Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Algebra 280 (2004) 635–654
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

The Jordan socle and finitary Lie algebras

Antonio Fernández López [a,*,1], Esther García [b,2],
Miguel Gómez Lozano [a,1]

[a] *Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, 29071 Málaga, Spain*
[b] *Departamento de Álgebra, Universidad Complutense de Madrid, 28040 Madrid, Spain*

Received 1 December 2003

Available online 13 August 2004

Communicated by Georgia Benkart

Abstract

In this paper we introduce the notion of Jordan socle for nondegenerate Lie algebras, which extends the definition of socle given in [A. Fernández López et al., 3-Graded Lie algebras with Jordan finiteness conditions, Comm. Algebra, in press] for 3-graded Lie algebras. Any nondegenerate Lie algebra with essential Jordan socle is an essential subdirect product of strongly prime ones having nonzero Jordan socle. These last algebras are described, up to exceptional cases, in terms of simple Lie algebras of finite rank operators and their algebras of derivations. When working with Lie algebras which are infinite dimensional over an algebraically closed field of characteristic 0, the exceptions disappear and the algebras of derivations are computed.

Keywords: Lie algebra; Grading; Finitary algebra; Jordan socle

* Corresponding author.
E-mail addresses: emalfer@agt.cie.uma.es (A. Fernández López), egarciag@mat.ucm.es (E. García), magomez@agt.cie.uma.es (M. Gómez Lozano).
[1] Partially supported by the MCYT, BFM2001-1938-C02-01, and the Junta de Andalucía FQM264.
[2] Partially supported by the MCYT, BFM2001-1938-C02-02 and by the RTN Network RTN2-2001-00193.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2004.06.013

1. Introduction

Let X be a vector space over a field F. Denote by $\mathfrak{fgl}(X)$ the Lie algebra of all finite rank linear operators on X. A Lie algebra is called *finitary* if it is isomorphic to a subalgebra of $\mathfrak{fgl}(X)$ for some vector space X. Finitary Lie algebras have received a considerable attention in the last years, motivated, in part, by their connection with the *finitary linear* groups, i.e., subgroups of $\mathrm{GL}(X)$ consisting of elements g such the endomorphism $1 - g$ has finite rank (see [20]). Particularly relevant for our purposes is the work by A.A. Baranov [2] classifying infinite dimensional central simple finitary Lie algebras over a field of characteristic 0.

In the present paper we approach simple finitary Lie algebras and their algebras of derivations from a Jordan point of view, we mean, by using techniques from the theory of Jordan systems. This is possible due to the fact that, in most cases, simple finitary Lie algebras admit a nontrivial 3-grading, and for any 3-graded Lie algebra $(L, \pi) = L_1 \oplus L_0 \oplus L_{-1}$, where $\pi = (\pi_1, \pi_0, \pi_{-1})$ denote the projections onto the subspaces L_1, L_0, L_{-1}, we have that $\pi(L) = (L_1, L_{-1})$ is a Jordan pair for the triple products defined by $\{x, y, z\} := [[x, y], z]$ for all $x, z \in L_\sigma$, $y \in L_{-\sigma}$, $\sigma = \pm 1$.

This idea of studying Lie algebras by means of Jordan methods is by no means a novelty; on the contrary, fundamental contributions to this topic can be found in papers like [1,4,19] and [22]. Let us say that in [19], the most related to our approach of these papers, E. Neher describes Lie algebras graded by a 3-graded root system. A Lie algebra L is graded by a 3-graded root system R if and only if it is a central extension of the Tits–Kantor–Koecher algebra of a Jordan pair V ($\mathrm{TKK}(V)$ for short) covered by a grid whose associated 3-graded root system is isomorphic to R. He gives the classification of Jordan pairs covered by a grid and describes their Tits–Kantor–Koecher algebras.

In recent years, a wealthy socle theory has been developed for nondegenerate Jordan pairs (see [15]) and, following the pattern of the structure of prime rings with minimal one sided ideals, strongly prime Jordan pairs with nonzero socle have been classified [9]. We note that any simple Jordan pair covered by a grid with division coordinate algebra coincides with its socle, so in this case the socle theory and the grid theory agree.

In our paper [7] we develop a similar socle theory for 3-graded Lie algebras making use of their close relationship with Jordan pairs, and describe nondegenerate 3-graded Lie algebras with *large* socles and their central extensions. Let (L, π) be a nondegenerate 3-graded Lie algebra. If $\pi(L)$ has socle $\mathrm{Soc}(\pi(L)) = (\mathrm{Soc}(\pi(L))^+, \mathrm{Soc}(\pi(L))^-)$, then

$$\mathrm{Soc}\big(\pi(L)\big)^+ \oplus \big[\mathrm{Soc}\big(\pi(L)\big)^+, \mathrm{Soc}\big(\pi(L)\big)^-\big] \oplus \mathrm{Soc}\big(\pi(L)\big)^-$$

turns out to be an ideal of L that we call the socle of (L, π) and denote by $\mathrm{Soc}_\pi(L)$.

It is natural to ask whether the socle of a 3-graded Lie algebra is independent of the grading. In the present paper we answer this question by proving that, in general, the socle depends on the grading (3.3), but for gradings which are effective (3.4), the socle turns to be independent (3.10). Nevertheless, it is still possible to extend the notion of socle to any nondegenerate Lie algebra L, not necessarily 3-graded, by taking $\mathrm{JSoc}(L) := \sum \mathrm{Soc}_\pi(I)$, where (I, π) ranges over all 3-graded ideals of L (cf. 4.1). We call $\mathrm{JSoc}(L)$ the Jordan socle of L.

In 4.3 we prove that any nondegenerate Lie algebra with essential Jordan socle is sandwiched, via the adjoint mapping, between the TKK-algebra $\mathrm{TKK}(V)$ of a nondegenerate Jordan pair V coinciding with its socle, and the algebra of derivations $\mathrm{Der}(\mathrm{TKK}(V))$. Moreover, in this case, $\mathrm{JSoc}(L) = \mathrm{ad}(\mathrm{TKK}(V)) = \bigoplus \mathrm{ad}(\mathrm{TKK}(V_i))$, where the V_i are simple Jordan pairs with minimal inner ideals (4.2). Thus, to describe the Jordan socle of a nondegenerate Lie algebra, it suffices to compute the TKK-algebras of the simple Jordan pairs with minimal inner ideals. This task is carried out in Section 5 where we prove (5.15) that, up to two exceptional cases (types E_6 and E_7), simple Lie algebras with nonzero Jordan socle are 3-graded Lie algebras of finite rank operators; the 3-gradings are also described. We complete the description of nondegenerate Lie algebras with essential Jordan socle by determining the algebra of derivations of the simple components of the Jordan socle. To do so, we consider in the last section simple finitary Lie algebras of infinite dimension over an algebraically closed field of characteristic 0. By using Baranov's classification [2, Corollary 1.2] together with De La Harpe's methods [5, I.8, Proposition 2], we compute their algebras of derivations. Any Lie algebra which is sandwiched between an infinite dimensional finitary simple Lie algebra, say M, and its algebra of derivations $\mathrm{Der}\, M$ is strongly prime and contains a reduced element (6.4). Conversely, any 3-graded Lie algebra which is strongly prime, infinite dimensional and whose associated Jordan pair contains a reduced element can be sandwiched between a finitary simple Lie algebra and its algebra of derivations (6.7). The question whether or not the 3-graded condition can be removed remains open.

2. Preliminaries on Lie algebras and Jordan pairs

2.1. Throughout this paper, we will be dealing with Lie algebras L and Jordan pairs $V = (V^+, V^-)$ over a ring of scalars Φ containing $1/6$. As usual, $[x, y]$ will denote the Lie product and ad_x the adjoint mapping determined by x. Jordan products will be denoted by $Q_x y$, for any $x \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$, with linearizations $Q_{x,z} y = \{x, y, z\} = D_{x,y} z$. The reader is referred to [12,14,19] for basic results, notation and terminology. Nevertheless, we will stress some notions and basic properties for both Jordan pairs and Lie algebras.

2.2. An element $x \in V^\sigma$ is called an *absolute zero divisor* if $Q_x = 0$. Then V is said to be *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $Q_{B^\pm} B^\mp = 0$ implies $B = 0$, and *prime* if $Q_{B^\pm} C^\mp = 0$ implies $B = 0$ or $C = 0$, for $B = (B^+, B^-)$, $C = (C^+, C^-)$ ideals of V. Similarly, $x \in L$ is an *absolute zero divisor* of L if $\mathrm{ad}_x^2 = 0$, and L is *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $[I, I] = 0$ implies $I = 0$, and *prime* if $[I, J] = 0$ implies $I = 0$ or $J = 0$, for I, J ideals of L. A Jordan pair or Lie algebra is *strongly prime* if it is prime and nondegenerate.

2.3. Ideals of nondegenerate (strongly prime) Jordan pairs inherit nondegeneracy (strong primeness) [14, JP3], [16]. The same is true for Lie algebras: every ideal of a nondegenerate (strongly prime) Lie algebra is nondegenerate (strongly prime) ([21, Lemma 4], [10, 0.4, 1.5]).

2.4. Given a subset S of L, the *annihilator* or *centralizer* of S in L, $\mathrm{Ann}_L(S)$, consists of the elements $x \in L$ such that $[x, S] = 0$. By the Jacobi identity, $\mathrm{Ann}_L(S)$ is a subalgebra of L and an ideal whenever S is so. Clearly, $\mathrm{Ann}_L(L) = Z(L)$, the center of L. Moreover, if an ideal E of L is semiprime (as an algebra), then E is *essential* ($E \cap I \neq 0$ for every nonzero ideal I of L) if and only if $\mathrm{Ann}_L(E) = 0$. Notice also that L is prime if and only if the annihilator of every nonzero ideal of L is zero.

The *annihilator* of a subset $X \subset V^\sigma$ is the set $\mathrm{Ann}_V(X)$ of all $a \in V^{-\sigma}$ satisfying: $Q_a x = 0$, $Q_x a = 0$, $Q_a Q_x = D_{a,x} = 0$, $Q_x Q_a = D_{x,a} = 0$ for every $x \in X$ (cf. [14,16]). In the linear case we are considering here, the annihilator can be more easily characterized [8, Lemma 1]: $a \in \mathrm{Ann}_V(X)$ if and only if $D_{a,x} = 0 = D_{x,a}$ for every $x \in X$. If $I = (I^+, I^-)$ is an ideal of V, then $\mathrm{Ann}_V(I) = (\mathrm{Ann}_V(I^-),\ \mathrm{Ann}_V(I^+))$ is also an ideal of I and has an easy expression when V is nondegenerate [16, Proposition 1.7]:

$$\mathrm{Ann}_V\big(I^\sigma\big) = \big\{a \in V^{-\sigma}\colon\ Q_a I^\sigma = 0\big\}. \tag{1}$$

A Lie analogue to (1) also holds, as can be seen in the following lemma.

2.5. Lemma. *Let I be a nondegenerate ideal of a Lie algebra L. Then* $\mathrm{Ann}_L(I) = \{a \in L \mid [a, [a, I]] = 0\}$. *Hence,* $I \cap \mathrm{Ann}_L(I) = 0$ *and, if* $\mathrm{Ann}_L(I) = 0$ *then the whole L is nondegenerate.*

Proof. If $a \in L$ is such that $[a, [a, I]] = 0$, then $[a, [a, I + \Phi a]] = 0$, i.e., a is also an absolute zero divisor of the subalgebra $I' = I + \Phi a$ of L. Therefore, $a \in K(I')$, where $K(\ )$ denotes the strongly degenerate radical or Kostrikin radical (cf. [21, p. 538]). Now $[a, I] \subset K(I') \cap I = K(I)$ by [21, Corollary 1, p. 543], and it is clear that $K(I) = 0$ because I is nondegenerate. We have shown that $[a, [a, I]] = 0$ implies that $[a, I] = 0$. For the last part, if $x \in I \cap \mathrm{Ann}_L(I)$ then $[x, [x, I]] = 0$, which implies $x = 0$ by nondegeneracy of I. We also have that if $\mathrm{Ann}_L(I) = 0$ then L is nondegenerate: if $[a, [a, L]] = 0$ for some $a \in L$, then $[a, [a, I]] = 0$, so $a \in \mathrm{Ann}_L(I) = 0$. □

2.6. A 3-*grading* of a Lie algebra L is a decomposition $L = L_1 \oplus L_0 \oplus L_{-1}$, where each L_i is a submodule of L satisfying $[L_i, L_j] \subset L_{i+j}$, and where $L_{i+j} = 0$ if $i + j \neq 0, \pm 1$. A Lie algebra is 3-*graded* if it has a 3-grading. We will write (L, π) to denote the Lie algebra L with the particular 3-grading $\pi = (\pi_1, \pi_0, \pi_{-1})$, where each π_i is the projection of L onto L_i, $i = 0, \pm 1$.

Given (L, π) we have that $\pi(L) := (L_1, L_{-1})$ is a Jordan pair for the triple products defined by $\{x, y, z\} := [[x,\ y],\ z]$ for all $x, z \in L_\sigma$, $y \in L_{-\sigma}$, $\sigma = \pm 1$, which is called the *associated Jordan pair* of (L, π). We note that if L is nondegenerate, so is $\pi(L)$ [22, Lemma 1.8]. A standard example of a 3-graded Lie algebra is that given by the TKK-algebra of a Jordan pair.

2.7. For any Jordan pair V, there exists a 3-graded Lie algebra $\mathrm{TKK}(V) = L_1 \oplus L_0 \oplus L_{-1}$, the *Tits–Kantor–Koecher algebra of V*, uniquely determined by the following conditions (cf. [19, 1.5(6)]):

(TKK1) The associated Jordan pair (L_1, L_{-1}) of L is isomorphic to V.
(TKK2) $[L_1, L_{-1}] = L_0$.
(TKK3) $[x_0, L_1 \oplus L_{-1}] = 0$ implies $x_0 = 0$, for any $x_0 \in L_0$.

In general, by a TKK-*algebra* we mean a Lie algebra of the form $\mathrm{TKK}(V)$ for some Jordan pair V.

2.8. Lemma. *Let (L, π) be a 3-graded Lie algebra with associated Jordan pair $\pi(L) \neq 0$. If $\pi(L)$ is perfect (i.e., $\{\pi_\sigma(L), \pi_{-\sigma}(L), \pi_\sigma(L)\} = \pi_\sigma(L)$ for both $\sigma = \pm 1$) and $[L, L]$ is simple, then* $\mathrm{TKK}(\pi(L)) \cong [L, L]$.

Proof. Since $\pi(L) \neq 0$ is perfect, $\pi_1(L) \oplus [\pi_1(L), \pi_{-1}(L)] \oplus \pi_{-1}(L)$ is a nonzero ideal of L by [7, Lemma 4.2], clearly contained in $[L, L]$. Then, by simplicity, $[L, L] = \pi_1(L) \oplus [\pi_1(L), \pi_{-1}(L)] \oplus \pi_{-1}(L)$, and the latter is isomorphic to $\mathrm{TKK}(\pi(L))$ by 2.7. □

3. The socle of a nondegenerate 3-graded Lie algebra

3.1. An *inner ideal* of a Jordan pair V is a Φ-submodule $K \subset V^\sigma$ such that $Q_K V^{-\sigma} \subset K$. Following [15], the *socle* of a nondegenerate Jordan pair V is defined by $\mathrm{Soc}(V) = (\mathrm{Soc}(V^+), \mathrm{Soc}(V^-))$, where $\mathrm{Soc}(V^\sigma)$ is the sum of all minimal inner ideals of V contained in V^σ. The socle is a von Neumann regular ideal and satisfies the descending chain condition on principal inner ideals.

3.2. The *socle* of a nondegenerate 3-graded Lie algebra (L, π) is defined as the ideal of L generated by the socle of the associated Jordan pair $\pi(L)$. Denoted by $\mathrm{Soc}_\pi(L)$ to show which grading we are considering, we have that $\mathrm{Soc}_\pi(L) = \mathrm{Soc}(\pi_1(L)) \oplus [\mathrm{Soc}(\pi_1(L)), \mathrm{Soc}(\pi_{-1}(L))] \oplus \mathrm{Soc}(\pi_{-1}(L))$ [7, 4.3]. Moreover, $\mathrm{Soc}_\pi(L)$ can be decomposed as a direct sum of simple ideals,

$$\mathrm{Soc}_\pi(L) = \bigoplus S^{(i)} = \bigoplus \mathrm{TKK}\big(\pi\big(S^{(i)}\big)\big),$$

where the $\pi(S^{(i)})$ are the simple components of $\mathrm{Soc}(\pi(L))$.

In general, the definition of the socle of a nondegenerate 3-graded Lie algebra depends on the 3-grading, as can be seen in the following example.

3.3. Example. Let V and W be two Jordan pairs coinciding with their socles, i.e., $V = \mathrm{Soc}(V)$ and $W = \mathrm{Soc}(W)$. Let L be the Lie algebra built as the direct sum of the TKK-algebras of V and W. Notice that L admits the gradings

$$\pi_1(L) = V^+, \qquad \pi_0(L) = \big[V^+, V^-\big] \oplus \mathrm{TKK}(W), \qquad \pi_{-1}(L) = V^-,$$
$$\pi'_1(L) = W^+, \qquad \pi'_0(L) = \big[W^+, W^-\big] \oplus \mathrm{TKK}(V), \qquad \pi'_{-1}(L) = W^-,$$
$$\pi''_1(L) = V^+ \oplus W^+, \qquad \pi''_0(L) = \big[V^+ \oplus W^+, V^- \oplus W^-\big], \qquad \pi''_{-1}(L) = V^- \oplus W^-,$$

which give three essentially different socles: $\mathrm{Soc}_{\pi}(L) = \mathrm{TKK}(V)$ while $\mathrm{Soc}_{\pi'}(L) = \mathrm{TKK}(W)$ and $\mathrm{Soc}_{\pi''}(L) = L$.

3.4. We will show that the socle is indeed independent of the grading of L when the grading is *effective* in the sense that there is no nonzero ideal contained in the zero part of L. Notice that this condition is satisfied when (L, π) has (TKK3), and, in particular, when L is graded as the TKK-algebra of a Jordan pair or when L is strongly prime.

3.5. Let (L, π) be a 3-graded Lie algebra with associated Jordan pair $\pi(L)$. For any ideal I of L, denote by $\pi_\sigma(I)$ the projection of I onto $\pi_\sigma(L)$, $\sigma = \pm 1$. We have the following relations:

(i) $[\pi_0(L), \pi_\sigma(I)] \subset \pi_\sigma(I)$, hence also $\{\pi_\sigma(L), \pi_{-\sigma}(L), \pi_\sigma(I)\} \subset \pi_\sigma(I)$, and $\{\pi_{-\sigma}(L), \pi_\sigma(I), \pi_{-\sigma}(L)\} \subset I \cap \pi_{-\sigma}(L)$. Therefore
(ii) $\pi(I) := (\pi_1(I), \pi_{-1}(I))$ is an ideal of $\pi(L)$. Moreover,
(iii) $\mathrm{id}_L(\pi(I)) = \pi_1(I) \oplus ([\pi_1(I), \pi_{-1}(L)] + [\pi_{-1}(I), \pi_1(L)]) \oplus \pi_{-1}(I)$, where by $\mathrm{id}_L(\pi(I))$ we denote the ideal of L generated by $\pi(I)$.

3.6. Lemma. *Let (L, π) be a nondegenerate 3-graded Lie algebra, and let I be an ideal of L. Then*

(i) $\mathrm{Ann}_{\pi(L)}(\pi_{-1}(I)) \oplus \mathrm{Ann}_{\pi(L)}(\pi_1(I)) \subset \mathrm{Ann}_L(\mathrm{id}_L(\pi(I)))$.
(ii) *If the grading is effective and I is nonzero, then there exists a nonzero element $x \in \pi_1(L) \cup \pi_{-1}(L)$ such that $[x, [x, I]] \neq 0$.*

Proof. (i) Let $x \in \mathrm{Ann}_{\pi(L)}(\pi_{-1}(I))$. Then $x \in \pi_1(L)$ satisfies

$$\big[\big[x, \pi_{-1}(I)\big], \pi_1(L) \oplus \pi_{-1}(L)\big] = \big\{x, \pi_{-1}(I), \pi_1(L)\big\} + \big\{\pi_{-1}(I), x, \pi_{-1}(L)\big\} = 0.$$

Hence $[x, \pi_{-1}(I)] = 0$ since the ideal $\pi_1(L) \oplus [\pi_1(L), \pi_{-1}(L)] \oplus \pi_{-1}(L)$ inherits nondegeneracy from $\pi(L)$ and therefore it is centerfree. Now it follows from 3.5(iii) that

$$\begin{aligned}\big[x, \mathrm{id}_L\big(\pi(I)\big)\big] &= \big[x, \big[\pi_1(I), \pi_{-1}(L)\big] + \big[\pi_{-1}(I), \pi_1(L)\big]\big] \\ &= \big\{\pi_1(I), \pi_{-1}(L), x\big\} + \big\{x, \pi_{-1}(I), \pi_1(L)\big\} = 0.\end{aligned}$$

Therefore $\mathrm{Ann}_{\pi(L)}(\pi_{-1}(I)) \subset \mathrm{Ann}_L(\mathrm{id}_L(\pi(I)))$. The other containment follows by symmetry.

(ii) Suppose on the contrary that $[x, [x, I]] = 0$ for every $x \in \pi_1(L) \cup \pi_{-1}(L)$. Then it follows from 2.5 that $\pi_1(L) \cup \pi_{-1}(L) \subset \mathrm{Ann}_L(\mathrm{id}_L(\pi(I)))$. But then $\mathrm{id}_L(\pi(I)) \subset \pi_1(L) \oplus [\pi_1(L), \pi_{-1}(L)] \oplus \pi_{-1}(L) \subset \mathrm{Ann}_L(\mathrm{id}_L(\pi(I)))$, which implies that $\mathrm{id}_L(\pi(I)) = 0$ by nondegeneracy of L. Then $I \subset \pi_0(L)$, which contradicts the effectiveness of the grading. □

3.7. In [3, 1.7(iii)] Benkart proves that any $x \in L$ with $\mathrm{ad}_x^3 = 0$ and any $y \in L$ satisfy the following identity

$$\mathrm{ad}^2_{\mathrm{ad}^2_x y} = \mathrm{ad}^2_x \, \mathrm{ad}^2_y \, \mathrm{ad}^2_x . \tag{1}$$

Notice that the condition $\mathrm{ad}_x^3 = 0$ is trivially fulfilled whenever x belongs to the 1 or -1 part of a 3-grading. In fact, if $x \in \pi_1(L)$ and denote by y_{-1} the projection of y onto $\pi_{-1}(L)$, this identity yields the fundamental Jordan identity on the Jordan pair $\pi(L)$:

$$\mathrm{ad}^2_{\mathrm{ad}^2_x y_{-1}} = \mathrm{ad}^2_{\mathrm{ad}^2_x y} = \mathrm{ad}^2_x \, \mathrm{ad}^2_y \, \mathrm{ad}^2_x = \mathrm{ad}^2_x \, \mathrm{ad}^2_{y_{-1}} \, \mathrm{ad}^2_x .$$

Identity (1) plays a fundamental role in the construction of minimal inner ideals.

A submodule B of a Lie algebra L is an *inner ideal* of L if $[B,[B,L]] \subset B$. An *abelian inner ideal* is an inner ideal B which is also an abelian subalgebra, i.e., $[B,B] = 0$.

3.8. Proposition.

(i) *Let L be a Lie algebra, B an inner ideal of L, and $c \in L$ be such that* $\mathrm{ad}_c^3 = 0$. *Then* $\mathrm{ad}_c^2 B$ *is an abelian inner ideal of L.*

Suppose for the rest of the proposition that L is nondegenerate.

(ii) *A nonzero abelian inner ideal B of L is minimal if and only if $B = \mathrm{ad}_b^2 L$ for every nonzero element b of B.*

(iii) *Let B be an abelian minimal inner ideal, and $c \in L$ be such that* $\mathrm{ad}_c^3 = 0$. *Then either* $\mathrm{ad}_c^2 B$ *is zero or an abelian minimal inner ideal.*

(iv) *Let $I = \pi_1(I) \oplus \pi_0(I) \oplus \pi_{-1}(I)$ be a* 3*-graded ideal of L. A submodule $B \subset \pi_\sigma(I)$, $\sigma = \pm 1$, is a minimal inner ideal of the Jordan pair $\pi(I)$ if and only if it is an* (*abelian*) *minimal inner ideal of L. In particular, an element $y \in \pi_\sigma(I)$ of* (*Jordan*) *rank one in $\pi(I)$ generates the abelian minimal inner ideal $[y,[y,L]]$ of L.*

Proof. For (i) use the same proof as that of [3, Lemma 1.8], while (ii) follows from [3, Theorem 1.12].

(iii) Suppose now that B is an abelian minimal inner ideal and that $\mathrm{ad}_c^2 b \neq 0$ for some $b \in B$. Then

$$0 \neq \mathrm{ad}^2_{\mathrm{ad}^2_c b} L = \mathrm{ad}^2_c \, \mathrm{ad}^2_b \, \mathrm{ad}^2_c L,$$

by (1) and nondegeneracy of L, which implies $\mathrm{ad}_b^2 \, \mathrm{ad}_c^2 L \neq 0$. But this is an inner ideal of L (contained in B) by (i), since $\mathrm{ad}_b^3 = \mathrm{ad}_c^3 = 0$. Thus, $\mathrm{ad}_b^2 \, \mathrm{ad}_c^2 L = B$ by minimality of B, and therefore,

$$\mathrm{ad}^2_{\mathrm{ad}^2_c b} L = \mathrm{ad}^2_c B,$$

which proves that $\mathrm{ad}_c^2 B$ is minimal.

(iv) Let B be a submodule of $\pi_\sigma(I)$ (say $\sigma = 1$), and therefore an abelian subalgebra of L. By (i) and nondegeneracy of I, if B is a minimal inner ideal of L, then

$B = \mathrm{ad}_b^2 I = \mathrm{ad}_b^2 \pi_{-1}(I) = \{b, \pi_{-1}(I), b\}$, for every $0 \neq b \in B$. Hence, B is a minimal inner ideal of $\pi(I)$. Suppose, conversely, that $B \subset \pi_1(I)$ is a minimal inner ideal of the Jordan pair $\pi(I)$. Note that, because of the grading, for any $b \in B$, $\mathrm{ad}_b^3 I = 0$. We claim that $\mathrm{ad}_b^3 L = 0$. Otherwise, there exists $a \in L$ such that $0 \neq c = \mathrm{ad}_b^3 a \in B$. But $\mathrm{ad}_b^4 L \subset \mathrm{ad}_b^3 I = 0$ hence by using a Kostrikin's result [13] (or [3, Proposition 1.5]), $\mathrm{ad}_c^3 L = 0$, giving that $B = \mathrm{ad}_c^2 I$ (by its minimality in $\pi(I)$) is an inner ideal of L by (i). But then $\mathrm{ad}_b^3 L = [b,[b,[b,L]]] \subset [b,B] = 0$, since $B \subset \pi_1(I)$ is clearly abelian. Therefore, $\mathrm{ad}_b^3 L = 0$ for every $b \in B$, so as soon as $b \neq 0$, $B = \mathrm{ad}_b^2 I$ (by its minimality in $\pi(I)$) is an abelian minimal inner ideal of L. The last assertion of (iv) follows from the first part of (iv) together with (ii). □

3.9. Theorem. *Let (L,π) be a nondegenerate 3-graded Lie algebra with an effective 3-grading π, and let I be an ideal of L which is graded with respect to a 3-grading π'. Then $\mathrm{Soc}_{\pi'}(I) \subset \mathrm{Soc}_\pi(L)$.*

Proof. If $\mathrm{Soc}_{\pi'}(I)$ is zero, there is nothing to prove. Assume then that $\mathrm{Soc}_{\pi'}(I) \neq 0$ and show that it is contained in $\mathrm{Soc}_\pi(L)$. By 3.6(ii), for any simple component $S'^{(i)}$ of $\mathrm{Soc}_{\pi'}(I)$, there exists a nonzero element, say $z \in \pi_1(L)$, such that $[z,[z,S'^{(i)}]] \neq 0$. Let us consider $0 \neq w = [z,[z,s']]$ for some $s' \in S'^{(i)}$. Notice that $w \in \pi_1(S'^{(i)})$ because $z \in \pi_1(L)$ and $S'^{(i)}$ is an ideal of L by simplicity. Therefore, the ideal of L generated by w, which coincides with $S'^{(i)}$ by simplicity, is π-graded, i.e., $S'^{(i)} = \pi_1(S'^{(i)}) \oplus [\pi_1(S'^{(i)}), \pi_{-1}(S'^{(i)})] \oplus \pi_{-1}(S'^{(i)})$.

Let us show that there exists $x \in \pi_1(S'^{(i)}) \cup \pi_{-1}(S'^{(i)})$ such that

$$\big[x,\big[x,\pi'_1\big(S'^{(i)}\big)\big]\big] + \big[x,\big[x,\pi'_{-1}\big(S'^{(i)}\big)\big]\big] \neq 0. \tag{2}$$

Otherwise, $[x,[x,\pi'_1(S'^{(i)})]] = 0 = [x,[x,\pi'_{-1}(S'^{(i)})]]$ for all $x \in \pi_1(S'^{(i)})$, so also $[x,[x,\pi_{-1}(\pi'_1(S'^{(i)}))]] = 0$, and $[x,[x,\pi_{-1}(\pi'_{-1}(S'^{(i)}))]] = 0$, giving

$$\pi_{-1}\big(\pi'_1\big(S'^{(i)}\big)\big) = 0 \quad \text{and} \quad \pi_{-1}\big(\pi'_{-1}\big(S'^{(i)}\big)\big) = 0 \tag{a}$$

by nondegeneracy of the pair $(\pi_1(S'^{(i)}), \pi_{-1}(S'^{(i)}))$. Similarly, if we supposed that $[x,[x,\pi'_1(S'^{(i)})]] = 0 = [x,[x,\pi'_{-1}(S'^{(i)})]]$ for all $x \in \pi_{-1}(S'^{(i)})$ we would obtain that

$$\pi_1\big(\pi'_1\big(S'^{(i)}\big)\big) = 0 \quad \text{and} \quad \pi_1\big(\pi'_{-1}\big(S'^{(i)}\big)\big) = 0. \tag{b}$$

Then (a) and (b) would lead to $\pi'_1(S'^{(i)}) + \pi'_{-1}(S'^{(i)}) \subset \pi_0(L)$, implying that the whole ideal $S'^{(i)}$, which is generated as a subalgebra by $\pi'_1(S'^{(i)}) + \pi'_{-1}(S'^{(i)})$, is contained in $\pi_0(L)$. This contradicts the hypothesis of π being an effective grading.

Since $\pi'_1(S'^{(i)})$ and $\pi'_{-1}(S'^{(i)})$ are generated by (Jordan) rank one elements of $\pi'(I)$ (therefore, by 3.8(iv), by elements $y \in \pi'(I)$ such that $\mathrm{ad}_y^2 L$ is an abelian minimal inner ideal of L), we can take by (2) an element, say $x \in \pi_1(S'^{(i)})$, and a (Jordan) rank one element, say $y \in \pi'_1(S'^{(i)})$, such that $\mathrm{ad}_x^2 y \neq 0$. Now it follows from (1) and 3.8(iii) that

$$0 \neq \mathrm{ad}^2_{\mathrm{ad}_x^2 y} L = \mathrm{ad}_x^2 \mathrm{ad}_y^2 \mathrm{ad}_x^2 L = \mathrm{ad}_x^2 \mathrm{ad}_y^2 L$$

is a minimal inner ideal of L, and since $\mathrm{ad}_x^2 y = [x, [x, \pi_{-1}(y)]] = -\{x, \pi_{-1}(y), x\} \in \pi_1(L)$,

$$B := \mathrm{ad}^2_{\mathrm{ad}_x^2 y} L$$

is also a minimal inner ideal of the Jordan pair $\pi(L)$. Therefore, $B \subset S'^{(i)} \cap \mathrm{Soc}_\pi(L)$, and since $S'^{(i)}$ is a simple ideal of L, $S'^{(i)} \subset \mathrm{Soc}_\pi(L)$, for any simple component $S'^{(i)}$ of $\mathrm{Soc}_{\pi'}(I)$. Thus $\mathrm{Soc}_{\pi'}(I) \subset \mathrm{Soc}_\pi(L)$. □

Therefore, as soon as a nondegenerate Lie algebra L has an effective 3-grading (L, π), its socle contains the socle of any other 3-grading of L.

3.10. Corollary. *Let L be a nondegenerate Lie algebra admitting an effective 3-grading (L, π). Then $\mathrm{Soc}_\pi(L) \supset \mathrm{Soc}_{\pi'}(L)$ for any other 3-grading (L, π') of L. In particular, the socle of L does not depend on the effective 3-grading considered.*

4. The Jordan socle

Motivated by Theorem 3.9 we are going to introduce a notion of socle, called the *Jordan socle*, for nondegenerate Lie algebras which are not necessarily 3-graded.

4.1. Given a nondegenerate Lie algebra L, we define its *Jordan socle* as the sum of the socles of (I, π), where I is any 3-graded ideal of L and π denotes any of its possible 3-gradings:

$$\mathrm{JSoc}(L) = \sum_{(I,\pi)} \mathrm{Soc}_\pi(I).$$

4.2. Theorem. *The Jordan socle of a nondegenerate Lie algebra L is an ideal of L. If $\mathrm{JSoc}(L) \neq 0$ then it is a direct sum of simple ideals each of which is the TKK-algebra of a simple Jordan pair with minimal inner ideals. Therefore, $\mathrm{JSoc}(L) \cong \mathrm{TKK}(V)$, where V is a nondegenerate Jordan pair coinciding with its socle.*

Proof. By 3.2, for any 3-graded ideal (I, π) of L,

$$\mathrm{Soc}_\pi(I) = \bigoplus S^{(i)} = \bigoplus \mathrm{TKK}\big(\pi\big(S^{(i)}\big)\big),$$

where, as pointed out before, the $S^{(i)}$ are actually ideals of L, and the $\pi(S^{(i)})$ are the simple components of $\mathrm{Soc}(\pi(I))$. Moreover, $\mathrm{JSoc}(L) = \bigoplus \mathrm{TKK}(\pi(S^{(i)})) \cong \mathrm{TKK}(\bigoplus \pi(S^{(i)}))$, and $\bigoplus \pi(S^{(i)})$ is a nondegenerate Jordan pair that coincides with its socle. □

Thus, the Jordan socle of a nondegenerate Lie algebra is the biggest 3-graded ideal of L that coincides with its socle. Moreover, if (L,π) is a nondegenerate 3-graded Lie algebra with effective 3-grading, then $\mathrm{Soc}_\pi(L) = \mathrm{JSoc}(L)$. We also have a structure theorem for nondegenerate Lie algebra with essential socle similar to that proved in [7] for 3-graded Lie algebras. Recall that an *essential subdirect product* of a family of algebras $\{L_\alpha\}$ is any subdirect product of the L_α containing an essential ideal of the full direct product $\prod A_\alpha$.

4.3. Theorem. *For a Lie algebra L, the following statements are equivalent*:

(i) *L is nondegenerate and has essential Jordan socle.*
(ii) *L is an essential subdirect product of a family of strongly prime Lie algebras L_α having nonzero Jordan socles.*
(iii) $\bigoplus \mathrm{ad}(\mathrm{TKK}(V_\alpha)) \lhd L \leqslant \prod \mathrm{Der}(\mathrm{TKK}(V_\alpha))$, *where each V_α is a simple Jordan pair with minimal inner ideals.*
(iv) *There exists a nondegenerate Jordan pair V that coincides with its socle such that* $\mathrm{ad}(\mathrm{TKK}(V)) \leqslant L \leqslant \mathrm{Der}(\mathrm{TKK}(V))$.

Moreover, in case (iv), $\mathrm{JSoc}(L) = \mathrm{ad}(\mathrm{TKK}(V))$, *and L is strongly prime if and only if* $\mathrm{JSoc}(L)$ *is simple, if and only if V is simple. We also have that* $\mathrm{Der}(\mathrm{TKK}(V))$ *is the largest strongly prime Lie algebra having Jordan socle equal to* $\mathrm{ad}(\mathrm{TKK}(V))$.

Proof. (i) ⇒ (ii). Let $\mathrm{JSoc}(L) = \bigoplus M_\alpha$ be essential, with the M_α being the simple components of $\mathrm{JSoc}(L)$, and take $L_\alpha := L/\mathrm{Ann}_L(M_\alpha)$ for each index α. By 2.4, $\bigcap \mathrm{Ann}_L(M_\alpha) = \mathrm{Ann}_L(\bigoplus M_\alpha) = \mathrm{Ann}_L(\mathrm{JSoc}(L)) = 0$, and therefore L is a subdirect product of the L_α. Moreover, by 2.5,

$$\overline{M_\alpha} := \frac{M_\alpha + \mathrm{Ann}_L(M_\alpha)}{\mathrm{Ann}_L(M_\alpha)} \cong \frac{M_\alpha}{M_\alpha \cap \mathrm{Ann}_L(M_\alpha)} = M_\alpha$$

is a nondegenerate simple ideal of L_α coinciding with its Jordan socle and having $\mathrm{Ann}_{L_\alpha}(\overline{M_\alpha}) = \bar{0}$. Hence, again by 2.5 and simplicity of $\overline{M_\alpha}$, L_α is a strongly prime Lie algebra with $\mathrm{JSoc}(L_\alpha) = \overline{M_\alpha}$. Finally, since the ideal $\bigoplus \overline{M_\alpha}$ is essential in $\prod L_\alpha$, we have that L is actually an essential subdirect product of the L_α.

(ii) ⇒ (iii). Assume now that L is an essential subdirect product of a family of strongly prime Lie algebras $\{L_\alpha\}$ having nonzero Jordan socles $\mathrm{JSoc}(L_\alpha) = M_\alpha \cong \mathrm{TKK}(V_\alpha)$, where the V_α are simple Jordan pairs with minimal inner ideals. Applying [6, Theorem 4.1] and taking into account that we can replace, for each index α, the maximal uniform component $\mathrm{Ann}_L(\mathrm{Ann}_L(M_\alpha))$ by M_α, since both have the same annihilator, we may assume that

$$\bigoplus M_\alpha \leqslant L \leqslant \prod L_\alpha,$$

with $L_\alpha \cong L/\mathrm{Ann}_L(M_\alpha)$ for each index α. Now, $L/\mathrm{Ann}_L(M_\alpha)$ can be regarded, via the adjoint mapping, as a subalgebra of $\mathrm{Der}(M_\alpha)$.

(iii) $\Rightarrow$ (iv). It follows setting $V = \bigoplus V_\alpha$ and observing that $\mathrm{Der}(\bigoplus \mathrm{TKK}(V_\alpha))$ is isomorphic to $\prod(\mathrm{Der}(\mathrm{TKK}(V_\alpha)))$.

(iv) $\Rightarrow$ (i). Suppose finally that L is as in (iv). By [7, Proposition 2.11], $\mathrm{TKK}(V)$ is a nondegenerate Lie algebra, which coincides with its Jordan socle. Moreover, $\mathrm{ad}(\mathrm{TKK}(V)) \cong \mathrm{TKK}(V)$ is essential in $\mathrm{Der}(\mathrm{TKK}(V))$ and therefore also in L. Hence L is nondegenerate, by 2.5, and has essential Jordan socle.

The remainder of the proof goes as that of [7, Theorem 5.4]. □

5. Simple Lie algebras coinciding with their Jordan socles

By 4.2, to describe the Jordan socle of a nondegenerate Lie algebra L, it suffices to compute the TKK-algebras of the simple Jordan pairs with minimal inner ideals. We begin by recalling some notation on pairs of dual vector spaces and related Jordan pairs of continuous operators.

5.1. Let $\mathcal{P} = (X, Y, g)$ and $\mathcal{P}' = (X', Y', g')$ be two pairs of dual vector spaces over the same division Φ-algebra Δ. A linear operator $a : X \to X'$ is *continuous* (relative to the pairs $\mathcal{P}$ and $\mathcal{P}'$) if there exists $a^\# : Y' \to Y$, necessarily unique, such that $g'(ax, y') = g(x, a^\# y)$ for all $x \in X$, $y' \in Y'$. Denote by $\mathcal{L}(X, X')$ the Φ-module of all continuous operators from X to X', and by $\mathcal{F}(X, X')$ the submodule of those continuous operators having finite rank.

5.2. Every $a \in \mathcal{F}(X, X')$ can be expressed as a sum $a = \sum y_i^* x_i'$, where both $\{x_i'\} \subset X'$ and $\{y_i\} \subset Y$ are linearly independent, and where $y^* x'$ is defined by $y^* x'(x) = g(x, y)x'$, $x \in X$.

5.3. $\mathcal{L}(\mathcal{P}, \mathcal{P}')^J := (\mathcal{L}(X, X'), \mathcal{L}(X', X))$ is a Jordan pair with Jordan products $\{a_1, b_1, a_2\} = a_1 b_1 a_2 + a_2 b_1 a_1$, $\{b_1, a_1, b_2\} = b_1 a_1 b_2 + b_2 a_1 b_1$, for $a_i \in \mathcal{L}(X, X')$, $b_i \in \mathcal{L}(X', X)$. This (*rectangular*) Jordan pair is strongly prime with socle the simple Jordan pair $\mathcal{F}(\mathcal{P}, \mathcal{P}')^J = (\mathcal{F}(X, X'), \mathcal{F}(X', X))$, if $\mathcal{P}$ and $\mathcal{P}'$ are both nonzero. Moreover, we have the following formulae:

(i) $a(y^* x') = y^*(a x')$, for all $x' \in X'$, $y \in Y$, and $a \in \mathrm{hom}_\Delta(X', X'')$,
(ii) $(y'^* x'')b = (b^\# y')^* x''$, for all $x'' \in X''$, $y' \in Y'$, and $b \in \mathcal{L}(X, X')$,
(iii) $(y'^* x'')(y^* x') = y^* g'(x', y') x''$,

where $\mathcal{P} = (X, Y, g)$ and $\mathcal{P}' = (X', Y', g')$ are pairs of dual vector spaces, and X'' is a vector space. When $\mathcal{P} = \mathcal{P}' = (X, Y, g)$, then $\mathcal{L}(X) := \mathcal{L}(X, X)$ is a primitive associative algebra with socle $\mathcal{F}(X) := \mathcal{F}(X, X)$. It follows from (i) and (ii) that

5.4. $\mathcal{F}(X)$ is a left ideal of $\mathrm{End}_\Delta(X)$, and a (two-sided) ideal of $\mathcal{L}(X)$.

5.5. Given a pair $\mathcal{P} = (X, Y, g)$ of *skew dual* vector spaces over a division algebra with involution $(\Delta, -)$ (cf. [9, 3.11]), we define the *opposite* of $\mathcal{P}$ as the pair of skew dual vector spaces $\mathcal{P}^{\mathrm{op}} := (Y, X, g^{\mathrm{op}})$, where $g^{\mathrm{op}}(y, x) := \overline{g(x, y)}$ for $x \in X$ and $y \in Y$. For any $a \in \mathcal{L}(X, Y)$, its adjoint $a^{\#}$ also lies in $\mathcal{L}(X, Y)$, so it makes sense to consider the *Hermitian* Jordan pair

$$\mathrm{Her}\big(\mathcal{L}(\mathcal{P}), \#\big) = \big(\mathrm{Her}\big(\mathcal{L}(X, Y), \#\big), \mathrm{Her}\big(\mathcal{L}(Y, X), \#\big)\big),$$

which is strongly prime with socle the simple Jordan pair $\mathrm{Her}(\mathcal{F}(\mathcal{P}), \#)$, if $\mathcal{P}$ is nonzero. Moreover, $\mathrm{Her}(\mathcal{F}(X, Y), \#)$ is additively generated by the rank one operators y^*y, for all $y \in Y$.

5.6. Similarly, given a pair of dual vector spaces $\mathcal{P} = (X, Y, g)$ over a field F, we may consider the *skew-symmetric* Jordan pair $\mathrm{Skew}(\mathcal{L}(\mathcal{P}), \#)$, which is strongly prime with socle $\mathrm{Skew}(\mathcal{F}(\mathcal{P}), \#)$. Note also that $\mathrm{Skew}(\mathcal{F}(X, Y), \#)$ is additively generated by the *traces* $y_1 y_2^* - y_2 y_1^*$, for all $y_1, y_2 \in Y$.

5.7. We also recall the so-called Clifford Jordan pairs. Let X be a vector space over a field F, and let $q : X \to F$ be a quadratic form on X with associated bilinear form $q(x, y) := q(x + y) - q(x) - q(y)$. Then (X, X) becomes a Jordan pair for the product given by $Q_x y = q(x, y)x - q(x)y$. It will be called the *Clifford pair* defined by q. If q is nondegenerate, then the Clifford pair, $C(X, q)$, is nondegenerate and coincides with its socle (see [14, 12.8]). Moreover, it is simple if $\dim X \neq 2$.

Strongly prime Jordan pairs with nonzero socle, and in particular simple Jordan pairs with minimal inner ideals, were classified in [9, 5.1]. Under our present restriction on the scalar ring Φ, the list reads as follows:

5.8. A Jordan pair V is simple with minimal inner ideals if and only if it is isomorphic to one of the following:

(1) A simple exceptional Jordan pair, which is finite dimensional over its centroid.
(2) A Jordan pair of Clifford type $C(X, q)$, where q denotes a nondegenerate quadratic form on a vector space X over a field F, with $\dim X \neq 2$.
(3) A Jordan pair of finite rank continuous operators $\mathcal{F}(\mathcal{P}, \mathcal{P}')^J$, where $\mathcal{P}, \mathcal{P}'$ are pairs of dual vector spaces over the same division algebra.
(4) A Jordan pair of Hermitian finite rank continuous operators $\mathrm{Her}(\mathcal{F}(\mathcal{P}), \#)$, where $\mathcal{P}$ is a pair of skew dual vector spaces over a division algebra with involution, and # is the adjoint involution.
(5) A Jordan pair of skew finite rank continuous operators $\mathrm{Skew}(\mathcal{F}(\mathcal{P}), \#)$, where $\mathcal{P}$ is a pair of dual vector spaces over a field F, and # is the adjoint involution.

If V in 5.8 is finite dimensional, it is a form of a pair covered by a grid, and its TKK-algebra has been described by E. Neher in [19]. In particular, the TKK-algebra of a simple exceptional Jordan pair is a Lie algebra of type E_6 or E_7. The study of the infinite dimensional cases will make use of finite rank continuous operators on dual vector spaces and a class of Lie algebras introduced by A.A. Baranov and known as finitary Lie algebras.

5.9. Associated to a pair of dual vector spaces $\mathcal{P} = (X, Y, g)$ over a division algebra Δ, we have the following Lie algebras:

(i) The *general linear algebra* $\mathfrak{gl}(\mathcal{P}) := \mathcal{L}(X)^{(-)}$.
(ii) The *general linear algebra of finite rank operators* $\mathfrak{fgl}(\mathcal{P}) := \mathcal{F}(X)^{(-)}$.
(iii) The *special linear algebra of finite rank operators* $\mathfrak{fsl}(\mathcal{P}) := [\mathfrak{fgl}(\mathcal{P}), \mathfrak{fgl}(\mathcal{P})]$.

Recall that a Lie algebra over a field F is *finitary* if it is isomorphic to a subalgebra of $\mathfrak{fgl}(X)$, for some vector space X over F. It is clear that if Δ is finite dimensional over a field F, then both $\mathfrak{fgl}(\mathcal{P})$ and $\mathfrak{fsl}(\mathcal{P})$ are finitary Lie algebras.

If $\dim_\Delta X > 1$, then $\mathfrak{fsl}(\mathcal{P})/\mathfrak{fsl}(\mathcal{P}) \cap Z$ is a simple Lie algebra (see [11, Theorem 1.12]), where Z denotes the center of the associative algebra $\mathcal{F}(X)$. In particular, if X is infinite dimensional over Δ or if Δ is finite dimensional over its center and has zero characteristic, $\mathfrak{fsl}(\mathcal{P}) \cap Z = 0$, so $\mathfrak{fsl}(\mathcal{P})$ is itself a simple Lie algebra.

Any decomposition of $\mathcal{P}$ as a direct sum of two subpairs (such a decomposition always exists, and we can even take one of the subpairs to be finite dimensional) induces a 3-grading in each of the Lie algebras $\mathfrak{gl}(\mathcal{P})$, $\mathfrak{fgl}(\mathcal{P})$, $\mathfrak{fsl}(\mathcal{P})$ (cf. [7, 5.8(1)]). Therefore, also in the Lie algebra $\mathfrak{fsl}(\mathcal{P})/\mathfrak{fsl}(\mathcal{P}) \cap Z$, since any central ideal is graded.

5.10. Let (X, h) be a nonsingular Hermitian or skew-Hermitian inner product over a division algebra with involution $(\Delta, -)$: $h(y, x) = \epsilon\overline{h(x, y)}$, $\epsilon = \pm 1$ (in fact, every skew-Hermitian inner product is alternate over a field F with the identity as involution, or becomes a Hermitian inner product). Denote by $*$ the adjoint involution ($h(ax, y) = h(x, a^* y)$, for all $x, y \in X$) of the associative algebra $\mathcal{L}(X)$.

Assume that Δ is a field F if characteristic not 2 with the identity as involution, and that $\dim_F X > 2$.

(i) If $\epsilon = 1$, i.e., h is a symmetric bilinear form, then $\mathrm{Skew}(\mathcal{L}(X), *)$ is the *orthogonal algebra* $\mathfrak{o}(X, h)$, and $\mathrm{Skew}(\mathcal{F}(X), *) = [\mathrm{Skew}(\mathcal{F}(X), *), \mathrm{Skew}(\mathcal{F}(X), *)]$ is the *finitary orthogonal algebra* $\mathfrak{fo}(X, h)$ [2].
(ii) If $\epsilon = -1$, i.e., h is alternate, then $\mathrm{Skew}(\mathcal{L}(X), *)$ is the *symplectic algebra* $\mathfrak{sp}(X, h)$, and $\mathrm{Skew}(\mathcal{F}(X), *) = [\mathrm{Skew}(\mathcal{F}(X), *), \mathrm{Skew}(\mathcal{F}(X), *)]$ is the *finitary symplectic algebra* $\mathfrak{fsp}(X, h)$ [2].

If $\dim_F X > 4$ (possibly infinite), both $\mathfrak{fo}(X, h)$ and $\mathfrak{fsp}(X, h)$ are simple by [11, Theorem 2.15], since in both cases $\mathrm{Skew}(\mathcal{F}(X), *) \cap Z = 0$, for Z the center of $\mathcal{F}(X)$.

Let us compute the TKK-algebras of the Jordan pairs (2), (3), (4) and (5) listed in 5.8. Notice that the nondegeneracy of these TKK-algebras is equivalent to the nondegeneracy of their associated Jordan pairs by [7, 2.11(ii)].

5.11. Let $V = C(X, q)$ be a Clifford pair as in 5.8(2) with $\dim_F X > 2$, and let $H = (x_+, x_-)$ be a hyperbolic plane. Set $Y = H \oplus X$ and define a quadratic form q' on Y by

$q'(x_+, x_-) = 1$, $q'|_X = q$ and $q'(x_+) = q'(x_-) = q'(H, X) = 0$. This way (cf. [7, 5.8(3)]), $\mathfrak{fo}(Y, q')$ is a simple 3-graded Lie algebra with associated Jordan pair V. Hence $\mathrm{TKK}(V) \cong \mathfrak{fo}(Y, q')$ by 2.8.

Conversely, any finitary orthogonal algebra $\mathfrak{fo}(X, h)$ such that X contains an isotropic vector and $\dim_F X > 4$ can be realized as the TKK-algebra of a simple Jordan pair of Clifford type (if X contains a nonzero isotropic vector, then $X = H \oplus H^{\perp}$ where H is a hyperbolic plane).

5.12. Let $V = \mathcal{F}(\mathcal{P}_1, \mathcal{P}_2)^J$ be a Jordan pair of finite rank continuous operators as in 5.8(3), with $\mathcal{P}_1 = (X_1, Y_1, g_1)$ and $\mathcal{P}_2 = (X_2, Y_2, g_2)$ pairs of dual vector spaces over the central division algebra Δ over F. Define $\mathcal{P} = (X, Y, g)$, where $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$ and $g = g_1 \oplus g_2$ (i.e., $g|_{X_1 \times Y_1} = g_1$, $g|_{X_2 \times Y_2} = g_2$, and $g|_{X_1 \times Y_2} = g|_{X_2 \times Y_1} = 0$). We have by 5.9 that $\mathfrak{fsl}(\mathcal{P})/\mathfrak{fsl}(\mathcal{P}) \cap Z$, with $Z = Z(\mathcal{F}(X))$, is a simple 3-graded Lie algebra with associated Jordan pair V. Hence, by 2.8, $\mathrm{TKK}(V) \cong \mathfrak{fsl}(\mathcal{P})/\mathfrak{fsl}(\mathcal{P}) \cap Z$.

Conversely, any simple Lie algebra $L = \mathfrak{fsl}(\mathcal{P})/\mathfrak{fsl}(\mathcal{P}) \cap Z$, where $\mathcal{P}$ is a pair of dual vector spaces over Δ, can be regarded as the TKK-algebra of a Jordan pair of finite rank operators $V = \mathcal{F}(\mathcal{P}_1, \mathcal{P}_2)^J$.

5.13. Let $V = \mathrm{Her}(\mathcal{F}(\mathcal{P}), \#)$ be a Jordan pair of Hermitian finite rank continuous operators as in 5.8(4), where $\mathcal{P} = (X, Y, g)$ is a pair of skew dual vector spaces over a division F-algebra with involution $(\Delta, -)$, $F = \mathrm{Sym}(Z(\Delta), -)$. On the vector space $X \oplus Y$ we define a nonsingular skew-Hermitian bilinear form, denoted by h, as $h(x \oplus y, x' \oplus y') = g(x, y') - g^{\mathrm{op}}(y, x') = g(x, y') - \overline{g(x', y)}$. Then the corresponding associative algebra $\mathcal{F}(X \oplus Y)$ can be represented as 2×2 matrices

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \begin{pmatrix} \mathcal{F}(X) & \mathcal{F}(X, Y) \\ \mathcal{F}(Y, X) & \mathcal{F}(Y) \end{pmatrix} \quad \text{with adjoint given by} \quad a^* = \begin{pmatrix} a_{22}^* & a_{12}^* \\ a_{21}^* & a_{11}^* \end{pmatrix}.$$

Now if $x_1, x_2 \in X$, we have

$$g^{\mathrm{op}}(a_{12}x_1, x_2) = -h(a_{12}x_1, x_2) = -h\big(x_1, a_{12}^* x_2\big) = g\big(x_1, -a_{12}^* x_2\big).$$

Therefore, the adjoint involution $*$ in $\mathcal{F}(X \oplus Y)$ corresponds to $-\#$ in the *associative pair* $\mathcal{F}(\mathcal{P}, \mathcal{P}^{\mathrm{op}})$. If $\dim_{Z(\Delta)}(X \oplus Y) > 4$ and we denote $\mathrm{Skew}(\mathcal{F}(X \oplus Y), *)$ by K and the center of $\mathcal{F}(X \oplus Y)$ by Z, then $[K, K]/[K, K] \cap Z$ is simple by [11, Theorem 2.15], and hence, by 2.8, isomorphic to the TKK-algebra of the Jordan pair V.

Conversely, let (X, h) be a nonsingular skew-Hermitian inner product vector space over a division algebra with involution $(\Delta, -)$, $\dim_{Z(\Delta)}(X) > 4$. If X is a direct sum of two totally isotropic vector subspaces, then the simple Lie algebra $[K, K]/[K, K] \cap Z$, where $K := \mathrm{Skew}(\mathcal{F}(X), *)$ and Z denotes the center of $\mathcal{F}(X)$, is 3-graded and isomorphic to the TKK-algebra of a Jordan pair of Hermitian finite rank continuous operators $V = \mathrm{Her}(\mathcal{F}(\mathcal{P}), \#)$.

Two particular cases of 5.13 deserve a special comment.

(i) If X is infinite dimensional over Δ, or Δ is finite dimensional over its center and has zero characteristic, then $Z = 0$ and $[K, K]$ is itself a simple Lie algebra. Moreover, if the involution of Δ is of second kind, then K is the *finitary unitary algebra* $\mathfrak{fu}(X \oplus Y, h)$, and $[K, K]$ is the *finitary special unitary algebra* $\mathfrak{fsu}(X \oplus Y, h)$.

(ii) If Δ is a field F with the identity as involution, then $V = \mathrm{Sym}(\mathcal{F}(\mathcal{P}), \#)$, with $\mathcal{P} = (X, Y, g)$ being a pair of dual vector spaces over F. In this case, $\mathrm{TKK}(V)$ is isomorphic to the finitary symplectic algebra $\mathfrak{fsp}(X \oplus Y, h)$: $h(x \oplus y, x' \oplus y') = g(x, y') - g(x', y)$.

5.14. Assume finally that $V = \mathrm{Skew}(\mathcal{F}(\mathcal{P}), \#)$ is a Jordan pair as in 5.8(5), where $\mathcal{P} = (X, Y, g)$ is a pair of dual vector spaces over a field F. Define a nonsingular symmetric bilinear form f on $X \oplus Y$ by $f(x \oplus y, x' \oplus y') = g(x, y') + g(x', y)$. We obtain as in 5.13 that if $\dim_F(X \oplus Y) > 4$, then $\mathrm{TKK}(V)$ is isomorphic to the finitary orthogonal algebra $\mathfrak{fo}(X \oplus Y, f)$.

Conversely, any finitary orthogonal algebra $\mathfrak{fo}(X, f)$, where X is a direct sum of two totally isotropic subspaces is 3-graded and isomorphic to the TKK-algebra of a Jordan pair $V = \mathrm{Skew}(\mathcal{F}(\mathcal{P}), \#)$, for $\mathcal{P} = (X, Y, g)$ a pair of dual vector spaces over a field F.

All these results can be summarized as follows.

5.15. Theorem. *A Lie algebra L (over a field F of characteristic different from* 2 *and* 3) *is simple, nondegenerate and has nonzero Jordan socle if and only if it is isomorphic to one of the following*:

(1) *an exceptional simple Lie algebra of type E_6 or E_7,*
(2) *a special linear algebra of finite rank operators $\mathfrak{fsl}(\mathcal{P})/\mathfrak{fsl}(\mathcal{P}) \cap Z$, where $\mathcal{P} = (X, Y, g)$ is a pair of dual vector spaces over a division F-algebra Δ, and where Z stands for the center of the associative algebra $\mathcal{F}(X)$,*
(3) *a finitary orthogonal algebra $\mathfrak{fo}(X, q)$ (over an extension field of F), where $X = H \oplus H^{\perp}$ with $H = (x, y)$ a hyperbolic plane, or $X = X_1 \oplus X_2$ a direct sum of two totally isotropic subspaces, and where, in both cases,* $\dim X > 4$,
(4) *a simple Lie algebra $[K, K]/[K, K] \cap Z$, with $K = \mathrm{Skew}(\mathcal{F}(X), *)$ and $Z = Z(\mathcal{F}(X))$ relative to a nonsingular skew-Hermitian inner product vector space over a division F-algebra with involution $(\Delta, -)$, and where $\dim_{Z(\Delta)}(X) > 4$ and $X = X_1 \oplus X_2$ is a direct sum of two totally isotropic subspaces.*

6. Simple finitary Lie algebras and their algebras of derivations

In this section we provide an intrinsic characterization of the simple finitary Lie algebras of infinite dimension over an algebraically closed field of characteristic 0 which can be equipped with a 3-grading, as well as of their algebras of derivations. We begin by recalling Baranov's classification of simple finitary Lie algebras [2, Corollary 1.2].

6.1. Theorem. *Let F be an algebraically closed field of characteristic* 0. *Then any infinite dimensional finitary simple Lie algebra over F is isomorphic to one of the following*:

(i) *a finitary special linear algebra $\mathfrak{fsl}(\mathcal{P})$,*
(ii) *a finitary orthogonal algebra $\mathfrak{fo}(X, q)$,*

(iii) *a finitary symplectic algebra* $\mathfrak{fsp}(X,h)$,

with $\mathcal{P}$ *and* X *being infinite dimensional over* F.

Inspired by De La Harpe's methods for classical Banach Lie algebras of compact operators on a Hilbert space (see [5, I.8, Proposition 2]), we compute the algebra of derivations of each one of the finitary simple Lie algebras listed above.

6.2. Theorem. *Let* F, $\mathcal{P}$, (X,q) *and* (X,h) *be as in* 6.1. *Then*

(i) $\mathrm{Der}(\mathfrak{fsl}(\mathcal{P})) \cong \mathfrak{gl}(\mathcal{P})/Z$, *where* $Z = F1_X$ *is the center of* $\mathfrak{gl}(\mathcal{P})$.
(ii) $\mathrm{Der}(\mathfrak{fo}(X,q)) \cong \mathfrak{o}(X,q)$.
(iii) $\mathrm{Der}(\mathfrak{fsp}(X,h)) \cong \mathfrak{sp}(X,h)$.

Proof. (i) Let $\mathcal{P} = (X,Y,g)$ be an infinite dimensional pair of dual vector spaces over F. Consider the adjoint mapping $a \mapsto \mathrm{ad}\,a$ from $\mathfrak{gl}(\mathcal{P})$ to $\mathrm{Der}(\mathfrak{fsl}(\mathcal{P}))$, which makes sense since $\mathfrak{fsl}(\mathcal{P})$ is an ideal of $\mathfrak{gl}(\mathcal{P})$. Clearly, its kernel consists of all scalar multiples of the identity on X, so we only need to show that it is onto. Fix a pair of vectors $(x_0,y_0) \in X \times Y$ such that $g(x_0,y_0) = 1$, and consider the family $\{\mathcal{P}_\lambda\}_{\lambda\in\Lambda}$ of all subpairs $\mathcal{P}_\lambda = (X_\lambda, Y_\lambda, g_\lambda)$ of dual vector subspaces of $\mathcal{P}$ such that (i) each $\mathcal{P}_\lambda$ has finite dimension $n_\lambda \geqslant 2$, and (ii) $(x_0,y_0) \in X_\lambda \times Y_\lambda$ for each index λ.

The $\mathcal{P}_\lambda$ form a directed set with respect to inclusion and $\mathcal{P} = \varinjlim \mathcal{P}_\lambda$, the direct limit of the $\mathcal{P}_\lambda$. Moreover, $\mathcal{P} = \mathcal{P}_\lambda \oplus \mathcal{P}_\lambda^\perp$ for each index λ, where $\mathcal{P}_\lambda^\perp = (Y_\lambda^\perp, X_\lambda^\perp, g_\lambda^\perp)$ (see [9, 3.17]). Consequently, for each $\lambda \in \Lambda$ the Lie algebra $L := \mathfrak{fsl}(\mathcal{P})$ contains a subalgebra $L_\lambda \cong \mathfrak{fsl}(\mathcal{P}_\lambda) = \mathfrak{sl}_{n_\lambda}(F)$ which is simple since $n_\lambda \geqslant 2$, the L_λ form a directed set with respect to inclusion, and $L = \varinjlim L_\lambda$.

Let $\Delta \in \mathrm{Der}(L)$. Since $L = \bigcup L_\lambda$, for each $\lambda \in \Lambda$ there exists $\mu_\lambda \in \Lambda$ such that both L_λ and $\Delta(L_\lambda)$ are contained in L_{μ_λ}. Denote by Δ_λ the restriction of Δ to L_λ into L_{μ_λ}. By [12, p. 80, Theorem 9], there exists $d_\lambda \in L_{\mu_\lambda}$ such that $\Delta_\lambda(a) = [d_\lambda, a]$ for all $a \in L_\lambda$. Moreover, the restriction of d_λ to X_λ is uniquely determined up to a scalar multiple of the identity on X_λ, so *uniquely determined* by the additional condition $g(d_\lambda x_0, y_0) = 0$.

Define $d : X \to X$ as $dx = d_\lambda x$ whenever $x \in X_\lambda$. Clearly, d is well defined, linear and satisfies $\Delta(a) = [d,a]$ for all $a \in \mathfrak{fsl}(\mathcal{P})$. Thus it remains to show that d is continuous, relative to the pair $\mathcal{P}$.

We have that $[d, \mathfrak{fsl}(\mathcal{P})] \subset \mathfrak{fsl}(\mathcal{P})$. Moreover, $d\mathcal{F}(X) \subset \mathcal{F}(X)$ by 5.4. Hence $ad \in \mathcal{F}(X)$ for all $a \in \mathfrak{fsl}(\mathcal{P})$. Let us see that this fact implies the continuity of d, i.e., for each $y \in Y$ there exists a *unique* vector $y' \in Y$ such that $g(dx,y) = g(x,y')$ for all $x \in X$. Indeed, given $y \in Y$ take a nonzero vector $x' \in X$ satisfying $g(x',y) = 0$, which clearly exists because the pair $\mathcal{P}$ has infinite dimension. Then $y^*x' \in \mathfrak{fsl}(\mathcal{P})$ and so $(y^*x')d \in \mathcal{F}(X)$, that is, $(y^*x')d = y'^*x'$ for a uniquely determined vector $y' \in Y$. Applying both terms of the equality to any $x \in X$, we get $x'g(dx,y) = x'g(x,y')$, so $g(dx,y) = g(x,y')$ since $x' \neq 0$, as required.

The proofs of (ii) and (iii) are similar, even easier than that of (i). To prove (ii), put (X,q) as a direct limit of nondegenerate inner subspaces (X_λ, q_λ) of finite dimension $2n \geqslant 8$, in order to assure that the corresponding subalgebras $L_\lambda \cong \mathfrak{fo}(X_\lambda, q_\lambda)$ of $\mathfrak{fo}(X,q)$ are simple of type D_n [12, pp. 139, 141]; to prove (iii), take the inner subspaces (X_λ, h_λ) of finite dimension $2n \geqslant 6$ [12, p. 140]. In both cases, the restrictions of the d_λ to the X_λ are uniquely determined, and the linear operator $d : X \to X$, locally defined by the d_λ, satisfies $d^* = -d$. Let us finally show that the adjoint mapping $a \mapsto \mathrm{ad}\, a$ is injective in both cases.

Let $a \in \mathfrak{o}(X, q)$ be such that $[a, \mathfrak{fo}(X, q)] = 0$. Given $x \in X$, take $y \in X$ such that y does not lie in the subspace of X generated by $\{x, ax\}$. Then there exists $z \in X$ such that $q(x, z) = 0 = q(ax, z)$ and $q(y, z) = 1$. The operator $y^*x - x^*y \in \mathfrak{fo}(X, q)$ and therefore $[a, y^*x - x^*y] = y^*ax - x^*ay + (ay)^*x - (ax)^*y = 0$. Evaluating this operator equality on z, we obtain $ax = -q(ay, z)x$, and hence a is a scalar multiple of the identity, which is a contradiction since $a^* = -a$ and $\mathrm{char}\, F = 0$.

Similarly, if $a \in \mathfrak{sp}(X, h)$ satisfies $[a, \mathfrak{fsp}(X, h)] = 0$, then for any $x \in X$ we have $[a, x^*x] = x^*ax + (ax)^*x = 0$. Assume $x \neq 0$ and let $y \in X$ be such that $h(y, x) = 1$. Evaluating the operator equality on y, we get $ax = -h(y, ax)x$, so $a = 0$ as before. □

6.3. Let L be a Lie algebra over F. An element $a \in L$ will be called *reduced* (over F) if $[a, [a, L]] = Fa$.

6.4. Proposition. *Let F be an algebraically closed field of characteristic* 0, *let M be an infinite dimensional finitary simple Lie algebra over F, and let L be a Lie algebra such that $M \lhd L \leqslant K$, where K is a Lie algebra isomorphic to* $\mathrm{Der}\, M$. *Then L contains a reduced element.*

Proof. We will use Theorems 6.1 and 6.2, and treat each case separately.

(i) If $M = \mathfrak{fsl}(\mathcal{P})$, take $a = y^*x$ where $x \in X$ and $y \in Y$ are nonzero vectors satisfying $g(x, y) = 0$.
(ii) If $M = \mathfrak{fo}(X, q)$, take $a = y^*x - x^*y$ where x, y are two nonzero, isotropic and mutually orthogonal vectors.
(iii) If $M = \mathfrak{fsp}(X, h)$, take $a = x^*x$ for a nonzero vector $x \in X$.

Since M is nondegenerate, it will suffice to verify that $[a, [a, M]] \subset Fa$. We also note that in all these cases, $a^2 = 0$ and hence $[a, [a, b]] = -2aba$ for all $b \in L$. If (i), $aba = (y^*x)b(y^*x) = g(bx, y)(y^*x) \in Fa$. If (ii), $aba = (y^*x - x^*y)b(y^*x - x^*y) = y^*q(y, bx)x + x^*q(x, by)y = q(bx, y)a \in Fa$ since $q(bx, x) = 0$ for all $b \in \mathfrak{o}(X, q)$ and $x \in X$. If (iii), $aba = (x^*x)b(x^*x) = h(x, bx)(x^*x) \in Fa$. Therefore, a is reduced in any of these cases. □

The structure theorem of strongly prime Jordan pairs with nonzero socle given in [9] admits the following refinement when the involved Jordan pairs do not merely have nonzero socle, but a reduced element over an algebraically closed field of characteristic 0.

6.5. Let V be a Jordan pair over a field F. An element $x \in V^\sigma$ is *reduced* if $Q_x V^{-\sigma} = Fx$.

Clearly, reduced elements in a Jordan pair V generate minimal inner ideals of V, so, if V is nondegenerate, then any reduced element lies in the socle.

6.6. Theorem. *Let F be an algebraically closed field of characteristic* 0. *A Jordan pair is simple and contains a reduced element over F if and only if it is isomorphic to one of the following*:

(i) *a simple exceptional Jordan pair, which is finite dimensional over F,*
(ii) *a Clifford Jordan pair $\mathcal{C}(X,q)$ determined by a nondegenerate quadratic form on a vector space X of dimension $\neq 2$ over F,*
(iii) *a rectangular Jordan pair $V = \mathcal{F}(\mathcal{P},\mathcal{P}')^J$,*
(iv) *a Hermitian Jordan pair $V = \mathrm{Sym}(\mathcal{F}(\mathcal{P}),\#)$, or*
(v) *an alternating Jordan pair $V = \mathrm{Skew}(\mathcal{F}(\mathcal{P}),\#)$,*

where $\mathcal{P} = (X,Y,g)$ and $\mathcal{P}' = (X',Y',g')$ are nonzero pairs of dual vector spaces over F.

Proof. As already mentioned, the Jordan pairs listed above are simple. Moreover, each of them contains a reduced element: see [17, 1.12(iv)–(vi)] for the exceptional and Clifford cases; for the rectangular case $V = \mathcal{F}(\mathcal{P},\mathcal{P}')^J$, take y'^*x where both $x \in X$, $y' \in Y'$ are nonzero; for the Hermitian case $V = \mathrm{Sym}(\mathcal{F}(\mathcal{P}),\#)$, take y^*y for a nonzero $y \in Y$; finally, for the alternating case, take $y_1{}^*y_2{}^* - y_2{}^*y_1{}^*$ where $y_1, y_2 \in Y$ are both nonzero.

Suppose then that V is a simple Jordan pair containing a reduced element over F. This implies that V has nonzero socle and that V is *central*, i.e., its centroid is the given ring of scalars F (cf. [18, 2.10]). This allows to us to refine the list given in 5.8. If V is exceptional, then it is actually finite dimensional over F [18, 5.9(V–VI)]; if V is of Clifford type, then V is the Jordan pair associated to a nondegenerate quadratic form of an F-space [18, 5.9(IV)]; if $V = \mathcal{F}(\mathcal{P},\mathcal{P}')^J$ then it follows as in [17, 1.12(ii)] that the coordinate division F-algebra Δ is equal to F. The same is true when V is Hermitian or alternating, by [17, 1.12(v)], since F is algebraically closed and therefore has no nontrivial quadratic extensions. □

We note that, because of the grading, any reduced element $a \in L_\sigma$ ($\sigma = \pm 1$) of the Jordan pair $V = (L_1, L_{-1})$ associated to a 3-graded Lie algebra L, is actually a reduced element of L. Such an element a will be called a *Jordan reduced element* of L. Although the existence of a Jordan reduced element involves the previous one of a 3-grading, we will make no particular mention of the 3-grading when speaking of Jordan reduced element in a Lie algebra.

6.7. Theorem. *Let F be an algebraically closed field of characteristic* 0. *A Lie algebra L is strongly prime, infinite dimensional and contains a Jordan reduced element over F if and only if it is, up to isomorphism, one of the following*:

(i) $\mathfrak{fsl}(\mathcal{P}) \lhd L \leqslant \mathfrak{gl}(\mathcal{P})/Z$,
(ii) $\mathfrak{fo}(X,q) \lhd L \leqslant \mathfrak{o}(X,q)$, *or*
(iii) $\mathfrak{fsp}(X,h) \lhd L \leqslant \mathfrak{sp}(X,h)$, *with $X = X_1 \oplus X_2$ being a direct sum of two totally isotropic subspaces,*

where $\mathcal{P} = (X, Y, g)$ is an infinite dimensional pair of dual vector spaces over F and $Z = F1_X$ is the center of $\mathfrak{gl}(\mathcal{P})$, and where q and h are nondegenerate bilinear forms (*the first one symmetric, the second one alternate*) *on an infinite dimensional vector space X over F.*

Proof. By 4.3 together with 6.2, any of the Lie algebras L listed above is strongly prime, and contains a reduced element (6.4), so we only need to show that any strongly prime 3-graded Lie algebra L over F which is infinite dimensional and whose associated Jordan pair contains a reduced element, is isomorphic to one of those listed above. By [7, 5.4], $\mathrm{ad}(\mathrm{TKK}(V)) \lhd L \leqslant \mathrm{Der}(\mathrm{TKK}(V))$, where V is a simple Jordan pair containing a reduced element. Moreover, V is infinite dimensional, since otherwise $\mathrm{Der}(\mathrm{TKK}(V)) = \mathrm{ad}(\mathrm{TKK}(V))$ would be finite dimensional. The computation of $\mathrm{TKK}(V)$ is a mere refinement of that carried out in 5.11–5.14, but now using 6.6 instead of 5.8. Finally, we can compute $\mathrm{Der}(\mathrm{TKK}(V))$ by 6.2. □

Open question

Does 6.7 remain true if the Jordan reduced element is replaced by a reduced element in the Lie sense, and the restriction $X = X_1 \oplus X_2$, with X_1 and X_2 being totally isotropic, is removed in (iii)? Recall that this last restriction is not required for the existence of reduced elements in the Lie algebra, but that Jordan reduced elements are necessary in order to apply Jordan techniques. On the other hand, it seems not to be known whether or not an alternating space (X, h) of uncountable dimension over F can be expressed as a direct sum of two totally isotropic subspaces.

Acknowledgment

The authors thank the referee for his/her valuable comments and remarks on a preliminary version of this paper.

References

[1] B.N. Allison, A construction of Lie algebras from J-ternary algebras, Amer. J. Math. 98 (1976) 285–294.
[2] A.A. Baranov, Finitary simple Lie algebras, J. Algebra 219 (1999) 299–329.
[3] G. Benkart, On inner ideals and ad-nilpotent elements of Lie algebras, Trans. Amer. Math. Soc. 232 (1977) 61–81.
[4] G. Benkart, E.I. Zelmanov, Lie algebras graded by finite root systems and intersection matrix algebras, Invent. Math. 126 (1996) 1–45.
[5] P. De La Harpe, Classical Banach Lie Algebras and Banach Lie Groups in Hilbert Space, in: Lecture Notes in Math., Springer-Verlag, New York, 1972, p. 285.
[6] A. Fernández López, E. García Rus, Algebraic lattices and nonassociative structures, Proc. Amer. Math. Soc. 126 (1998) 3211–3221.
[7] A. Fernández López, E. García, M. Gómez Lozano, 3-Graded Lie algebras with Jordan finiteness conditions, Comm. Algebra, in press.

[8] A. Fernández López, E. García Rus, O. Jaa, Towards a Goldie theory for Jordan pairs, Manuscripta Math. 95 (1998) 70–99.
[9] A. Fernández López, M. Tocón, Strongly prime Jordan pairs with nonzero socle, Manuscripta Math. 111 (2003) 321–340.
[10] E. García, Inheritance of primeness by ideals in Lie algebras, Int. J. Math. Game Theory Algebra, in press.
[11] I.N. Herstein, Topics in Ring Theory, in: Chicago Lectures in Math., The University of Chicago Press, 1969.
[12] N. Jacobson, Lie Algebras, Interscience Publishers, New York, 1962.
[13] A.I. Kostrikin, On the Burnside problem, Izv. Acad. Nauk SSSR Ser. Mat. 23 (1959) 3–34, English translation in: Amer. Math. Soc. Transl. 36 (2) (1964) 63–99.
[14] O. Loos, Jordan Pairs, in: Lecture Notes in Math., Springer-Verlag, New York, 1975, p. 460.
[15] O. Loos, On the socle of a Jordan pair, Collect. Math. 40 (2) (1989) 109–125.
[16] K. McCrimmon, Strong prime inheritance in Jordan systems, Algebras Groups Geom. 1 (1984) 217–234.
[17] K. McCrimmon, Reduced elements in Jordan systems, J. Algebra (1985) 540–564.
[18] K. McCrimmon, Jordan centroids, Comm. Algebra (1999) 933–954.
[19] E. Neher, Lie algebras graded by 3-graded root systems and Jordan pairs covered by grids, Amer. J. Math. 118 (2) (1996) 439–491.
[20] R.E. Phillips, Finitary linear groups: a survey, in: B. Hartley, G.M. Seitz, A.V. Borovik, R.M. Bryant (Eds.), Finite and Locally Finite Groups, in: NATO ASI Series C, vol. 471, Kluwer Academic Publishers, Dordrecht, the Netherlands, 1995, pp. 111–146.
[21] E.I. Zelmanov, Lie algebras with an algebraic adjoint representation, Math. USSR Sb. 49 (2) (1984) 537–552.
[22] E.I. Zelmanov, Lie algebras with a finite grading, Math. USSR Sb. 52 (2) (1985) 537–552.