pair of both L and $L/Z(L)$, the socle of V is essential by (5.4)(ii) $\Rightarrow$ (i).

(ii) $\Rightarrow$ (i) We claim that $\overline{L} = L/Z(L)$ has (TKK3). Indeed, if $\bar{x}_0 \in \overline{L}_0$ satisfies $[\bar{x}_0, \overline{L}_1 \oplus \overline{L}_{-1}] = \overline{0}$, then $[x_0, L_1 \oplus L_{-1}] \subset Z(L)$, so $\{[x_0, L_\sigma], L_{-\sigma}, L_\sigma\} = 0$ implies by non-degeneracy of V that $[x_0, L_\sigma] = 0$ for both $\sigma = \pm 1$. Now, by weak-TKK(3), $x_0 \in Z(L)$, i.e., $\bar{x}_0 = \overline{0}$. Since V has essential socle, $L/Z(L)$ is non-degenerate with essential socle by (5.4)(i) $\Rightarrow$ (ii).

In our paper (see Fernández López, García and Gómez Lozano, to appear) we determine the TKK-algebras of the simple Jordan pairs with minimal inner ideals.

These TKK-algebras provide examples of simple infinite-dimensional 3-graded Lie algebras coinciding with their socle. Some of these examples will be nevertheless treated below.

5.8. Examples. (1) Let X be a vector space over a division algebra Δ, decomposed into the direct sum of two subspaces, $X = X_1 \oplus X_2$. Then

$$L = \mathscr{L}(X_1 \oplus X_2)^{(-)} = \begin{pmatrix} \mathscr{L}(X_1) & \mathscr{L}(X_2, X_1) \\ \mathscr{L}(X_1, X_2) & \mathscr{L}(X_2) \end{pmatrix}$$

is a 3-graded Lie algebra whose associated Jordan pair is the rectangular Jordan pair $V = (\mathscr{L}(X_1, X_2), \mathscr{L}(X_2, X_1))$ (cf. 4.1(1)). It is not difficult to see that L satisfies weak-(TKK3), and since V is strongly prime with $\mathrm{Soc}(V) = (\mathscr{F}(X_1, X_2), \mathscr{F}(X_2, X_1))$ (4.1(1)), we have by (5.7) that $L/Z(L)$ is (strongly prime) with essential socle: $\mathrm{Soc}(L/Z(L)) \cong \mathrm{TKK}(\mathrm{Soc}(V))$ (4.3). We also have that $\mathrm{Soc}(L)$ coincides with the derived ideal $[\mathscr{F}(X), \mathscr{F}(X)]$ (take matrix representations of elements and argue as in Neher (1996, 3.4.3) if $\dim(X) \geq 3$; the two-dimensional case can be treated separately), so $\mathrm{Soc}(L)$ does not depend on the chosen decomposition $X = X_1 \oplus X_2$. If X is infinite-dimensional, the $\mathrm{Soc}(L)$ is centerfree and hence $\mathrm{Soc}(L) \cong \mathrm{TKK}(\mathrm{Soc}(V))$ is simple. If additionally Δ is finite-dimensional over its center, then $\mathrm{Soc}(L)$ is the *central finitary simple Lie algebra* $\mathfrak{fsl}(X)$ (see Baranov, 1999, 6.19(i)).

(2) Let X be a vector space over a field F with a non-degenerate quadratic form $q : X \to F$. The *orthogonal Lie algebra* (of q) is the following subalgebra of $\mathscr{L}(X)^{(-)} : \mathfrak{o}(X, q) = \{a \in \mathscr{L}(X) \mid q(ax, x) = 0 \text{ for all } x \in X\}$.

For $x, y, z \in X$, let x^* be the F-linear form on X given by $x^*(z) = q(x, z)$, and denote by yx^* the linear mapping on X given by $yx^*(z) = yq(x, z)$. Clearly, $yx^* - xy^* \in \mathfrak{o}(X, q)$.

Suppose that X contains a hyperbolic plane H (for instance, F is an algebraically closed field and $\dim X > 1$). Then $X = H \oplus H^\perp$ and the elements of $\mathfrak{o}(X, q)$ can be represented as 3×3-matrices (see Neher, 1996, 5.1) in the following way:

$$a = \begin{pmatrix} \alpha & -y^* & 0 \\ x & b & y \\ 0 & -x^* & -\alpha \end{pmatrix},$$

for all $\alpha \in F, x, y \in H^\perp$, and $b \in \mathfrak{o}(H^\perp, q)$. This representation defines the following 3-grading on $L = \mathfrak{o}(X, q)$:

$$L_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & -x^* & 0 \end{pmatrix} \right\}, \quad L_0 = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -\alpha \end{pmatrix} \right\},$$

$$L_{-1} = \left\{ \begin{pmatrix} 0 & -y^* & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

whose associated Jordan pair $V = (L_1, L_{-1})$ is isomorphic to the Clifford pair $(H^\perp, H^\perp)$ (with the Jordan product given in (4.1)(2)), and hence it is non-degenerate and coincides with its socle. Since $\mathrm{char}(F) \neq 2$, L satisfies (TKK3) and hence it is non-degenerate with essential socle (5.4). In fact, $\mathrm{Soc}(\mathfrak{o}(X,q))$ is the *central finitary simple Lie algebra* $\mathfrak{fo}(X,q)$ (see Baranov, 1999) if $\dim X > 4$.

(3) Let X be a vector space over a field F with a non-degenerate alternating bilinear form $h : X \times X \to F$. The *symplectic Lie algebra* (of h) is the following subalgebra of $\mathscr{L}(X)$: $\mathfrak{sp}(X,h) = \{a \in \mathfrak{L}(X) \mid h(ax,y) + h(x,ay) = 0 \text{ for all } x, y \in X\}$.

Suppose that $X = \bigoplus_{i\in I} H_i$ is an orthogonal sum of hyperbolic planes (for instance, X is finite-dimensional or even is of countable dimension, Kaplansky, 1974, p. 45, example 2), $H_i = \langle x_i, y_i\rangle$. Then the elements of $\mathfrak{sp}(X,h)$ can be represented as 2×2 matrices (see Neher, 1996, 4.2)

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11}^* \end{pmatrix},$$

where $a_{11} \in \mathscr{L}(Y)$ and $a_{12}, a_{21} \in H(\mathscr{L}(Y), *)$, and where $Y := \bigoplus_{i\in I} Fy_i$ is endowed with the non-degenerate symmetric bilinear form given by $g(y_i, y_j) = \delta_{ij}$. This representation defines a 3-grading on $L = \mathfrak{sp}(X,h)$ whose associated Jordan pair $V = (L_1, L_{-1})$ is isomorphic to the hermitian pair $(H(\mathscr{L}(Y),*), H(\mathscr{L}(Y),*))$. Thus $\mathrm{Soc}(V) \cong (H(\mathscr{F}(Y),*), H(\mathscr{F}(Y),*))$ (cf. 4.1(3)). As above, it is not difficult to see that L satisfies (TKK3) and hence L is strongly prime with non-zero socle. In fact, $\mathrm{Soc}(L)$ is isomorphic to the central finitary simple Lie algebra $\mathfrak{fsp}(X,h)$ (see Baranov, 1999).

5.9. Banach Lie algebras. (1) Following De La Harpe (1972) or Strasek and Zalar (2002), we denote by $\mathfrak{gl}(H, \mathscr{C}_\infty)$ the classical Banach Lie algebra of compact operators on an infinite dimensional complex Hilbert space H. Any decomposition of H as a Hilbert direct sum of closed subspaces, $H = X \oplus Y$, defines a 3-grading on $\mathfrak{gl}(H, \mathscr{C}_\infty)$ with associated Jordan pair that given by the generalized Cartan factor $\mathrm{KL}(X,Y)$ of compact operators from X into Y. The 3-graded Lie algebra $\mathfrak{gl}(H, \mathscr{C}_\infty)$ is strongly prime with socle equal to $[\mathrm{FBL}(H), \mathrm{FBL}(H)]$, the derived ideal of finite rank bounded linear operators.

(2) The analytic counterpart of $\mathfrak{fo}(X,q)$ is the classical orthogonal Banach Lie algebra of compact operators on an infinite dimensional complex Hilbert space H (see De La Harpe 1972 or Strasek and Zalar, 2002), given by

$$o(H, \theta, \mathscr{C}_\infty) = \{a \in \mathrm{KL}(H) \mid \theta^* a \theta = -a\},$$

where θ is a *conjugation* on H, i.e., $\theta : H \to H$ is an involutive conjugate linear mapping satisfying $\langle \theta_x, y\rangle = \langle \theta_y, x\rangle$ for all $x, y \in H$. Then the formula $q(x,y) := \langle x, \theta y\rangle$ defines a non-degenerate symmetric bilinear form on the vector space H, and $o(H, \theta, \mathscr{C}_\infty)$ can be regarded as a 3-graded subalgebra of $o(H,q)$ having the same socle, $\mathrm{Soc}(o(H, \theta, \mathscr{C}_\infty)) = \mathfrak{fo}(H,q)$.

(3) Suppose now that H is endowed with an anticonjugation θ, i.e., $\theta : H \to H$ is a conjugate linear mapping satisfying $\langle \theta_x, y \rangle = -\langle \theta_y, x \rangle$ for all $x, y \in H$, and $\theta^2 x = -x$ for all $x \in H$, and define the transpose of an operator $a \in \mathrm{BL}(H)$ by $a^{\#} := \theta a^* \theta$. Following De La Harpe (1972), the classical symplectic Banach Lie algebra of bounded operators is defined by

$$\mathfrak{sp}(H, \theta, \mathscr{C}_\infty) = \{a \in \mathrm{BL}(H) : a^{\#} = a\}.$$

Then θ yields a Hilbert decomposition $H = X \oplus Y$ such that $Y = \theta(X)$. Hence a 3-grading can be defined on $\mathfrak{sp}(H, \theta, \mathscr{C}_\infty)$ with associated Jordan pair given by $V = (\mathrm{Sym}(\mathrm{BL}(X, Y), \#), \mathrm{Sym}(\mathrm{BL}(Y, X), \#))$. This Jordan pair V is strongly prime with socle $(\mathrm{Sym}(\mathrm{FBL}(X, Y), \#), \mathrm{Sym}(\mathrm{FBL}(Y, X), \#))$, and $\mathfrak{sp}(H, \theta, \mathscr{C}_\infty) \cap \mathrm{FBL}(H)$. Moreover, the closure of $\mathrm{Soc}(\mathfrak{sp}(H, \theta, \mathscr{C}_\infty))$ coincides with the classical symplectic Banach Lie algebra of compact operators on H.

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