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Maximal algebras of Martindale-like quotients of strongly prime linear Jordan algebras

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Abstract

In this paper we prove the existence and give precise descriptions of maximal algebras of Martindale like quotients for arbitrary strongly prime linear Jordan algebras. As a consequence, we show that Zelmanov's classification of strongly prime Jordan algebras can be viewed exactly as the description of their maximal algebras of Martindale-like quotients. As a side result, we show that the Martindale associative algebra of symmetric quotients can be expressed in terms of the symmetrized product, i.e., in purely Jordan terms.

Keywords: Martindale associative algebra of quotients; Jordan algebra; Central extensions

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Introduction

The structure theories of important types of associative algebras involve different notions of algebras of quotients. Indeed, certain classes of algebras are too broad to allow a full description of their objects, while their algebras of quotients can be more precisely characterized. The choice of the notion of algebra of quotients is given by the nature of the class of algebras under study. Goldie Theory [19, Chapter 3] is an example of such a situation: though there is not an explicit description of semiprime noetherian rings, we have a precise description of their classical rings of quotients and, on the other hand, the information on the ring of quotients $Q(R)$ of a noetherian ring R reverts to R itself.

Martindale rings of quotients were introduced by Martindale in 1969 [10] to study prime rings satisfying a generalized polynomial identity. In 1989, McCrimmon [14] modified Martindale's construction to obtain what he called the Martindale ring of symmetric quotients and also its triple system version. These objects play a central role in McCrimmon–Zelmanov's classification of strongly prime Jordan algebras [16].

In [6], definitions of linear Jordan systems of Martindale-like quotients are given. When $1/3$ and $1/2$ are assumed to be in the ring of scalars, the existence of maximal systems of quotients is obtained by using Lie algebras of quotients [20] through the Tits–Kantor–Koecher construction. The notions of Jordan system of quotients in [6] follow the pattern of McCrimmon [14], and are given with respect to filters of ideals, so that some general properties are obtained when the filters are sufficiently regular.

A different notion of Jordan algebra of quotients in which the denominators are inner ideals instead of ideals has been recently given by Montaner and Paniello in [18]. Indeed, they go through the list of strongly prime Jordan algebras [16, 15.2], showing the existence and giving precise descriptions of the maximal algebras of quotients in each case. We follow some of their ideas to obtain analogous results for Martindale-like quotients in the linear case (indeed, some of the arguments given in [18, Chapter 4] apply almost verbatim in our setting).

The main result of our paper establishes the existence and gives precise descriptions of the maximal algebras of Martindale-like quotients of strongly prime linear Jordan algebras with respect to the filter of all nonzero ideals. As a consequence, we show that Zelmanov's classification [21, Theorem 3] of strongly prime Jordan algebras can be viewed exactly as the description of their maximal algebras of Martindale-like quotients.

The paper is divided into four sections. After listing some basic facts, mainly about Jordan algebras, we recall in Section 1 the basic definitions of algebras of quotients, and establish their universal properties in order to be able to identify them later on in the paper. In Section 2, we study the interaction of the notions of algebra of quotients and associative envelope when dealing with special Jordan algebras. As a consequence, we show in Section 3 that the symmetrization $Q(R)^{(+)}$ of the Martindale associative algebra of quotients $Q(R)$ given in [14] is just the maximal Jordan algebra of quotients of $R^{(+)}$, when R is a prime associative algebra such that $R^{(+)}$ is not PI. A similar description is obtained for Jordan algebras of symmetric elements $H(R,*)$ of an associative algebra with involution. In the final section we obtain the description of the maximal algebra of Martindale-like quotients of a strongly prime PI Jordan algebra, as well as study the interaction of Jordan algebras of quotients and ideals. These together with the results of the previous section yield our main result mentioned above.

0. Preliminaries

0.1. We will deal with associative and Jordan algebras. The reader is referred to [7, 16,21] for definitions and properties of Jordan algebras not explicitly mentioned or proved in this section. We will deal with algebras over a ring of scalars Φ such that $1/2 \in \Phi$. A Jordan algebra over such a ring of scalars is called *linear* since the products $U_x y$, x^2 which define its algebraic structure can be expressed in terms of the linear product $x \circ y = V_x y := (x+y)^2 - x^2 - y^2$:

$$U_x y = \frac{1}{2}\big((x \circ y) \circ x - x^2 \circ y\big), \qquad x^2 = \frac{1}{2}(x \circ x).$$

We will also use the linearization $\{x, y, z\}$ of the U-product:

$$\{x, y, z\} = U_{x,z} y = V_{x,y} z := U_{x+z} y - U_x y - U_z y,$$

so that $U_x y = \frac{1}{2}\{x, y, x\}$.

0.2. The usual notions of Jordan algebras simplify when restricting to linear objects. Thus an *ideal* I of a linear Jordan algebra J is just a Φ-submodule of J satisfying $I \circ J \subseteq I$, while a *homomorphism* $f : J \to \tilde{J}$ between two linear Jordan algebras J and $\tilde{J}$ is a linear map preserving squares or linear products: $f(x \circ y) = f(x) \circ f(y)$, for any $x, y \in J$, or, equivalently, $f(x^2) = f(x)^2$, for any $x \in J$. We remark the fact that the Jordan cube $U_I I$ of an ideal I is also an ideal [12]. The intersection of all nonzero ideals of J is called the *heart* of J and it is denoted by $\mathrm{Heart}(J)$.

0.3. Given a Jordan algebra J, we can consider a Jordan algebra structure on $\hat{J} := J \oplus \Phi 1$ so that J is an ideal of $\hat{J}$ and 1 is the unit element in $\hat{J}$. In our linear setting, we just need to define squares in $\hat{J}$ by $(\alpha 1 \oplus x)^2 = \alpha^2 1 \oplus (2\alpha x + x^2)$, for any $\alpha \in \Phi$, and any $x \in J$.

0.4. A Jordan algebra J is said to be *nondegenerate* if zero is the only *absolute zero divisor*, i.e., zero is the only $x \in J$ such that $U_x = 0$. We say that J is *semiprime* if $U_I I \neq 0$, for any nonzero ideal I of J, and say that J is *prime* if $U_I L \neq 0$, for any nonzero ideals I, L of J. Every nondegenerate Jordan algebra is semiprime. A nondegenerate prime Jordan algebra is said to be *strongly prime*. Notice that, in a prime Jordan algebra J, $I \cap L \neq 0$, for any nonzero ideals I, L of J.

0.5. Given a subset X of a Jordan algebra J, the *annihilator of X in J* [12, 1.2] is given by

$$\begin{aligned}\mathrm{Ann}(X) = \mathrm{Ann}_J(X) = \big\{z \in J \;\big|\; \{z, X, \hat{J}\} = \{X, z, \hat{J}\} = \{z, \hat{J}, x\} = U_X z = U_z X \\ = U_z U_X \hat{J} = U_X U_z \hat{J} = U_z U_{\hat{J}} X = U_X U_{\hat{J}} z = 0\big\}.\end{aligned}$$

If I is an ideal of J, then $\mathrm{Ann}_J(I)$ is also an ideal of J. Moreover, if J is nondegenerate and I is an ideal of J, then $\mathrm{Ann}_J(I) = \{x \in J \mid U_x I = 0\}$ [12, 1.7]. If J is prime, then $\mathrm{Ann}_J(I) = 0$ for any nonzero ideal I of J [12, 1.6]. In general, an ideal I of J will be called *sturdy* if $\mathrm{Ann}_J(I) = 0$. An ideal I of J will be called *essential* if it hits every nonzero ideal of J ($I \cap L \neq 0$ for any nonzero ideal L of J).

0.6. The *centroid* $\Gamma(J)$ of a Jordan algebra J is the set of linear maps $T : J \to J$ such that

$$TU_x = U_xT, \qquad TV_{x,y} = V_{x,y}T, \qquad TV_x = V_xT,$$
$$T^2(x^2) = (T(x))^2, \qquad T^2U_x = U_{T(x)},$$

for any $x, y \in J$ [15]. In the linear case, the above just amounts to saying $TV_x = V_xT$, for any $x \in J$. Following [5], the (*weak*) *center* of J is the set $C(J)$ of all elements $z \in J$ such that $U_z, V_z \in \Gamma(J)$. In the linear case, $C(J)$ is just the set of elements $z \in J$ satisfying $V_z \in \Gamma(J)$.

0.7. One can obtain Jordan systems from associative systems by symmetrization: If R is an associative algebra, we can obtain a Jordan algebra denoted by $R^{(+)}$, over the same Φ-module, with products built out of the associative product by $x^2 = xx$, $U_x y = xyx$, for any $x, y \in R$.

A Jordan algebra is said to be *special* if it is a subalgebra of $R^{(+)}$ for some associative algebra R. A particularly important example of special Jordan algebras can be obtained out of an associative algebra R with involution $*$ by taking the set $H(R, *)$ of symmetric elements of R. If J is a subalgebra of $R^{(+)}$ (respectively, $H(R, *)$) and R is generated by J, then R is said to be an (*associative*) *envelope* (respectively, $*$-*envelope*) of J. We say that an envelope (respectively, $*$-envelope) R is *tight* (respectively, $*$-*tight*) if every nonzero ideal (respectively, $*$-ideal) of R hits J.

0.8. We will need the following identities which are valid for arbitrary, not necessarily linear Jordan algebras.

(i) $\{a, y \circ x, b\} = \{a \circ x, y, b\} + \{a, y, x \circ b\} - x \circ \{a, y, b\}$,
(ii) $U_x(y \circ z) = \{x \circ y, z, x\} - y \circ U_x z$,
(iii) $(U_x y) \circ y = (U_y x) \circ x$,
(iv) $U_{U_x y} = U_x U_y U_x$, $U_{x^2} = (U_x)^2$,
(v) $U_x(y^2) = (x \circ y)^2 - (U_y x) \circ x - U_y(x^2)$,
(vi) $(U_x(y \circ x))^2 = U_x U_y U_x(x^2) + (U_x y) \circ (U_x U_x y) + U_x U_x U_y(x^2)$,
(vii) $U_x U_y(x^2) = (U_x y)^2$,
(viii) $U_x(x^2) = x^4 = (x^2)^2$,
(ix) $U_x U_x(U_y x \circ x) = U_x(\{x, x, y\} \circ U_x y) - ((U_x y) \circ (U_x U_x y))$,
(x) $(U_x(\{x, x, y\})) \circ (U_x U_x y) = U_x(U_x U_x y \circ U_x y) + \{U_x y, x^2, U_x U_x y\}$,
(xi) $\{y^2, x, z\} = \{y, y \circ x, z\} - U_y x \circ z$,
(xii) $\{y_1 \circ y_2, x, y_3\} = \{y_1, y_2 \circ x, y_3\} + \{y_2, x \circ y_1, y_3\} - \{y_1, x, y_2\} \circ y_3$,

(xiii) $\{x, z, \{y_1, y_2, y_3\}\} = \{x, \{z, y_1, y_2\}, y_3\} + \{x, \{z, y_3, y_2\}, y_1\} - \{x, y_2, \{y_1, z, y_3\}\}$,
(xiv) $(x \circ y) \circ z = \{x, y, z\} + \{y, x, z\}$.

Indeed, (i) is the linearization of (ii), (ii)–(xi) and (xiv) follow from Macdonald's Theorem [8], (xii) is the linearization of (xi), and (xiii) is the linearization of [9, JP10].

0.9. Proposition. *If J is a strongly prime Jordan algebra that has a nonzero PI ideal, then J is PI.*

Proof. Let I be a nonzero PI ideal of J, so that I satisfies a multilinear identity $f(x_1, \dots, x_n)$. Notice that I is also strongly prime by [12, 2.5], hence there exists $0 \neq z \in I$ such that U_z, V_z are in the centroid of I [4, 3.6]. Thus, for any $y_1, \dots, y_n \in J$, $U_z y_1, \dots, U_z y_n \in I$, and $f(U_z y_1, \dots, U_z y_n) = 0$. But $f(U_z y_1, \dots, U_z y_n) = U_z^n f(y_1, \dots, y_n)$ because U_z, V_z are in the centroid of J by [4, 3.2], which yields $U_z^n f(y_1, \dots, y_n) = 0$, hence $f(y_1, \dots, y_n) = 0$ using $U_z \neq 0$ by nondegeneracy and the fact that the centroid acts faithfully on J [15, 2.8]. □

1. Universal properties of algebras of quotients

Through this section, unless explicitly stated, "algebra" will stand for an associative or Jordan algebra over Φ. In the case of associative algebras, the results remain valid without assuming that $1/2 \in \Phi$.

1.1. In [6,14] algebras of quotients in the Jordan and associative cases are introduced and studied. When dealing with associative algebras, *algebra of quotients* will mean Martindale algebra of symmetric quotients [14]. When dealing with Jordan algebras, *algebra of quotients* will mean Jordan algebra of Martindale-like quotients [6].

1.2. Given an algebra J, a *filter* of ideals of J is a nonempty set $\mathcal{F}$ of nonzero ideals of J such that for any $I_1, I_2 \in \mathcal{F}$, there exists $L \in \mathcal{F}$ such that $L \subseteq I_1 \cap I_2$. A filter $\mathcal{F}$ will be called a *power* filter if for any $I \in \mathcal{F}$, there exists $L \in \mathcal{F}$ such that $L \subseteq I'$, where $I' = II$ in the associative setting, and $I' = I \circ I$ in the Jordan setting.

If $\mathcal{F}$ is a power filter of a Jordan algebra, then for any $I \in \mathcal{F}$ there exists $K \in \mathcal{F}$ such that $K \subseteq U_I I$: by definition, there exists $L \in \mathcal{F}$ such that $L \subseteq I \circ I$ and $K \in \mathcal{F}$ such that $K \subseteq L \circ L \subseteq (I \circ I) \circ (I \circ I) \subseteq (I \circ I) \circ I$, but $(I \circ I) \circ I \subseteq U_I I$ by 0.8(xiv).

If J is a semiprime Jordan algebra, the set $\mathcal{F}$ of all sturdy ideals of J coincides with the set of all essential ideals of J and is a power filter of sturdy ideals [6, 2.1]. When J is strongly prime, the above filter is just the set of all nonzero ideals of J.

1.3. Unlike [6,14], we will outline the monomorphism linking an algebra with its algebra of quotients. Thus, given an algebra J, an algebra of quotients for J with respect to a filter of ideals $\mathcal{F}$ is (Q, τ), such that Q is an algebra, $\tau : J \to Q$ is an algebra monomorphism, and, for any $0 \neq q \in Q$, there exists $I \in \mathcal{F}$ such that

- $0 \neq q \circ \tau(I) \subseteq \tau(J)$, when J is a Jordan algebra.
- $0 \neq q\tau(I) + \tau(I)q \subseteq \tau(J)$, when J is an associative algebra.

1.4. Notice that in the associative case, the definition given in [14, 1.2, 1.3] includes the fact that the annihilator of $\tau(K)$ in Q is zero, for any $K \in \mathcal{F}$, rather than $0 \neq q\tau(I) + \tau(I)q$. However, both facts are equivalent when dealing with a filter $\mathcal{F}$ of sturdy ideals. Indeed, assuming our definition, for any $K \in \mathcal{F}$, if $q \in Q$ satisfies $q\tau(K) + \tau(K)q = 0$, then we can take $I \in \mathcal{F}$ such that $0 \neq q\tau(I) + \tau(I)q \subseteq \tau(J)$, and $q\tau(I)\tau(K) + \tau(K)q\tau(I) \subseteq q\tau(K) + \tau(K)q\tau(I) = 0$, which implies $q\tau(I) = 0$ by sturdiness of K in J, and similarly $\tau(I)q = 0$, which is a contradiction. The converse is clear, since $0 \neq q\tau(I) + \tau(I)q$ just means that q does not belong to the annihilator of $\tau(I)$ in Q. Anyway, if $\mathcal{F}$ is a filter of sturdy ideals, $K \in \mathcal{F}$, and $0 \neq q \in Q$, then $q\tau(K) + \tau(K)q \neq 0$.

An analogous fact for Jordan algebras is stated in the following result.

1.5. Lemma. *Let J be a Jordan algebra and let (Q, τ) be an algebra of quotients of J with respect to a power filter $\mathcal{F}$ of sturdy ideals of J. Then for any nonzero $q \in Q$ and any ideal $I \in \mathcal{F}$ we have $0 \neq q \circ \tau(I)$. If J is nondegenerate, then also $\{\tau(I), q, \tau(I)\} \neq 0$.*

Proof. By [6, 4.2, 5.2], $0 \neq \{q, \tau(I), \tau(I)\} + \{\tau(I), q, \tau(I)\} \subseteq (q \circ \tau(I)) \circ \tau(I) + q \circ (\tau(I) \circ \tau(I))$ (see 0.1) $\subseteq (q \circ \tau(I)) \circ \tau(I) + q \circ \tau(I)$, which implies $0 \neq q \circ \tau(I)$.

Let us assume now that J is nondegenerate. Since $0 \neq q \in Q$, there exists an element $x \in \tau(J)$ such that $0 \neq q \circ x \in \tau(J)$. Using [17, 1.3], nondegeneracy of $J \cong \tau(J)$, and sturdiness of I in J, $0 \neq \{\tau(I), q \circ x, \tau(I)\} \subseteq \{x \circ \tau(I), q, \tau(I)\} + x \circ \{\tau(I), q, \tau(I)\}$ (by 0.8(i)) $\subseteq \{\tau(I), q, \tau(I)\} + x \circ \{\tau(I), q, \tau(I)\}$, hence $\{\tau(I), q, \tau(I)\} \neq 0$. □

1.6. Going through the proofs of [6, 3.2, 5.5], [20, 3.6], [14, 1.3], one can obtain (assuming $1/3 \in \Phi$ in the Jordan case) the following result. Let J be an algebra, and $\mathcal{F}$ be a power filter of sturdy ideals of J, then there exists an algebra of quotients (Q, τ) with respect to $\mathcal{F}$ such that

(1) for any algebra of quotients $(\widetilde{Q}, \tilde{\tau})$ of J with respect to the same filter, there is an algebra homomorphism $f : \widetilde{Q} \to Q$ such that $f\tilde{\tau} = \tau$.

1.7. Given an algebra J, and a filter $\mathcal{F}$ of ideals of J, an algebra of quotients (Q, τ) of J with respect to $\mathcal{F}$ will be said to be *maximal* if it satisfies 1.6(1).

We will show that one gets uniqueness of f in 1.6(1) for free.

1.8. Lemma. *Let J be an algebra, (Q, τ), $(\widetilde{Q}, \tilde{\tau})$ be algebras of quotients of J respect to a power filter $\mathcal{F}$ of sturdy ideals of J. If $f : Q \to \widetilde{Q}$ and $g : Q \to \widetilde{Q}$ are algebra homomorphisms such that $f\tau = \tilde{\tau}$ and $g\tau = \tilde{\tau}$, then $f = g$.*

Proof. Assume, for example, that we are in the Jordan algebra case. If there is $q \in Q$ such that $f(q) \neq g(q)$, then we can find $I \in \mathcal{F}$ such that $0 \neq (f(q) - g(q)) \circ \tilde{\tau}(I) \subseteq \tilde{\tau}(J)$, and also $q \circ \tau(I) \subseteq \tau(J)$, $f(q) \circ \tilde{\tau}(I) \subseteq \tilde{\tau}(J)$ and $g(q) \circ \tilde{\tau}(I) \subseteq \tilde{\tau}(J)$ (1.5). Thus, we can find $y \in I$ such that $(f(q) - g(q)) \circ \tilde{\tau}(y) \neq 0$. But also

$$\begin{aligned}
&\big(f(q)-g(q)\big)\circ\tilde\tau(y)\\
&\quad= f(q)\circ\tilde\tau(y)-g(q)\circ\tilde\tau(y)\\
&\quad= f(q)\circ f\big(\tau(y)\big)-g(q)\circ g\big(\tau(y)\big)\quad \big(\text{using } f\tau=\tilde\tau,\ g\tau=\tilde\tau\big)\\
&\quad= f\big(q\circ\tau(y)\big)-g\big(q\circ\tau(y)\big)\quad (\text{using } f,\ g \text{ are algebra homomorphisms})\\
&\quad= 0,
\end{aligned}$$

since $q\circ\tau(y)\in\tau(J)$, leading to a contradiction. □

1.9. Theorem (Universal Property for Maximal Algebras of Quotients). *Let J be an algebra, (Q,τ) be a maximal algebra of quotients with respect to a power filter $\mathcal{F}$ of sturdy ideals of J, and $(\widetilde{Q},\tilde\tau)$ be any algebra of quotients of J with respect to $\mathcal{F}$. Then there exists a unique algebra homomorphism $f:\widetilde{Q}\to Q$ such that $f\tilde\tau=\tau$.*

1.10. Remark. Using 1.9, maximal algebras of quotients of J with respect to a power filter $\mathcal{F}$ of sturdy ideals of J are unique up to isomorphism. From now on, $(Q_{\mathcal{F}}(J),\tau_J)$, or simply $(Q(J),\tau_J)$ will denote such a maximal algebra of quotients.

We can also obtain injectivity of f in 1.9 for free.

1.11. Lemma. *Let J be an algebra, (Q,τ), $(\widetilde{Q},\tilde\tau)$ be algebras of quotients of J with respect to a power filter $\mathcal{F}$ of sturdy ideals of J. If $f:Q\to\widetilde{Q}$ is an algebra homomorphism satisfying $f\tau=\tilde\tau$, then f is injective.*

Proof. Notice that by 1.3 every nonzero ideal of Q hits $\tau(J)$. On the other hand, $f\tau=\tilde\tau$ implies that $\tau(J)\cap\operatorname{Ker} f=0$ by injectivity of $\tilde\tau$, hence $\operatorname{Ker} f=0$, i.e., f is injective. □

1.12. Corollary. *Under the conditions of* 1.9, *f is injective.*

2. Associative envelopes and algebras of quotients

2.1. Lemma. *Let J be a subalgebra of a special Jordan algebra Q. Suppose that for an element $x\in Q$ there exists an ideal I of J with $x\circ I\subseteq J$. Then*

(i) $x(y_1\circ y_2)\in Jy_1+Jy_2-\{y_1,x,y_2\}$ *for any $y_1,y_2\in I$, where the associative products are taken in any associative envelope of Q, and*
(ii) $\{I\circ I,x,I\}\subseteq\{I,J,I\}+J\circ I\subseteq I$.

Proof. (i) It is straightforward to check that $x(y_1\circ y_2)=(x\circ y_2)y_1+(x\circ y_1)y_2-\{y_1,x,y_2\}\in Jy_1+Jy_2-\{y_1,x,y_2\}$.

(ii) By 0.8(xii), for any $y_1,y_2,y_3\in I$ we have that $\{y_1\circ y_2,x,y_3\}=\{y_1,y_2\circ x,y_3\}+\{y_2,x\circ y_1,y_3\}-\{y_1,x,y_2\}\circ y_3\in\{I,J,I\}+J\circ I\subseteq I$. □

The following result and its proof are patterned out of [18, 4.3.13].

2.2. Proposition. *Let (Q,τ) be an algebra of quotients of a strongly prime Jordan algebra J with respect to the filter of all nonzero ideals of J. Suppose that Q is a special Jordan algebra, let S be any associative envelope* (*respectively, $*$-envelope*) *of Q, and T be the associative subalgebra of S generated by $\tau(J)$. Then*

(1) *for any $s\in S$ there exists a nonzero ideal K of J such that $s\tau(K)+\tau(K)s\subseteq T$.*

Moreover, if S is tight (*respectively, $*$-tight*) *over Q and $0\neq s\in S$, then $\tau(I)s\tau(I)\neq 0$ for any nonzero ideal I of J.*

Proof. Since S is an envelope of Q, its elements are generated by elements of Q, hence $s=\sum q_{i_1}\cdots q_{i_{n_k}}$ with $q_{i_j}\in Q$. We will prove (1) for monomials of the form $q_1\cdots q_n$ by induction on n (once we find ideals for the summands of s, their intersection, which is a nonzero ideal by strong primeness of J, satisfies (1) for s).

If $s=q\in Q$, there exists a nonzero ideal L of J with $q\circ\tau(L)\subseteq\tau(J)$. Since J is strongly prime, $K=\{L,L,L\}=U_LL$ is a nonzero ideal of J. Then

$$\begin{aligned} q\tau(K)=q\tau\bigl(\{L,L,L\}\bigr)&=q\bigl\{\tau(L),\tau(L),\tau(L)\bigr\}\subseteq q\bigl(\tau(L)\circ\tau(L)\bigr)\\ &\subseteq\tau(J)\tau(L)+\bigl\{\tau(L),q,\tau(L)\bigr\}\quad\bigl(\text{by 2.1(i)}\bigr)\\ &\subseteq\tau(J)\tau(L)+\bigl(\tau(L)\circ q\bigr)\circ\tau(L)+q\circ\bigl(\tau(L)\circ\tau(L)\bigr)\\ &\subseteq\tau(J)\tau(L)+\tau(J)\circ\tau(L)+q\circ\tau(L)\\ &\subseteq\tau(J)\tau(L)+\tau(L)+\tau(J)\subseteq T. \end{aligned}$$

By symmetry, the ideal K also satisfies $\tau(K)q\subseteq T$.

Now suppose that (1) is true for any monomial of length less than or equal to n. If $s=s_nq$, for a monomial s_n of S of length n and $q\in Q$, by the induction hypothesis there exists a nonzero ideal K_1 of J such that $s_n\tau(K_1)\subseteq T$. Moreover, since $q\in Q$ there also exists a nonzero ideal K_2 of J with $q\circ\tau(K_2)\subseteq\tau(J)$. Now, since J is strongly prime, $K_3=K_1\cap K_2$ is nonzero and also $L=\{K_3,K_3,K_3\}$ is a nonzero ideal of J. Then

$$\begin{aligned} s\tau(L)=s_nq\tau(L)=s_nq\bigl\{\tau(K_3),\tau(K_3),\tau(K_3)\bigr\}&\subseteq s_nq\bigl(\bigl(\tau(K_3)\circ\tau(K_3)\bigr)\circ\tau(K_3)\bigr)\\ &\subseteq s_n\bigl(\tau(J)\tau(K_3)\bigr)+s_n\bigl\{\bigl(\tau(K_3)\circ\tau(K_3)\bigr),q,\tau(K_3)\bigr\}\quad\bigl(\text{by 2.1(i)}\bigr)\\ &\subseteq s_n\bigl(\bigl(\tau(J)\circ\tau(K_3)\bigr)+\tau(K_3)\tau(J)\bigr)+s_n\tau(K_3)\quad\bigl(\text{by 2.1(ii)}\bigr)\\ &\subseteq s_n\tau(K_3)+s_n\tau(K_3)\tau(J)\subseteq T. \end{aligned}$$

Similarly, since s also can be written as $q's'_n$ for a monomial s'_n of S of length n and $q'\in Q$, there exists a nonzero ideal M of J such that $\tau(M)s\subseteq T$. The nonzero ideal $K=L\cap M$ satisfies $\tau(K)s+s\tau(K)\subseteq T$.

Now, let us show the last assertion when S is $*$-tight over Q (the case without involution follows analogously, with obvious changes). If $0\neq s\in S$, let us consider the nonzero $*$-ideal $I_s=\widehat{S}s\widehat{S}+\widehat{S}s^*\widehat{S}$ of S generated by s and s^*. Since S is $*$-tight over Q, $I_s \cap Q \neq 0$ and there exists a finite number of elements $a_i, b_i, c_j, d_j \in \widehat{S}$ with $0 \neq q = \sum_i a_i s b_i + \sum_j c_j s^* d_j \in Q$. Using (1), the fact that (Q, τ) is an algebra of quotients of J, and strong primeness of J, we can find a nonzero ideal K of J such that $\tau(K)a_i + \tau(K)c_j \subseteq T$, $b_i\tau(K) + d_j\tau(K) \subseteq T$, for every i, j, and $\tau(K) \circ q \subseteq \tau(J)$. By strong primeness of J, we can find a nonzero ideal L of J contained into $(K \cap I) \circ (K \cap I)$. Then, by 1.5,

$$\begin{aligned} 0 \neq \{\tau(L), q, \tau(L)\} &\subseteq \sum_i \tau(I)\tau(K)a_i s b_i \tau(K)\tau(I) + \sum_j \tau(I)\tau(K)c_j s^* d_j \tau(K)\tau(I) \\ &\subseteq \tau(I)TsT\tau(I) + \tau(I)Ts^*T\tau(I) \\ &\subseteq \big(T\tau(I) + \tau(I)\big)s\big(\tau(I)T + \tau(I)\big) \\ &\quad + \big(T\tau(I) + \tau(I)\big)s^*\big(\tau(I)T + \tau(I)\big), \end{aligned}$$

getting that $\tau(I)s\tau(I) \neq 0$ (notice that $\tau(I)T \subseteq T\tau(I) + \tau(I)$ and $T\tau(I) \subseteq \tau(I)T + \tau(I)$ by induction because $hy = h \circ y + yh$ for every $h \in \tau(J)$ and $y \in \tau(I)$). □

2.3. Theorem. *Let (Q, τ) be an algebra of quotients of a strongly prime Jordan algebra J with respect to the filter of all nonzero ideals of J. Suppose that Q is a special Jordan algebra, let S be any tight* (*respectively, $*$-tight*) *associative envelope of Q, and T be the associative subalgebra of S generated by $\tau(J)$. Let $j : Q \to S$, $\mu : T \to S$ be the inclusion maps. Then T is a tight* (*respectively, $*$-tight*) *envelope of $\tau(J)$, hence T is prime* (*respectively, $*$-prime*), *and (S, μ) is an algebra of quotients of T with respect to the filter of all nonzero ideals* (*respectively, $*$-ideals*) *of T. Moreover, in the case with involution, μ is a $*$-homomorphism.*

Proof. We will prove the theorem in the case with involution (the proof holds also without involution, with obvious changes).

Notice that T is a $*$-subalgebra of S since it is generated by the elements of $\tau(J) \subseteq Q$, which are $*$-symmetric, so that μ is a $*$-homomorphism.

To show that T is a $*$-tight envelope of $\tau(J)$, we will proceed as in [18, 4.3.13(2)]. We just need to show that every nonzero $*$-ideal of T hits $\tau(J)$. Indeed, if I is a nonzero $*$-ideal of T, then $\widehat{S}I\widehat{S}$ is a nonzero $*$-ideal of S, hence it hits Q by $*$-tightness. Thus, there exists a finite number of elements $a_i, b_i \in \widehat{S}$, $y_i \in I$ such that $0 \neq q = \sum_i a_i y_i b_i \in Q$. Using the fact that (Q, τ) is an algebra of quotients of J and 2.2, together with strong primeness of J, we can find a nonzero ideal K of J such that $q \circ \tau(K) \subseteq \tau(J)$, and $\tau(K)a_i \subseteq T$, $b_i\tau(K) \subseteq T$ for every i. On the one hand, $\{\tau(K), q, \tau(K)\} \subseteq \tau(K)q\tau(K) \subseteq \sum_i \tau(K)a_i y_i b_i \tau(K) \subseteq \sum_i T y_i T \subseteq I$ since I is an ideal of T. But, on the other hand, by 1.5, $0 \neq \{\tau(K), q, \tau(K)\} \subseteq (\tau(K) \circ q) \circ \tau(K) + q \circ (\tau(K) \circ \tau(K)) \subseteq \tau(J)$, i.e., we have that $0 \neq \{\tau(K), q, \tau(K)\} \subseteq I \cap \tau(J)$.

By 2.2(1), given $0 \neq s \in S$, there exists a nonzero ideal K of J such that

$$s\tau(K) + \tau(K)s \subseteq T. \tag{1}$$

Notice that the ideal $\widetilde{K}$ of T generated by $\tau(K)$ satisfies

$$\widetilde{K} = \tau(K)\widehat{T} = \widehat{T}\tau(K) \tag{2}$$

(iterate the fact that $\tau(K)\tau(J) = \tau(K) \circ \tau(J) + \tau(J)\tau(K) = \tau(K \circ J) + \tau(J)\tau(K) \subseteq \tau(K) + \tau(J)\tau(K)$ and the analogous $\tau(J)\tau(K) \subseteq \tau(K) + \tau(K)\tau(J)$). Now, $s\mu(\widetilde{K}) + \mu(\widetilde{K})s = s\widetilde{K} + \widetilde{K}s \subseteq T$ by (1) and (2). Moreover, $s\widetilde{K} + \widetilde{K}s \neq 0$ since $\widetilde{K}s\widetilde{K} \supseteq \tau(K)s\tau(K) \neq 0$ by 2.2. □

3. Maximal algebras of quotients of symmetrizations of associative algebras

3.1. Let R be a prime associative algebra, $(Q(R), \tau_R)$ be the maximal algebra of quotients of R with respect to the filter of all nonzero ideals of R. Notice that $R^{(+)}$ is strongly prime by [11, p. 384], [1, 1.2(ii)].

3.2. Let R be a $*$-prime associative algebra with involution $*$, $(Q(R), \tau_R)$ be the maximal algebra of quotients of R with respect to the filter of all nonzero $*$-ideals of R. By [14, 1.10], there exists a unique involution (also denoted by $*$) on $Q(R)$ extending the involution of R, so that τ_R is a $*$-homomorphism. Notice that $H(R,*)$ is strongly prime by [1, 2.7(i)]).

3.3. Theorem.

- (i) *Under the conditions of* 3.1:
  - (a) $(Q(R)^{(+)}, \tau_R)$ *is an algebra of quotients of* $R^{(+)}$ *with respect to the filter of all nonzero ideals of* $R^{(+)}$.
  - (b) *If* $R^{(+)}$ *is not PI, then we have that* $(Q(R)^{(+)}, \tau_R)$ *is the maximal algebra of quotients of* $R^{(+)}$ *with respect to the filter of all nonzero ideals of* $R^{(+)}$.
- (ii) *Under the conditions of* 3.2:
  - (a) $(H(Q(R),*), \tau)$ *is an algebra of quotients of* $H(R,*)$ *with respect to the filter of all nonzero ideals of* $H(R,*)$, *where* τ *denotes the restriction of* τ_R.
  - (b) *If* R *is a* $*$*-tight associative envelope of* $H(R,*)$, *and* $H(R,*)$ *is not PI, then* $(H(Q(R),*), \tau)$ *is the maximal algebra of quotients of* $H(R,*)$ *with respect to the filter of all nonzero ideals of* $H(R,*)$.

Proof. (i)(a) Let $0 \neq q \in Q(R)$. There exists an ideal I of R such that $0 \neq q\tau_R(I) + \tau_R(I)q \subseteq \tau_R(R)$. Since R is semiprime, $K = II$ is a nonzero ideal of R, hence it is a nonzero ideal of $R^{(+)}$. Clearly $q \circ \tau_R(I) \subseteq \tau_R(R)$, but also $q \circ \tau_R(I) \neq 0$: otherwise, $qx = -xq$ for any $x \in \tau_R(I)$ and, in particular, for any $x, y \in \tau_R(I)$,

$$\begin{aligned} -q^2xy &= qxyq \quad \bigl(\text{since } xy \in \tau_R(I)\bigr) \\ &= -qxqy \quad \bigl(\text{since } y \in \tau_R(I)\bigr) \\ &= q^2xy \end{aligned}$$

since $x \in \tau_R(I)$, hence $qxyq = 0$ using $1/2 \in \Phi$, and we have shown $q\tau_R(K)q = 0$, which contradicts the algebra version without involution of [3, 2.8].

(ii)(a) is just [6, 2.11, 4.1, 5.2].

(i)–(ii)(b) We just need to prove $(Q(R)^{(+)}, \tau_R)$ (respectively, $(H(Q(R),*), \tau)$) satisfies 1.6(1). Let $(\widetilde{Q}, \tilde{\tau})$ be an algebra of quotients of $R^{(+)}$ (respectively, $H(R,*)$). Since $R^{(+)}$ (respectively, $H(R,*)$) is strongly prime, $\widetilde{Q}$ is also strongly prime [6, 4.4, 5.2], and since $R^{(+)}$ (respectively, $H(R,*)$) is not PI, $\widetilde{Q}$ is not PI, hence it is special by [21, Theorem 3].

Let S be a tight (respectively, $*$-tight) envelope of $\widetilde{Q}$, and T be the subalgebra of S generated by $\tilde{\tau}(R^{(+)})$ (respectively, $\tilde{\tau}(H(R,*))$). As in 2.3, let $j : \widetilde{Q} \to S$, $\mu : T \to S$ be the inclusion maps, and $\tau' : R^{(+)} \to T$ (respectively $\tau' : H(R,*) \to T$) be the restriction of $j\tilde{\tau}$. Since T is a tight (respectively $*$-tight) envelope of $\tilde{\tau}(R^{(+)})$ (respectively, $\tilde{\tau}(H(R,*)) = \tau'(H(R,*))$) by 2.3, we can use [13, 3.1] to find, replacing R by its opposite if it is necessary, an associative algebra isomorphism $g : R \to T$ extending τ', i.e., $g = \tau'$ (respectively, we can use [13, 2.3] to find an associative algebra $*$-isomorphism $g : R \to T$ extending τ', i.e., $g|_{H(R,*)} = \tau'$). Now, $\mu g : R \to S$ is an algebra homomorphism (respectively $*$-homomorphism) such that $(S, \mu g)$ is an algebra of quotients of R with respect to the filter of all nonzero ideals (respectively, $*$-ideals), using the corresponding fact for (S, μ), established in 2.3. Thus, by the universal property of $(Q(R), \tau_R)$, there exists an associative algebra homomorphism $f : S \to Q(R)$ such that

$$f\mu g = \tau_R. \tag{1}$$

Hence, in case (i), we can restrict f to the Jordan algebra homomorphism $h : \widetilde{Q} \to Q(R)^{(+)}$ which satisfies $h\tilde{\tau} = \tau_R$: for any $x \in R$, $h\tilde{\tau}(x) = f\tilde{\tau}(x) = f\tau'(x) = f\mu\tau'(x) = f\mu g(x) = \tau_R(x)$ by (1). In case (ii), f is a $*$-homomorphism by [14, 3.20], hence, we have $f(\widetilde{Q}) \subseteq f(H(S,*)) \subseteq H(Q(R),*)$, and we can restrict f to the Jordan algebra homomorphism $h : \widetilde{Q} \to H(Q(R),*)$ which satisfies $h\tilde{\tau} = \tau$: for any $x \in H(R,*)$, $h\tilde{\tau}(x) = f\tilde{\tau}(x) = f\tau'(x) = f\mu\tau'(x) = f\mu g(x) = \tau_R(x) = \tau(x)$ by (1). $\square$

4. The general case

We begin with the study of strongly prime PI Jordan algebras. The description of their maximal algebras of quotients is based on the fundamental fact that nonzero ideals contain nonzero central elements [4, 3.6], a result that was extended in [18, 4.7.4] to essential inner ideals. Our result is based on [18, 4.7.7], though in our proof we have extracted the work with weak centers, which gives rise to the following result of independent interest, valid for arbitrary nondegenerate algebras (not necessarily PI).

4.1. Proposition. *Let J be a nondegenerate Jordan algebra, $\mathcal{F}$ be a power filter of sturdy ideals of J, and (Q, τ) be an algebra of quotients of J with respect to $\mathcal{F}$. Then $\tau(C(J)) \subseteq C(Q)$.*

Proof. Replacing J by its isomorphic image $\tau(J)$, we can assume that τ is the inclusion map and prove that $C(J) \subseteq C(Q)$. Let $z \in C(J)$.

(I) For any $q \in Q$ and any $x \in J$ such that $x \circ q \in J$, $z \circ (x \circ q) = (z \circ x) \circ q$: Take $I \in \mathcal{F}$ such that $I \circ q \subseteq J$. For any $y_1, y_2, y_3 \in I$, and $t \in \hat{J}$,

$$\begin{aligned}
\{z \circ t, q, \{y_1, y_2, y_3\}\} &= \{z \circ t, \{q, y_1, y_2\}, y_3\} + \{z \circ t, \{q, y_3, y_2\}, y_1\} \\
&\quad - \{z \circ t, y_2, \{y_1, q, y_3\}\} \quad (\text{by 0.8(xiii)}) \\
&= z \circ \{t, \{q, y_1, y_2\}, y_3\} + z \circ \{t, \{q, y_3, y_2\}, y_1\} \\
&\quad - z \circ \{t, y_2, \{y_1, q, y_3\}\} \\
&= z \circ \{t, q, \{y_1, y_2, y_3\}\} \quad (\text{by 0.8(xiii)})
\end{aligned} \tag{1}$$

since $\{q, y_1, y_2\}, \{q, y_3, y_2\}, \{y_1, q, y_3\} \in J$, $z \in C(J)$, and $C(J) \subseteq C(\hat{J})$ [5, Corollary 1]. Now, given $K \in \mathcal{F}$ such that $K \subseteq U_I I = \{I, I, I\}$, and any $y \in K$,

$$\begin{aligned}
U_y((z \circ x) \circ q) &= \{y \circ (z \circ x), q, y\} - (z \circ x) \circ U_y q \quad (\text{by 0.8(ii)}) \\
&= \{z \circ (y \circ x), q, y\} - z \circ (x \circ U_y q) \quad (\text{since } x, y, U_y q \in J \text{ and } z \in C(J)) \\
&= z \circ \{y \circ x, q, y\} - z \circ (x \circ U_y q) \quad (\text{by (1)}) \\
&= z \circ (U_y(x \circ q)) \quad (\text{by 0.8(ii)}) \\
&= U_y(z \circ (x \circ q))
\end{aligned}$$

since $y, x \circ q \in J$ and $z \in C(J)$. We have shown that $U_K((z \circ x) \circ q - z \circ (x \circ q)) = 0$, which implies $(z \circ x) \circ q - z \circ (x \circ q) = 0$ by 1.5.

(II) Assume $q \in Q$, $I \in \mathcal{F}$ and $I \circ q \subseteq J$. Then $(z \circ q) \circ x = z \circ (q \circ x)$ for any $x \in U_I I$:

$$\begin{aligned}
(z \circ q) \circ x &= 2\{z, q, x\} - z \circ (q \circ x) + (z \circ x) \circ q \quad (\text{see 0.1}) \\
&= 2\{z, q, x\} \quad (\text{by (I)}) \\
&= \{z \circ 1, q, x\} = z \circ \{1, q, x\} \quad (\text{by (1)}) \\
&= z \circ (q \circ x).
\end{aligned}$$

(III) For any $p, q \in Q$, $(z \circ p) \circ q = z \circ (p \circ q)$, i.e., $z \in C(Q)$. Indeed, let $I_1, I_2, I_3 \in \mathcal{F}$ such that $p \circ I_1 + q \circ I_2 + (p \circ q) \circ I_3 \subseteq J$. Let $I \in \mathcal{F}$ satisfy $I \subseteq I_1 \cap I_2 \cap I_3$, and let $K, L \in \mathcal{F}$ satisfy $K \subseteq U_I I$ and $L \subseteq U_K K$. Notice that

$$\begin{aligned}
L \circ q \subseteq U_K K \circ q &\subseteq \{K \circ q, K, K\} + U_K(K \circ q) \quad (\text{by 0.8(ii)}) \\
&\subseteq \{J, K, K\} + U_K J \subseteq K
\end{aligned} \tag{2}$$

and

$$\begin{aligned}
U_L q &\subseteq (L \circ q) \circ L + L^2 \circ q \quad (\text{see 0.1}) \\
&\subseteq J \circ L + L \circ q \subseteq K
\end{aligned} \tag{3}$$

by (2). Now, for any $y \in L$,

$$\begin{aligned}
U_y\big((z \circ p) \circ q\big) &= \big\{y \circ (z \circ p), q, y\big\} - (z \circ p) \circ U_y q \quad \big(\text{by 0.8(ii)}\big) \\
&= \{z \circ (y \circ p), q, y\} - z \circ (p \circ U_y q) \quad \big(\text{using (II) since } y, U_y q \in K \text{ by (3)}\big) \\
&= z \circ \{y \circ p, q, y\} - z \circ (p \circ U_y q) \quad \big(\text{by (1)}\big) \\
&= z \circ U_y (p \circ q) \quad \big(\text{by 0.8(ii)}\big) \\
&= z \circ \frac{1}{2}\big[\big(y \circ (p \circ q)\big) \circ y - y^2 \circ (p \circ q)\big] \quad (\text{by 0.1}) \\
&= \frac{1}{2}\big[\big(y \circ \big(z \circ (p \circ q)\big)\big) \circ y - y^2 \circ \big(z \circ (p \circ q)\big)\big] \quad \big(\text{by (II)}\big) \\
&= U_y\big(z \circ (p \circ q)\big),
\end{aligned}$$

and we have shown $U_L((z \circ p) \circ q - z \circ (p \circ q)) = 0$, which implies $(z \circ p) \circ q - z \circ (p \circ q) = 0$ by 1.5. □

4.2. Proposition. *Let J be a strongly prime PI Jordan algebra, Γ the centroid of J, and $\tau_J : J \to \Gamma^{-1}J$ the natural injection. Then $(\Gamma^{-1}J, \tau_J)$ is the maximal Jordan algebra of quotients of J with respect to the filter of all nonzero ideals of J.*

Proof. Given $0 \neq q = \gamma^{-1}x \in \Gamma^{-1}J$, we have that $I = \gamma J$ is a nonzero ideal of J such that $0 \neq I \circ q = \tau_J(I) \circ q \subseteq J = \tau_J(J)$. Thus, $(\Gamma^{-1}J, \tau_J)$ is a Jordan algebra of quotients of J with respect to the set of all nonzero ideals of J.

Let (Q, τ) be an algebra of quotients of J. For every $q \in Q$ there exists a nonzero ideal I of J such that $\tau(I) \circ q \subseteq \tau(J)$. By [4, 3.6] I contains a nonzero element z such that $z \in C(J)$; moreover $0 \neq U_z$ by nondegeneracy of J. This allows us to define a map $f : Q \to \Gamma^{-1}J$ given by

$$f(q) = \gamma^{-1}\tau^{-1}(U_{\tau(z)}q), \tag{1}$$

where $\gamma = U_z$ satisfies

$$U_{\tau(z)}q \in \tau(J), \tag{2}$$

$$0 \neq z \in C(J). \tag{3}$$

Let us show that f is well defined. If z' also satisfies (2) and (3), and write $\delta = U_{z'}$, then we have $\gamma^{-1}\tau^{-1}(U_{\tau(z)}q) = \delta^{-1}\tau^{-1}(U_{\tau(z')}q)$ using injectivity of τ since

$$\begin{aligned}
&\tau\big(\gamma\delta\big[\gamma^{-1}\tau^{-1}(U_{\tau(z)}q) - \delta^{-1}\tau^{-1}(U_{\tau(z')}q)\big]\big) \\
&\quad = \tau\big(\big[\delta\tau^{-1}(U_{\tau(z)}q) - \gamma\tau^{-1}(U_{\tau(z')}q)\big]\big) = \tau\big(\big[U_{z'}\tau^{-1}(U_{\tau(z)}q) - U_z\tau^{-1}(U_{\tau(z')}q)\big]\big) \\
&\quad = U_{\tau(z')}U_{\tau(z)}q - U_{\tau(z)}U_{\tau(z')}q = 0
\end{aligned}$$

using $\tau(z), \tau(z') \in C(Q)$ by 4.1. The map f clearly satisfies $f\tau = \tau_J$: for any $x \in J$,

$$f\big(\tau(x)\big) = \gamma^{-1}\tau^{-1}\big(U_{\tau(z)}\tau(x)\big) = \gamma^{-1}\tau^{-1}\big(\tau(U_z x)\big) = \gamma^{-1}U_z x = \tau_J(x).$$

We will finally show that f is an algebra homomorphism. Given $p, q \in Q$, we can use strong primeness of J to find a nonzero ideal I satisfying $\tau(I)\circ p + \tau(I)\circ q + \tau(I)\circ(p^2) \subseteq \tau(J)$; by [4, 3.6] I contains a nonzero element z such that $z \in C(J)$, so that z satisfies (2) and (3) for p, q, p^2 and all their Φ-multiples at the same time, and also z^2 satisfies (2) and (3) for p^2 ($z^2 \neq 0$ since $0 \neq U_{z^2}$ because Γ is a domain [15, 2.8], and $U_{z^2} = (U_z)^2$ by 0.8(iv)); thus, if $\gamma = U_z$, $\delta = \gamma^2 = U_{z^2}$, $\alpha \in \Phi$,

$$\begin{aligned}
f(\alpha p) &= \gamma^{-1}\tau^{-1}\big(U_{\tau(z)}(\alpha p)\big) = \alpha\gamma^{-1}\tau^{-1}(U_{\tau(z)}p) = \alpha f(p),\\
f(p+q) &= \gamma^{-1}\tau^{-1}\big(U_{\tau(z)}(p+q)\big) = \gamma^{-1}\tau^{-1}(U_{\tau(z)}p) + \gamma^{-1}\tau^{-1}(U_{\tau(z)}q)\\
&= f(p) + f(q),\\
f(p^2) &= \delta^{-1}\tau^{-1}\big(U_{\tau(z^2)}\big(p^2\big)\big) = \delta^{-1}\tau^{-1}\big(U_{\tau(z)^2}\big(p^2\big)\big)\\
&= \delta^{-1}\tau^{-1}\big(U_{\tau(z)}U_{\tau(z)}\big(p^2\big)\big) \quad \big(\text{by 0.8(iv)}\big)\\
&= \delta^{-1}\tau^{-1}\big((U_{\tau(z)}p)^2\big) \quad \big(\text{since } \tau(z) \in C(Q) \text{ by 4.1}\big)\\
&= \gamma^{-1}\gamma^{-1}\big(\tau^{-1}(U_{\tau(z)}p)\big)^2 = \big(\gamma^{-1}\tau^{-1}(U_{\tau(z)}p)\big)^2 = \big(f(p)\big)^2. \qquad \square
\end{aligned}$$

4.3. Proposition. *Let J be a strongly prime Jordan algebra and let I be a nonzero ideal of J. If $j : I \to J$ denotes the inclusion, (J, j) is an algebra of quotients of I with respect to the filter of all nonzero ideals of I. Moreover, if $(\widetilde{Q}, \tilde{\tau})$ is an algebra of quotients of J with respect to the filter of all nonzero ideals of J, then $(\widetilde{Q}, \tilde{\tau}j)$ is an algebra of quotients of I with respect to the filter of all nonzero ideals of I.*

Proof. Let $0 \neq x \in J$. Clearly $x \circ j(I) = x \circ I \subseteq I$. Moreover, $x \circ I \neq 0$ since $U_I x \neq 0$ by strong primeness of J and [17, 1.3].

For any $q \in \widetilde{Q}$, there exists a nonzero ideal L of J such that $q \circ \tilde{\tau}(L) \subseteq \tilde{\tau}(J)$. Let $K = U_{L\cap I}(L \cap I)$ which is a nonzero ideal of I and J. For any $a, b \in I \cap L$,

$$\begin{aligned}
q \circ \tilde{\tau}j(U_a b) &= q \circ \big(U_{\tilde{\tau}(a)}\tilde{\tau}(b)\big) = \big\{\tilde{\tau}(a)\circ q, \tilde{\tau}(b), \tilde{\tau}(a)\big\} - U_{\tilde{\tau}(a)}\big(q \circ \tilde{\tau}(b)\big) \quad \big(\text{by 0.8(ii)}\big)\\
&\in \big\{\tilde{\tau}(J), \tilde{\tau}(I), \tilde{\tau}(I)\big\} + U_{\tilde{\tau}(I)}\tilde{\tau}(J) \subseteq \tilde{\tau}(I) = \tilde{\tau}j(I),
\end{aligned}$$

which shows $q \circ \tilde{\tau}j(K) \subseteq \tilde{\tau}j(I)$. Moreover, by 1.5, $0 \neq q \circ \tilde{\tau}(K) = q \circ \tilde{\tau}j(K)$. $\square$

4.4. Let J be a strongly prime Jordan algebra and let I be a nonzero ideal of J. Let $(Q(I), \tau_I)$ be a maximal algebra of quotients of I with respect to the filter of all nonzero ideals of I (notice that I is strongly prime by [12, 2.5]). If $j : I \to J$ denotes the inclusion, by 4.3 and 1.9 there exists a unique algebra homomorphism $f : J \to Q(I)$ such that $fj = \tau_I$ and, moreover, f is injective by 1.11. Let

$$Q = \big\{q \in Q(I) \bigm| q \circ f(L) \subseteq f(J), \text{ for some nonzero ideal } L \text{ of } J\big\}.$$

Clearly $f(J) \subseteq Q$ (for any $x \in J$, $x \circ I \subseteq I$ implies $f(x) \circ f(I) \subseteq f(I) \subseteq f(J)$), so that f can be restricted to $\tau : J \to Q$.

4.5. Proposition. *Under the conditions of* 4.4, (Q, τ) *is a maximal algebra of quotients of* J *with respect to the filter of all nonzero ideals of* J. *Moreover, if* J *has a simple ideal* (*equivalently,* $\mathrm{Heart}(J) \neq 0$ [2, 2.6]), *then* $Q = Q(I)$.

Proof. Notice that (Q, τ) is an algebra of quotients of J with respect to the filter of all nonzero ideals of J: for any $q \in Q$, there exists an ideal L of J such that $q \circ \tau(L) \subseteq \tau(J)$, but $0 \neq I \cap L$ is a nonzero ideal of I, hence, by 1.5, $0 \neq q \circ \tau_I(I \cap L) \subseteq q \circ \tau(L)$.

Let $(\widetilde{Q}, \tilde{\tau})$ be an algebra of quotients of J with respect to the filter of all nonzero ideals of J. By 4.3, $(\widetilde{Q}, \tilde{\tau} j)$ is an algebra of quotients of I, hence, there exists an algebra homomorphism $h : \widetilde{Q} \to Q(I)$ such that $h\tilde{\tau} j = \tau_I$. We claim that $h(\widetilde{Q}) \subseteq Q$. Indeed, for any $q \in \widetilde{Q}$, there exists a nonzero ideal L of J such that $q \circ \tilde{\tau}(L) \subseteq \tilde{\tau}(J)$. Let $K = U_{L\cap I}(L \cap I)$, which is a nonzero ideal of J and I. Notice that

$$q \circ \tilde{\tau}(K) \subseteq \tilde{\tau}(L \cap I) \tag{1}$$

(for any $a, b \in \tilde{\tau}(L \cap I)$, 0.8(ii) yields $q \circ U_a b = -U_a(q \circ b) + \{a \circ q, b, a\} \in U_{\tilde{\tau}(L\cap I)}\tilde{\tau}(J) + \{\tilde{\tau}(J), \tilde{\tau}(L \cap I), \tilde{\tau}(L \cap I)\} \subseteq \tilde{\tau}(L \cap I)$ since $L \cap I$ is an ideal of J). But

$$\begin{aligned} h(q) \circ f(K) &= h(q) \circ fj(K) = h(q) \circ \tau_I(K) = h(q) \circ h\tilde{\tau} j(K) = h\big(q \circ \tilde{\tau} j(K)\big) \\ &= h\big(q \circ \tilde{\tau}(K)\big) \subseteq h\tilde{\tau}(L \cap I) \quad \big(\text{by } (1)\big) \\ &= h\tilde{\tau} j(L \cap I) = \tau_I(L \cap I) = fj(L \cap I) = f(L \cap I) \subseteq f(J). \end{aligned}$$

Now we can restrict h to an algebra homomorphism $g : \tilde{\tau}(J) \to Q$ which satisfies $g\tilde{\tau} = \tau$ ($h\tilde{\tau} j = \tau_I = fj$ implies $h\tilde{\tau} = f$ by uniqueness in 4.4, hence, for any $x \in J$, $g\tilde{\tau}(x) = h\tilde{\tau}(x) = f(x)$).

If I_0 is a simple ideal of J, then I_0 is contained in I by strong primeness of J. Thus I_0 is a simple ideal of I, and, since I is strongly prime, I_0 is contained in any nonzero ideal of I. Thus, any $q \in Q(I)$ satisfies $q \circ f(I_0) = q \circ fj(I_0) = q \circ \tau_I(I_0) \subseteq \tau_I(I) = fj(I) = f(I) \subseteq f(J)$, hence $q \in Q$, and we have shown $Q = Q(I)$. □

4.6. Theorem. *Let* J *be a strongly prime Jordan algebra. Then, there exists a maximal algebra of quotients* $(Q(J), \tau_J)$ *of* J *with respect to the filter of all nonzero ideals of* J. *Up to isomorphism,* $(Q(J), \tau_J)$ *can be obtained as follows*:

(i) *When* J *is PI,* $Q(J) = \Gamma^{-1}J$, *where* Γ *is the centroid of* J, *and* τ_J *is the natural injection of* J *in* $\Gamma^{-1}J$.

(ii) *When* J *is not PI, then there exists a nonzero ideal of* J *of the form* $H(R, *)$, *where* R *is a* $*$*-prime associative algebra which can be assumed to be a* $*$*-tight envelope of* $H(R, *)$. *If* $(H(Q(R), *), \tau)$ *is as in* 3.2, *then* $Q(J) = \{q \in H(Q(R), *) \mid q \circ f(L) \subseteq f(J)$ *for some nonzero ideal* L *of* $J\}$, *where* $f : J \to H(Q(R), *)$ *denotes the unique* *algebra homomorphism such that* $f|_{H(R,*)} = \tau$, *and* τ_J *is the restriction of* f. *Moreover, if* $\mathrm{Heart}(J) \neq 0$, *then* $Q(J) = H(Q(R), *)$ *and* $\tau_J = f$.

Proof. Use 4.2, 3.3, 4.5, and [21, Theorem 3], together with the fact that ideals of a non PI strongly prime Jordan algebra are non PI (0.9). □

4.7. Final remarks. (i) Notice that 4.6 shows that Zelmanov's classification of strongly prime linear Jordan algebras [21, Theorem 3] is given in terms of the maximal Jordan algebras of Martindale-like quotients.

(ii) The existence of maximal algebras of quotients established in 4.6 extends [6, 5.4] to rings of scalars not necessarily having $1/3$, when dealing with strongly prime Jordan algebras.

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