Polynomial Identities and Speciality of Martindale-like Covers of Jordan Algebras

José A. Anquela [1,2], Teresa Cortés [2]
anque@pinon.ccu.uniovi.es, cortes@pinon.ccu.uniovi.es
Departamento de Matemáticas, Universidad de Oviedo,
C/ Calvo Sotelo s/n, 33007 Oviedo, Spain
Fax number: ++34 985 102 886

Esther García [2]
egarciag@mat.ucm.es
Departamento de Álgebra, Universidad Complutense de Madrid,
28040 Madrid, Spain

Miguel Gómez-Lozano [3]
magomez@agt.cie.uma.es
Departamento de Álgebra, Geometría, y Topología,
Facultad de Ciencias, Universidad de Málaga,
Campus de Teatinos, 29071 Málaga, Spain

Abstract: In this paper we prove the inheritance of polynomial identities by covers of nondegenerate Jordan algebras satisfying certain ideal absorption properties. As a consequence we obtain the inheritance of speciality by Martindale-like covers, proving, in particular, that a Jordan algebra having a nondegenerate essential ideal which is special must be special.

Keywords: Jordan algebra, Martindale-like cover, polynomial identity, speciality.

MSC2000: 17C10.

Introduction

In the classification of the maximal algebras of Martindale-like quotients of strongly prime linear Jordan algebras [2], one can readily notice that speciality is inherited by the algebra of quotients. As a consequence, any algebra of Martindale-like quotients of a strongly prime special linear Jordan algebra is also special. The

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natural extension of this property to nondegenerate Jordan algebras finds the apparently unavoidable obstacle of the bad behavior of quotients with respect to subdirect products.

In this paper we present a different approach to that subject through polynomial identities, which allows us to solve the problem in the even more general setting of (quadratic) Martindale-like covers [1, Sect. 2]. On the other hand, we prove results of independent interest on the inheritance of polynomial identities. These results, involving polynomial identities, together with that of the inheritance of speciality, apply to the particular situation of a Jordan algebra having a nondegenerate essential ideal. From our results one can conclude that (admissible) polynomial identities and speciality lift from such an ideal to the whole algebra.

The paper is divided into three sections plus a preliminary one devoted to recalling the basic known facts. Section 1 deals with polynomial identities in Jordan algebras, including the fundamental fact that speciality of nondegenerate Jordan algebras can be expressed in terms of satisfying certain polynomial identities. In Section 2 we show that a polynomial vanishing strictly on a PI nondegenerate Jordan algebra vanishes strictly on any of its Martindale-like covers (indeed, on more general covers), even if the polynomial is not admissible in the sense of Jordan identities. In the proof of these results we use the abundance of central elements in nondegenerate PI Jordan algebras [3, 3.6]. The final section is devoted to studying the inheritance of speciality. The results of the previous sections settle the problem in the PI case. From that, the general solution is given by studying the evaluation of the ideal of s-identities in arbitrary nondegenerate Jordan algebras, together with some additional work on the interaction of Martindale-like covers and ideals.

0. Preliminaries

0.1 We will deal with Jordan algebras over a ring of scalars Φ. The reader is referred to [6, 7, 12] for definitions and basic properties not explicitly mentioned or proved in this section. Given a Jordan algebra J, its products will be denoted x^2, $U_x y$, for $x, y \in J$. They are quadratic in x and linear in y and have linearizations denoted $V_x y = x \circ y$, $U_{x,z} y = \{x, y, z\} = V_{x,y} z$, respectively.

0.2 Given an associative algebra R over Φ, a Jordan algebra $R^{(+)}$, called the *symmetrization* of R, can be built on the linear structure of R by taking the products $U_x y = xyx$, $x^2 = xx$, where juxtaposition denotes the associative product of R. AJordan algebra is said to be *special* if it is a subalgebra of $R^{(+)}$, for some associative algebra R, and it is said to be *exceptional* otherwise.

0.3 A Jordan algebra J is said to be *nondegenerate* if zero is the only *absolute zero divisor*, i.e., zero is the only $x \in J$ such that $U_x = 0$.

0.4 We recall that an *ideal* I of a Jordan algebra J is just a Φ-submodule of J satisfying $U_I J + I^2 + U_J I + I \circ J \subseteq I$, which implies $\{I, J, J\} \subseteq I$.

0.5 We say that a Jordan algebra J is *semiprime* if $U_I I \neq 0$, for any nonzero ideal I of J, and say that J is *prime* if $U_I L \neq 0$, for any nonzero ideals I, L of J. Every nondegenerate Jordan algebra is semiprime. A nondegenerate prime Jordan algebra is said to be *strongly prime*. Notice that, in a prime Jordan algebra J, $I \cap L \neq 0$, for any nonzero ideals I, L of J. An ideal I of J is said to be *essential* if $I \cap L \neq 0$, for any nonzero ideal L of J.

0.6 In a Jordan algebra J, the *annihilator* $\mathrm{Ann}_J(I)$ of an ideal I of J is an ideal of J which, when J is nondegenerate, is given by

$$\mathrm{Ann}_J(I) = \{x \in J \mid U_x I = 0\} = \{x \in J \mid U_I x = 0\}$$

[8, 1.3, 1.7; 13, 1.3]. An ideal I of J will be called *sturdy* if $\mathrm{Ann}_J(I) = 0$. It is easy to prove that essential ideals coincide with sturdy ideals in any semiprime Jordan algebra. In a prime Jordan algebra J, $\mathrm{Ann}_J(I) = 0$ for any nonzero ideal I of J [8, 1.6].

0.7 The *centroid* $\Gamma(J)$ of a Jordan algebra J is the set of linear maps acting "scalarly" in Jordan products [10]:

$$\Gamma(J) = \{T \in \mathrm{End}_\Phi(J) \mid TU_x = U_x T,\ TV_x = V_x T,\\ T^2(x^2) = (T(x))^2,\ T^2 U_x = U_{T(x)},\ \text{for any } x \in J\}.$$

It can be proved that $TV_{x,y} = V_{x,y}T$, $TU_{x,y} = U_{x,y}T$, for any $T \in \Gamma(J)$, and any $x, y \in J$. Clearly, $\Phi \mathrm{Id}_J \subseteq \Gamma(J)$. By [10, 2.5], when the extreme radical of J is zero (for example when J is nondegenerate), $\Gamma(J)$ is a unital associative commutative Φ-algebra and J can be seen as a Jordan algebra over $\Gamma(J)$.

0.8 [1, 0.6] If J is a nondegenerate Jordan algebra and $T \in \Gamma(J)$, then

(i) $\mathrm{Ker}\, T = \mathrm{Ker}\, T^n$ for any positive integer n,

(ii) $\mathrm{Ker}\, T$ is an ideal of J.

0.9 Following [4], the (*weak*) *center* of J is the set $C(J)$ of all elements $z \in J$ such that $U_z, V_z \in \Gamma(J)$, which is a subalgebra of J when J is nondegenerate [4, Th. 1, 2].

0.10 When J and Q are Jordan algebras such that J is a subalgebra of Q, we will say that Q is a *cover* of J. Following [1], we will consider the following *ideal absorption properties* of a cover Q of J:

the *outer* ideal absorption properties

(IA1) for any $0 \neq q \in Q$ there exists an essential ideal I of J such that $0 \neq U_I q \subseteq J$,

(IA2) for any $q \in Q$ there exists an essential ideal I of J such that $I \circ q \subseteq J$,

and the *inner* ideal absorption property

(IA3) for any $q \in Q$ there exists an essential ideal I of J such that $U_q I \subseteq J$.
A cover Q of J will be called a *Martindale-like cover* if it satisfies (IA1–3).

0.11 Remark: Notice that any cover Q of J satisfying (IA1) is tight over J, i.e., any nonzero ideal of Q hits J. As a consequence, if J is nondegenerate then Q is also nondegenerate (cf. [9, 2.9(iii)]).

1. Polynomial Identities in Jordan Algebras

1.1 Recall [6, Ch. 3; 11, IV.B.1] that the *free Jordan algebra* $\mathrm{FJ}[X]$ on a set of variables X is spanned over Φ by the so-called *Jordan algebra monomials* (on X), defined inductively as follows: the elements in X are monomials, and, given monomials a, b, c, the elements a^2, $a \circ b$, $U_a b$ and $U_{a,b}c$ are also Jordan algebra monomials. We will say that the elements in X are Jordan algebra monomials of degree 1, and, in general, we will say that a Jordan algebra monomial a has degree $n > 1$ if one of the following holds:

(i) $a = b^2$, where b is a Jordan algebra monomial of degree $n/2$ (only if n is even),

(ii) $a = b \circ c$, where b, c are Jordan algebra monomials of degree k, l, respectively, and $k + l = n$,

(iii) $a = U_b c$, where b, c are Jordan algebra monomials of degree k, l, respectively, and $2k + l = n$,

(iv) $a = U_{b,c}d$, where b, c, d are Jordan algebra monomials of degree k, l, m, respectively, and $k + l + m = n$.

Although the set of Jordan algebra monomials is not a basis of $\mathrm{FJ}[X]$ over Φ (for example, $U_{x,x}y = 2U_x y$ for any $x, y \in X$), the above grading on monomials induces a grading on $\mathrm{FJ}[X]$ and the corresponding notion of homogeneity.

The elements of $\mathrm{FJ}[X]$ will be called *Jordan polynomials.*

1.2 Let $\mathrm{FAss}[X]$ be the free associative algebra over X. The subalgebra of $\mathrm{FAss}[X]^{(+)}$ generated by X is called the *free special Jordan algebra over* X, and it is denoted by $\mathrm{FSJ}[X]$.

1.3 The algebra $\mathrm{FJ}[X]$ is free over the set X in the sense that, for any map $\varphi : X \longrightarrow J$ from X to a Jordan algebra J, there exists a unique Jordan algebra homomorphism $\tilde{\varphi} : \mathrm{FJ}[X] \longrightarrow J$ extending φ. Given $p(X_1, \ldots, X_n) \in \mathrm{FJ}[X]$, $a_1, \ldots, a_n \in J$, the *evaluation* $p(a_1, \ldots, a_n)$ of p is just the image $\tilde{\varphi}(p)$ when $\varphi(X_1) = a_1, \ldots, \varphi(X_n) = a_n$. In particular we have the *canonical specialization* $\psi : \mathrm{FJ}[X] \longrightarrow \mathrm{FSJ}[X]$, which is the unique Jordan algebra homomorphism from $\mathrm{FJ}[X]$ to $\mathrm{FSJ}[X]$ fixing the elements of X.

1.4 A Jordan polynomial p will be said *admissible* if its image by the canonical specialization is a monic associative polynomial (i.e., it has a leading monomial with coefficient 1).

1.5 We will say that a polynomial $p \in \mathrm{FJ}[X]$ *vanishes on* the Jordan algebra J if $p(a_1, \ldots, a_n) = 0$, for any $a_1, \ldots, a_n \in J$. We will say that $p \in \mathrm{FJ}[X]$ *vanishes strictly on* J if p vanishes on any scalar extension $J \otimes_\Phi \Omega$, equivalently, if p vanishes on $J \otimes_\Phi \Phi[T]$, where $\Phi[T]$ is the unital commutative associative algebra of polynomials on an infinite set T of variables, equivalently all partial linearizations (including the homogeneous components) of p vanish on J.

1.6 A Jordan algebra J will be called PI if there is an admissible Jordan polynomial p vanishing strictly on J.

1.7 A Jordan polynomial $p \in \mathrm{FJ}[X]$ is called an *s-identity* if it vanishes in every special Jordan algebra, equivalently, if it lies in the kernel of the canonical specialization ψ of (1.3). Hence, the set s-Id$[X]$ of all s-identities is an ideal of $\mathrm{FJ}[X]$. The fact that speciality is preserved by free scalar extensions implies that s-identities vanish strictly on any special Jordan algebra, so that s-Id$[X]$ is a homogeneous ideal, invariant under partial linearizations (cf. (1.5)). Given an arbitrary Jordan algebra J, s-Id(J) will denote the set of all evaluations of s-identities on J, which is an ideal of J. If s-Id$(J) = 0$, J will be said to be *i-special* (any polynomial vanishing on all special Jordan algebras vanishes on J).

1.8 Obviously, special Jordan algebras are always i-special, but the converse is false (see [11, IV.A.3.3]). However, if we restrict to nondegenerate algebras i-speciality and speciality turn out to be equivalent [12, 15.4].

1.9 Strongly prime exceptional Jordan algebras are always PI. Moreover, the admissible Jordan polynomial

$$p(x, y, z) := \mathcal{S}_4(V_{x^3,y}, V_{x^2,y}, V_{x,y}, V_{1,y})(z),$$

for the alternating standard identity

$$\mathcal{S}_4(x_1, x_2, x_3, x_4) := \sum_\pi (-1)^\pi x_{\pi(1)} x_{\pi(2)} x_{\pi(3)} x_{\pi(4)}$$

summed over all permutations on 4 letters, vanishes on any Albert form (cf. [11, p. 112]), hence, it vanishes strictly on any strongly prime exceptional Jordan algebra (see [12, 15.2]).

2. Extending Polynomial Identities to Covers

The goal of this section consists of lifting identities from PI Jordan algebras to a certain type of covers. Although we are mainly concerned with Martindale-like covers, we state our results in a more general context.

2.1 Lemma. *Let J be a nondegenerate PI Jordan algebra, and Q be a cover of J satisfying (IA1) and such that $C(J) \subseteq C(Q)$. Then, for any $0 \neq q \in Q$, and any essential ideal I of J, there exists $z \in I \cap C(J)$ such that $U_z q \neq 0$.*

Proof: Let $X = I \cap C(J)$, and take

$$A_J(X) = \{y \in J \mid U_x y = 0, \text{ for any } x \in X\} = \bigcap_{x \in X} \operatorname{Ker}_J U_x,$$

which is an ideal of J by (0.8). If $A_J(X) \neq 0$, then $I \cap A_J(X) \neq 0$ by essentialness of I, so $A_J(X) \cap X = A_J(X) \cap I \cap C(J) \neq 0$ by [3, 3.6]. Then we can take $0 \neq x \in A_J(X) \cap X$, which satisfies $U_x x = 0$, hence $U_x^3 = U_{U_x x}$ (by [12, 0.3]) $= 0$. But this is a contradiction by [10, 2.8] since, due to nondegeneracy, $0 \neq U_x \in \Gamma(J)$. Therefore $A_Q(X) \cap J = A_J(X) = 0$. But $A_Q(X) = \bigcap_{x \in X} \operatorname{Ker}_Q U_x$ is an ideal of Q by (0.8) since Q is nondegenerate by (0.11), and $C(J) \subseteq C(Q)$. By tightness of Q over J (0.11), $A_Q(X) = 0$, which implies $q \notin A_Q(X)$, i.e., there exists $z \in I \cap C(J)$ such that $U_z q \neq 0$.

2.2 Theorem. *Let J be a nondegenerate PI Jordan algebra, and let $p \in \mathrm{FJ}[X]$ be a homogeneous Jordan polynomial vanishing on J. Then p vanishes on any cover Q of J satisfying (IA1) and such that $C(J) \subseteq C(Q)$.*

Proof: Put $p = p(X_1, \ldots, X_n)$ and assume that p has degree m. Given $q_1, \ldots, q_n \in Q$, we only have to prove that $p(q_1, \ldots, q_n) = 0$. By the hypothesis (IA1), there exist essential ideals $I_1, \ldots, I_n$ of J such that $U_{I_i} q_i \subseteq J$, for $i = 1, \ldots, n$. Now $I = \cap_{i=1}^n I_i$ is an essential ideal of J satisfying $U_I q_i \subseteq J$ for any $i = 1, \ldots, n$. If $p(q_1, \ldots, q_n) \neq 0$, then there exists $z \in I \cap C(J) \subseteq I \cap C(Q)$ such that $U_z p(q_1, \ldots, q_n) \neq 0$, by (2.1). Hence $U_z^m p(q_1, \ldots, q_n) \neq 0$ by (0.8), since Q is nondegenerate by (0.11), but $U_z^m p(q_1, \ldots, q_n) = p(U_z q_1, \ldots, U_z q_n) \in p(J, \ldots, J) = 0$, which is a contradiction.

2.3 Corollary. *Let J be a nondegenerate PI Jordan algebra, and let $p \in \mathrm{FJ}[X]$ be a Jordan polynomial vanishing strictly on J. Then p vanishes strictly on any cover Q of J satisfying (IA1) and such that $C(J) \subseteq C(Q)$.*

Proof: Notice that all linearizations of p are homogeneous polynomials vanishing on J (see (1.5)), which then vanish on Q by (2.2), i.e., p vanishes strictly on Q.

By restricting (2.3) to admissible polynomials, we get

2.4 Corollary. *Let J be a nondegenerate Jordan algebra and Q be a cover of J satisfying (IA1) and such that $C(J) \subseteq C(Q)$. If J is PI, then Q is PI.*

When not only (IA1) but also (IA2) and (IA3) are satisfied, i.e., when Q is a Martindale-like cover of J, $C(J) \subseteq C(Q)$ [1, 3.16], so that we obtain the following result.

2.5 Corollary. *A Martindale-like cover Q of a nondegenerate PI Jordan algebra J is PI. In this situation,*

(i) *if* $p \in \mathrm{FJ}[X]$ *is a homogeneous Jordan polynomial vanishing on* J*, then* p *vanishes on* Q*, and*

(ii) *if* J *is PI and* $p \in \mathrm{FJ}[X]$ *is a Jordan polynomial vanishing strictly on* J*, then* p *vanishes strictly on* Q*.*

2.6 Remark: The above result (2.5) applies to the following particular cases of Martindale-like covers:

(i) J is a linear Jordan algebra and Q is a Jordan algebra of Martindale-like quotients with respect to a power filter of sturdy ideals of J,

(ii) J is an essential ideal of Q.

Indeed, in case (i), we only have to recall [1, 2.6], while, in case (ii), (IA1) follows from (0.6) whereas (IA2) and (IA3) are obvious.

3. Speciality of Martindale-like Covers

We will show that under analogous conditions as in the previous section, speciality is inherited by covers. We start with the PI case.

3.1 Proposition. *Let* J *be a nondegenerate special PI Jordan algebra. Then any cover* Q *of* J *satisfying (IA1) and such that* $C(J) \subseteq C(Q)$ *is special.*

Proof: Since J is special, it satisfies every s-identity. By (2.2), Q also satisfies every homogeneous s-identity, hence every s-identity (because the ideal of all s-identities is homogeneous), i.e., Q is i-special. Thus Q is special by (1.8), since it is nondegenerate by (0.11).

3.2 Proposition. *If* J *is a strongly prime exceptional Jordan algebra, then any nonzero ideal of* J *is a strongly prime exceptional Jordan algebra.*

Proof: Let I be a nonzero ideal of a strongly prime exceptional Jordan algebra J. It is immediate that J is a cover of I. Moreover, it satisfies (IA1) by (0.6), and $C(I) \subseteq C(J)$ by [3, 3.2]. On the other hand, I is PI since J is PI by (1.9), and I is strongly prime by [8, 2.5]. Finally, if I were special, J would be special by (3.1), which would be a contradiction.

3.3 Lemma. *Let* Q *be a nondegenerate Jordan algebra. Then the ideal* s-Id(Q) *of* Q *is either zero (hence* Q *is special) or an exceptional PI-algebra.*

Proof: If s-Id$(Q) = 0$, then Q is i-special, hence special by (1.8). Otherwise, s-Id(Q) is a nonzero ideal of Q. Since Q is nondegenerate, it is a subdirect product of strongly prime Jordan algebras [14, Cor. 4]. So there exists a family $\{I_\alpha\}_{\alpha\in A}$ of ideals of Q such that $Q \subset \prod_{\alpha\in A} Q/I_\alpha$ and Q/I_α is strongly prime for any $\alpha \in A$. Moreover,

$$\text{s-Id}(Q) \subseteq \prod_{\alpha\in A} \text{s-Id}(Q/I_\alpha) \cong \prod_{\alpha\in B} \text{s-Id}(Q/I_\alpha) \subseteq \prod_{\alpha\in B} Q/I_\alpha, \tag{1}$$

where $B = \{\alpha \in A \mid \text{s-Id}(Q/I_\alpha) \neq 0\}$. Notice that, for any $\alpha \in B$, Q/I_α is exceptional by (1.8) since it is not i-special, hence the admissible Jordan polynomial p given in (1.9) vanishes strictly on Q/I_α. Thus p vanishes strictly on $\Pi_{\alpha\in B} Q/I_\alpha$, therefore on s-Id($Q$), which means s-Id($Q$) is PI.

Furthermore, s-Id(Q) is exceptional. Indeed $B \neq \emptyset$ by (1) since s-Id$(Q) \neq 0$. Now, for any $\alpha \in B$, let $f_\alpha : Q \to Q/I_\alpha$ denote the canonical projection of Q onto Q/I_α (so that $0 \neq \text{s-Id}(Q/I_\alpha) = f_\alpha(\text{s-Id}(Q))$). Since Q/I_α is strongly prime exceptional, we have that the nonzero ideal s-Id(Q/I_α) is a strongly prime exceptional Jordan algebra by (3.2), hence it is not i-special by (1.8), i.e., $\text{s-Id}(\text{s-Id}(Q/I_\alpha)) \neq 0$, and

$$f_\alpha(\text{s-Id}(\text{s-Id}(Q))) = \text{s-Id}(\text{s-Id}((Q/I_\alpha))) \neq 0,$$

which implies $\text{s-Id}(\text{s-Id}(Q)) \neq 0$, i.e., s-Id($Q$) is not i-special, hence it is exceptional.

3.4 Lemma. *Let Q be a nondegenerate Jordan algebra, $q \in Q$, and I be a Φ-submodule of Q such that $U_I q = 0$. Then $U_I U_q I = 0$.*

Proof: If $U_I q = 0$, then $U_{I[t]} q = 0$ in the algebra $Q[t]$ of polynomials over Q which is also nondegenerate. For any $h \in I[t]$, let $a := U_h U_q h \in Q[t]$. By [12, 0.3],

$$U_a Q[t] = U_h U_q U_h U_q U_h Q[t] = U_{U_h q} U_q U_h Q[t] = 0,$$

hence $a = 0$ by nondegeneracy. For $x, y \in I$, the coefficient of t in $U_{x+ty} U_q (x + ty)$ is $U_x U_q y + U_{x,y} U_q x$, which is then zero. But, on the other hand, $U_{x,y} U_q x = \{U_x q, q, y\}$ (by [7, JP2]) $= 0$, hence we obtain $U_x U_q y = 0$, for any $x, y \in I$.

3.5 Lemma. *Let J be a nondegenerate Jordan algebra and let Q be a cover of J satisfying (IA1). If $\tilde{K}$ is an ideal of Q, then $\tilde{K}$ is a cover of $K := \tilde{K} \cap J$ satisfying (IA1). Moreover,*

(i) *if Q satisfies (IA2) and/or (IA3) over J, then $\tilde{K}$ satisfies (IA2) and/or (IA3) over K, respectively,*

(ii) *if $C(J) \subseteq C(Q)$, then $C(K) \subseteq C(\tilde{K})$.*

Proof: We claim that

(1) if I is an essential ideal of J, then $I \cap K$ is an essential ideal of K.

Indeed, K is also nondegenerate since it is an ideal of J, hence (0.6) applies, and we just need to show that $\text{Ann}_K(I \cap K) = 0$: If $k \in \text{Ann}_K(I \cap K)$, then, for any $y \in I$,

$$U_{U_k y} I = U_k U_y U_k I \subseteq U_k (I \cap K) = 0,$$

which shows $U_k y \in \text{Ann}_J(I) = 0$ since I is sturdy in J by (0.6); therefore $U_k I = 0$, which means $k \in \text{Ann}_J(I) = 0$. This proves (1).

For any $0 \neq q \in \tilde{K}$, there exists an essential ideal I of J such that $0 \neq U_I q \subseteq J$, but also $U_I q \subseteq U_Q \tilde{K} \subseteq \tilde{K}$, hence

$$0 \neq U_I q \subseteq J \cap \tilde{K} = K. \tag{2}$$

In particular $U_{I\cap K}q \subseteq U_I q \subseteq K$. Moreover,

(3) $U_q(L \cap K) \neq 0$, for any essential ideal L of J.

Indeed, $I \cap L$ is an essential ideal of J, hence, by [1, 2.4], $U_{I\cap L}q \neq 0$, and we can find $y \in I \cap L$ such that $0 \neq U_y q \in U_I q \subseteq K$ by (2). Since $\mathrm{Ann}_K(L \cap K) = 0$ by (1), $0 \neq U_{U_y q}(L \cap K) = U_y U_q U_y (L \cap K) \subseteq U_y U_q (L \cap K)$, which implies (3).

Finally $U_{I\cap K}q \neq 0$: otherwise,

$$U_{I\cap K}U_q(I \cap K) = 0 \tag{4}$$

by (3.4). On the other hand, (3) implies that there exists $x \in I \cap K$ such that $U_q x \neq 0$, hence (3) applied to $0 \neq U_q x \in \tilde{K}$ implies that $0 \neq U_{U_q x}(I \cap K) = U_q U_x U_q (I \cap K) \subseteq U_q U_{I\cap K} U_q (I \cap K)$, which would contradict (4). Thus, $\tilde{K}$ satisfies (IA1) over K.

(i) If Q satisfies (IA2) (resp., (IA3)) over J, then, for any $q \in \tilde{K}$, there exists an essential ideal I of J such that $q \circ I \subseteq J$ (resp., $U_q I \subseteq J$), but also $q \circ I \subseteq \tilde{K}$ (resp., $U_q I \subseteq \tilde{K}$) as in (2) above, hence

$$q \circ (I \cap K) \subseteq q \circ I \subseteq J \cap \tilde{K} = K \qquad (\text{resp., } U_q(I \cap K) \subseteq U_q I \subseteq J \cap \tilde{K} = K).$$

(ii) Notice that $C(K) \subseteq C(J)$ (by [3, 3.2]) $\subseteq C(Q)$, which obviously implies $C(K) \subseteq C(\tilde{K})$.

3.6 Theorem. *Let J be a nondegenerate Jordan algebra and Q be a cover of J satisfying (IA1) and such that $C(J) \subseteq C(Q)$. If J is special, then Q is special.*

Proof: If s-Id$(Q) = 0$, then Q is i-special, hence special by (1.8), since it is nondegenerate by (0.11). Otherwise s-Id(Q) is an exceptional PI algebra by (3.3). On the other hand, s-Id(Q) is a cover of s-Id$(Q) \cap J$ satisfying (IA1) and $C($s-Id$(Q) \cap J) \subseteq C($s-Id$(Q))$ by (3.5), and s-Id$(Q) \cap J$ is special and PI, so that (3.1) yields s-Id(Q) is special, which is a contradiction.

As in (2.5) and (2.6), we immediately obtain:

3.7 Corollary. *A Martindale-like cover Q of a nondegenerate special Jordan algebra J is special. In particular, this holds when J is nondegenerate special and either*

(i) *J is a linear Jordan algebra and Q is a Jordan algebra of Martindale-like quotients with respect to a power filter of sturdy ideals of J, or*

(ii) *J is an essential ideal of Q.*

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