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THE MAXIMAL LEFT QUOTIENT RINGS OF ALTERNATIVE RINGS[#]

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We study the notion of a (general) left quotient ring of an alternative ring and show the existence of a maximal left quotient ring for every alternative ring that is a left quotient ring of itself.

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INTRODUCTION

The theory of rings of quotients has its origins between 1930 and 1940, in the works of Ore and Osano on the construction of the total ring of fractions (Lam, 1998, Ch. 4). In that decade Ore proved that a necessary and sufficient condition for a ring R to have a (left) classical ring of quotients is that for any regular element $a \in R$, and any $b \in R$ there exist a regular $c \in R$ and $d \in R$ such that $cb = da$ (left Ore condition). At the end of the 1950s, Goldie et al. characterized the (associative) rings that are classical left orders in semiprime and left artinian rings; this is known as Goldie's theorem (Lam, 1998, Ch. 4).

Utumi (1956) introduced the notion of general left quotient ring and proved that the rings without right zero divisors are precisely those that have maximal left quotient rings.

Following Goldie's idea of characterizing certain types of rings via a suitable envelope, Johnson characterized those rings whose maximal left quotient rings are von Neumann regular; see Lam (1998), 13.36. Gabriel specialized it further by giving characterizations for those rings whose maximal left quotient rings are semisimple, i.e., isomorphic to a finite direct product of rings of the form $End_{\Delta}(V)$ for suitable finite dimensional left vector spaces V over division rings Δ (Lam, 1998, 13.40).

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Fountain and Gould (1990), based on ideas from semigroup theory, introduced a notion of order in a ring, which need not have an identity, and in the following year gave a Goldie-like characterization of two-sided orders in semiprime rings with descending chain condition on principal one-sided ideals (equivalently, coinciding with their socles). Ánh and Márki (1991) extended this result to one-sided orders, and more recently (1994) the same authors developed a general theory of Fountain–Gould left quotient rings (we point out that the maximal left quotient ring plays a fundamental role in this work).

It is natural to ask whether similar notions (and results) can be obtained for alternative rings.

Beidar and Mikhalev (1989), interested in the structure of nondegenerate and purely alternative algebras, introduced what they referred to as the almost classical localization of an algebra and described, using the theory of orthogonally complete algebraic systems, the structure of this type of algebras; see their Section 2.12. This construction, which only works when the center associative coincides with the center (which is a property of nondegenerate and purely alternative rings), coincides with our maximal left quotient ring in these particular conditions, see (2.15)(5).

The question of Goldie's theorems for alternative algebras was posed by Essannouni and Kaidi (1994) for Noetherian alternative rings. In 1994 the same authors established a Goldie-like theorem for alternative rings without elements of order three in its associator ideal. Gómez Lozano and Siles Molina (Preprint) introduced Fountain–Gould left orders in alternative rings and gave a Goldie-like characterization of alternative rings that are Fountain–Gould left orders in nondegenerate alternative rings that coincide with their socle (this result generalizes the classical Goldie theorems for alternative rings without additional conditions). In this work the authors introduced, as a tool, the notion of general left quotient rings and related properties of a ring to any of its general rings of quotients.

In this paper, we construct the maximal left quotient ring of any alternative ring that is a left quotient ring of itself and prove that this is an alternative ring when $D(R)$ is semiprime or 2-torsion free.

In a preliminary section we introduce basic definitions, properties, and notations. In Corollary 1.9 we show nondegenerate alternative rings as semiprime rings in which every nonzero (left or right) ideal has nonzero associative center.

We construct the maximal left quotient ring of any alternative ring that is a left quotient ring of itself. We follow Utumi (1956) where a maximal left quotient ring is constructed as a direct limit of partially defined homomorphisms from left ideals of R to R. From point (2.2) to (2.7) we give the filter of "dense" left ideals. In (2.8) we define the type of partially defined homomorphism we need in this work. In (2.9) and (2.10) we construct a unital ring that will be the maximal left quotient ring. Theorem (2.11) proves that the associator of this ring is always a skew-symmetric map of its arguments and under some additional conditions on the associator ideal of R, it will be an alternative ring. Now, from (2.12) to (2.14), we show that it is the maximal left quotient ring (therefore the maximal left quotient ring of an alternative ring is always unital). We finish giving explicitly the maximal left quotient ring of particular alternative rings; see (2.15).

In a forthcoming paper we will give a Johnson and a Gabriel-like theorem, i.e., we will characterize those alternative rings that are left nonsingular (via their maximal left quotient rings) or whose maximal left quotient ring is nondegenerate and artinian, respectively.

1. PRELIMINARIES

1.1. The following three basic central subsets can be considered in a ring R: the *associative center* $N(R)$, the *commutative center* $K(R)$, and the *center* $Z(R)$, defined by

$$N(R) = \{x \in R \mid (x, R, R) = (R, x, R) = (R, R, x) = 0\},$$
$$K(R) = \{x \in R \mid [x, R] = 0\},$$
$$Z(R) = N(R) \cap K(R),$$

where $[x, y] = xy - yx$ denotes the *commutator* of two elements $x, y \in R$ and $(x, y, z) = (xy)z - x(yz)$ is the *associator* of three elements x, y, z of R.

1.2. The defining axioms for an *alternative* ring R are the *left* and the *right alternative laws*:

$$(x, x, y) = 0 = (y, x, x)$$

for every $x, y \in R$. As a consequence, we have the fact that the associator is an alternating function of its arguments. The standard reference for alternative rings is Zhevlakov et al. (1982).

1.3. The associative nucleus and the associator ideal of an alternative ring will be very important notions in this theory. Given a ring R, every ideal contained in the associative center of R will be called a *nuclear ideal*. The largest nuclear ideal of R will be the *associative nucleus*, denoted by $U(R)$. By $D(R)$ we will mean the *associator ideal*, i.e., the ideal of R generated by the set (R, R, R) of all associators.

1.4. From now on, for a ring R, R^1 will denote its unitization, that is, R if the ring is unital, or $R \times \mathbb{Z}$ with product $(x, m)(y, n) := (xy + nx + my, mn)$ if R has no unity.

1.5. A ring without nonzero *trivial ideals* (i.e., ideals with zero multiplication) is called *semiprime*. By Zhevlakov et al. (1982), Exercise 9.1.8, every semiprime alternative ring does not contain nonzero trivial left (right) ideals. An element a of an alternative ring R is called an *absolute zero divisor* if $aRa = \{0\}$. The ring R is called *nondegenerate* (or *strongly semiprime*) if R does not contain nonzero absolute zero divisors.

1.6. We recall that for every nonempty subset X of an alternative ring R, the *left annihilator* of X is defined as the set $lan(X) = \{a \in R \mid ax = 0 \text{ for all } x \in X\}$, written $lan_R(X)$ when it is necessary to emphasize the dependence on R. Similarly, the *right annihilator* of X, $ran(X) = ran_R(X)$, is defined by $ran(X) = \{a \in R \mid xa = 0 \text{ for all } x \in X\}$. We also write $ann(X) = ann_R(X) := lan(X) \cap ran(X)$ to denote the *annihilator* of X. In general the left (right) annihilator of a subset X of an alternative ring R does not have to be a left (right) ideal, but it is true if X is a right (left) ideal of R or if $X \subset N(R)$.

Let R be an alternative ring and consider $X \subset R$. We will mean by $_R(X]$ (respectively by $[X)_R$) the left (right) ideal of R generated by X.

The next proposition is a generalization of Zhevlakov et al. (1982), Theorem 9.1.1.

1.7. Proposition. *Let I be a semiprime ideal of an alternative ring R. Then $N(I) = I \cap N(R)$.*

Proof. Let $ann_R(I)$ be the annihilator of I in R, which is an ideal by (1.6). Since I is a semiprime ideal of R, $I \cap ann_R(I) = 0$. We denote by $\overline{R} := R/ann_R(I)$ and by $\pi : R \to R/ann_R(I)$ the canonical projection from R on $R/ann_R(I)$.

Now, I can be seen as an ideal of $\overline{R}$. Let us show that it is essential in $\overline{R}$: let $\overline{K}$ be an ideal of $\overline{R}$ such that $\overline{K} \cap I = 0$, then $\overline{K}I = \overline{0} = I\overline{K}$, which implies that $\pi^{-1}(\overline{K})I$ and $I\pi^{-1}(\overline{K})$ are contained in $ann_R(I) \cap I = 0$, i.e., $\pi^{-1}(\overline{K}) \subset ann_R(I)$, so $\overline{K} = \overline{0}$. Now $\overline{R}$ is semiprime because it has an essential ideal that is semiprime, and therefore, by Zhevlakov et al. (1982), Theorem 9.1.1, $N(I) \subset N(\overline{R})$. Finally, if $\lambda \in N(I)$ and $a, b \in R$ we have $(\lambda, a, b) \in ann_R(I) \cap I = 0$, since $(\overline{\lambda}, \overline{a}, \overline{b}) = \overline{0}$, i.e., $\lambda \in N(R)$. Now it is trivial that $N(I) = I \cap N(R)$. □

1.8. Proposition. *Let R be an alternative ring such that $D(R)$ is semiprime. Then*

(i) *The linear span of all elements $\{x \in D(R) \mid xRx = 0\}$ is an ideal of R, denoted by $\mathcal{R}$, such that $N(\mathcal{R}) = 0$.*

(ii) *Moreover, if R is semiprime, the linear span of all absolute zero divisors of R, denoted by $\mathcal{R}'$, is an ideal of R such that $\mathcal{R}' \subset ann_R(U)$ and $N(\mathcal{R}') = 0$.*

Proof. (i). By definition $\mathcal{R}$ is a subgroup of R. Now, if $x, z \in R$, $y \in D(R)$, and $yRy = 0$, by McCrimmon (1971), Formula (8), we have

$$\begin{aligned} (xy)z(xy) &= x[y(zx)y] = 0 \\ (yx)z(yx) &= [y(xz)y]x = 0, \end{aligned} \tag{1}$$

which implies that $\mathcal{R}$ is an ideal of R.

By Zhevlakov et al. (1982), Theorem 9.1.4, $\mathcal{R}$ and $U(\mathcal{R})$ are semiprime rings. Let us show that $\mathcal{R}$ is a purely alternative ring: Given $x \in U(\mathcal{R})$ and an element y of $D(R)$ such that $yRy = 0$ we have, by (1), that xy is an absolute zero divisor of the semiprime associative ring $U(\mathcal{R})$, so $xy = 0$. Hence $U(\mathcal{R})\mathcal{R} = 0$, since $\mathcal{R}$ is generated by all absolute zero divisors of R, which implies that $U(\mathcal{R}) = 0$, since $\mathcal{R}$ is semiprime.

Now, by Zhevlakov et al. (1982), Theorem 8.3.11, $N(\mathcal{R}) = Z(\mathcal{R})$. Let us show that $N(\mathcal{R}) = 0$. Otherwise, let n be the smallest natural number such that a sum of n absolute zero divisors of R is nonzero and belongs to $N(\mathcal{R})$, and let $0 \neq \lambda = \sum_{i=1}^{n} x_i \in N(\mathcal{R}) = Z(\mathcal{R})$, with $x_iRx_i = 0$, $i = 1, 2, \ldots, n$. Then $x_1 + \cdots + x_{n-1} = \lambda - x_n$, and if we multiply it by $y = \lambda^2 + \lambda x_n + x_n^2$ we obtain $x_1y + \cdots + x_{n-1}y = (\lambda - x_n)y = \lambda^3$, a contradiction with the choice of n, because $\lambda^3 \neq 0$, since $\mathcal{R}$ is semiprime and $\lambda \in Z(\mathcal{R})$, and for each i, x_iy is an absolute zero divisor of R.

(ii). This follows as in (i). □

Every nontrivial ideal (left ideal or right ideal) of a nondegenerate alternative ring has a nontrivial associative center, see Gómez Lozano and Siles Molina (Preprint), (1.2) (v). The previous proposition has showed that this property, in fact, characterizes the nondegenerancy of a semiprime alternative ring.

1.9. Corollary. *A semiprime alternative ring R is nondegenerate if and only if every nonzero* (*left or right*) *ideal of R has a nonzero associative center.*

The next lemma has the same proof as Gómez Lozano and Siles Molina (Preprint), (1.3) (iii), since in this item the semiprimeness of the alternative ring is not necessary.

1.10. Lemma. *Let R be an alternative ring*; *then $U(R) + D(R)$ is an essential* (*left and right*) *ideal of R.*

1.11. The notion of left quotient ring in the setting of alternative rings was introduced by Gómez Lozano and Siles Molina (Preprint), where the relationship among classical, Fountain–Gould, and this type of ring of quotients was established.

Let R be a subring of an alternative ring Q. We recall that Q is a *left quotient ring* of R, denoted by $R \leq_q Q$, if

(1) $N(R) \subset N(Q)$, and
(2) For every $p, q \in Q$, with $p \neq 0$, there exists an $r \in N(R)$ such that $rp \neq 0$ and $rq \in R$.

Note that R and Q can be seen as left $N(R)$-modules, and that condition (2) of the previous definition means that R is a dense left $N(R)$-submodule of Q, see Lam (1998), (8.2).

1.12. Proposition. *Let R be a subring of an alternative ring Q.*

(i) *If $R \leq_q Q$ and we take $q_1, q_2, \ldots, q_n \in Q$, with $q_1 \neq 0$, then there exists an $r \in N(R)$ such that $rq_1 \neq 0$ and $rq_i \in R$ for $i = 1, 2, \ldots, n$.*
(ii) *Let $R \subset S \subset Q$ be three alternative rings. Then $R \leq_q Q$ if and only if $R \leq_q S$ and $S \leq_q Q$.*

Proof. (i). The proof follows as in the associative case, Utumi (1956), (1.4).

(ii). Suppose that $R \leq_q Q$. Then $N(R) \subset N(Q) \cap S \subset N(S)$, and given $n \in N(S)$, with $(n, p, q) \neq 0$ for $p, q \in Q$, there exist $n_1, n_2 \in N(R)$ such that $n_1 p \in R \subset S$ and $n_2 q \in R \subset S$, with $0 \neq n_2 n_1 (n, p, q) = (n, n_1 p, n_2 q) = 0$, a contradiction. So $N(S) \subset N(Q)$. Now it is straightforward that $R \leq_q S$ and $S \leq_q Q$.

Conversely, suppose that $R \leq_q S$ and $S \leq_q Q$. Then $N(R) \subset N(S) \subset N(Q)$. Moreover, given $p, q \in Q$, with $p \neq 0$, by (i), there exists an $s \in N(S)$ such that $sp, sq \in S$, with $sp \neq 0$, and there exists $n \in N(R)$ such that $nsp \neq 0$, $nsq \in R$, and $ns \in R$. So, $ns \in R \cap N(S) \subset N(R)$, and verifies $nsp \neq 0$ and $nsq \in R$. □

2. CONSTRUCTION OF THE MAXIMAL LEFT QUOTIENT RING OF AN ALTERNATIVE RING

The notion of a maximal left quotient ring, in the setting of associative rings, was studied by Utumi (1956); he proved that rings that are left quotient rings of themselves (equivalently, rings without total right zero divisors) have a unique maximal left quotient ring.

Following the categorical definition of Utumi (1956), we define the notion of maximal left quotient ring of an alternative ring.

2.1. Definition. We will say that an alternative ring R has a *maximal left quotient ring* if there exists a ring Q such that

(i) Q is a left quotient ring of R and
(ii) If S is a left quotient ring of R, there exists a unique monomorphism of rings $f : S \to Q$ with $f(r) = r$ for every $r \in R$.

Clearly, this definition implies that the maximal left quotient ring of a ring R, if it exists, is unique up to isomorphisms. We will denote it by $Q^l_{max}(R)$. Moreover, if R has a maximal left quotient ring, by (1.2(ii)) it is a left quotient ring of itself. Now we are going to prove the reciprocal of this fact.

2.2. Definition. We will say that a left ideal I of an alternative ring R is *dense* if for every $p, q \in R$, with $p \neq 0$, there exists an $a \in N(R)$ such that $ap \neq 0$ and $aq \in I$.

2.3. Lemma. *A left ideal I of an alternative ring R is dense if and only if R is a left quotient ring of I.*

Proof. Suppose that I is a dense left ideal of R. On the one hand, given $n \in N(I)$, if there exist $p, q \in R$ such that $(n, p, q) \neq 0$, then there exist $n_1, n_2 \in N(R)$ such that $n_1 p \in I$, $n_2 q \in I$ and $0 \neq n_2 n_1 (n, p, q) = (n, n_1 p, n_2 q) = 0$, a contradiction. On the other hand, given $p, q \in R$, with $p \neq 0$, there exists $n \in N(R)$ such that $np \neq 0$ and $nq \in I$ and there exists $s \in N(R)$ such that $snp \neq 0$ and $sn \in I$. So, $sn \in N(R) \cap I \subset N(I)$ and $snp \neq 0$ and $snq \in I$. The reciprocal is trivial. □

2.4. Definition. Let R be an alternative ring. We denote by $\mathcal{F}^*$ the set of all left ideals A of $N(R)$ such that for every $0 \neq x \in R$ and $\mu \in N(R)$, there exists $\lambda \in N(R)$ such that $\lambda x \neq 0$ and $\lambda\mu \in A$.

2.5. Remark. It is easy to prove that λ can be taken in A and that the intersection of a finite family of elements of $\mathcal{F}^*$ is an element of $\mathcal{F}^*$.

2.6. Proposition. *Let R be an alternative ring. Then*

(i) *If I is a dense left ideal of R, then $N(I) \in \mathcal{F}^*$.*
(ii) *If $A \in \mathcal{F}^*$, then $I := R^1 A$ is a dense left ideal of R.*

Proof. (i). It is straightforward.

(ii). Let $x, y \in R$, with $x \neq 0$. By hypothesis there exists $\lambda \in A$ such that $\lambda x \neq 0$. By Zhevlakov et al. (1982), Corollary 1 of Lemma 7.1.3, $[\lambda, y] := \lambda y - y\lambda \in N(R)$, so if we apply the hypothesis again, there exists $\mu \in N(R)$ such that $\mu\lambda x \neq 0$ and $\mu[\lambda, y] \in A$. Therefore $\mu\lambda y = \mu[\lambda, y] + \mu y\lambda \in R^1 A$, which completes the proof. □

2.7. Notation. *We call* $\mathcal{F} := \{R^1 A \mid A \in \mathcal{F}^*\}$. *Now, given* $I = R^1 A \in \mathcal{F}$, *we have that* $A \subset N(I)$, *hence* $I = R^1 A \subset R^1 N(I) \subset I$. *Therefore* $I = R^1 N(I)$, *with* $N(I) \in \mathcal{F}^*$. *Moreover, by* (2.5), *the intersection of a finite family of elements of* $\mathcal{F}$ *contains an element of* $\mathcal{F}$.

2.8. Let us consider

$$S := \{(I, f) \mid I \in \mathcal{F} \quad \text{and} \quad f \in Hom^*_{N(R)}(I, R)\}$$

where $Hom^*_{N(R)}(I, R)$ denotes the set of all homomorphisms of left $N(R)$–modules from I to R such that for every $x \in R$ and $\lambda, \mu \in N(I)$, $(x\lambda)f = x(\lambda)f$ and $([\lambda, \mu])f \in N(R)$.

2.9. The following relation on S is an equivalence relation: $(I, f) \approx (I', f')$ if and only if there exists $I'' \in \mathcal{F}$ such that $f\,|_{I''} = f'|_{I''}$. We denote by $[I, f]$ the equivalence class of (I, f) and let $Q := S/\approx$.

Abusing notation, given an element $q \in Q$, we will denote by A_q any element of $\mathcal{F}^*$ and by f_q any element of $Hom^*_{N(R)}(R^1 A_q, R)$ such that $q = [R^1 A_q, f_q]$. The dense left ideal $R^1 A_q$ will be denoted by I_q.

2.10. Let us define an $N(R)$-algebra structure on Q: Let $q, q' \in Q$ and $\lambda, \mu \in N(R)$:

(1) We define the sum $q + q' := [R^1(A_q \cap A_{q'}), f_q + f_{q'}]$.
(2) We define the structure of the left $N(R)$-module: $\lambda q := [R^1 A_{\lambda q}, \rho_\lambda f_q]$, where $A_{\lambda q} := \{a \in N(R) \mid a\lambda \in A_q\} \in \mathcal{F}^*$ and ρ denotes right multiplication.
(3) We define the structure of the right $N(R)$-module: $q\lambda := [I_q, f_q \rho_\lambda]$.
(4) We define a product on Q: We denote by

$$A_{qq'} := \{\lambda \in A_q \text{ such that } (\lambda)f_q \in I_{q'}\}.$$

Let us show that $A_{qq'} \in \mathcal{F}^*$. It is clear that it is a left ideal of R. Now, given $0 \neq x \in R$ and $\mu \in N(R)$, there exists $\lambda \in N(R)$ such that $\lambda x \neq 0$ and $\lambda\mu \in A_q$, and there exists $\gamma \in N(R)$ such that $\gamma\lambda x \neq 0$ and $\gamma(\lambda\mu)f_q \in I_{q'}$, because $I_{q'}$ is a dense left ideal of R. So $\gamma\lambda \in N(R)$ and verifies that $\gamma\lambda x \neq 0$ and $\gamma\lambda\mu \in A_{qq'}$. Now, we can define the product

$$qq' := [R^1 A_{qq'}, f_{qq'}], \qquad \text{where} \qquad \left(\sum x_i a_i\right) f_{qq'} := \sum x_i\,((a_i)f_q)f_{q'}$$

for every $x_i \in R^1$ and $a_i \in A_{qq'}$. Let us show that it is well defined. Suppose that $\sum x_i a_i = 0$, where $x_i \in R^1$ and $a_i \in A_{qq'}$ but $\sum x_i\,((a_i)f_q)f_{q'} \neq 0$. By hypothesis, there exists $\mu \in A_{qq'}$ such that $\mu \sum x_i\,((a_i)f_q)f_{q'} \neq 0$. Then

$$\begin{aligned}
&\mu \sum x_i\,((a_i)f_q)f_{q'}\\
&\quad = \sum([\mu, x_i] + x_i\mu)((a_i)f_q)f_{q'}\\
&\quad =^{(1)} \big(\big(\sum[\mu, x_i]a_i\big)f_q\big)f_{q'} + \sum x_i((\mu a_i)f_q)f_{q'}\\
&\quad = \big(\big(\mu \sum x_i a_i\big)f_q\big)f_{q'} - \big(\big(\sum x_i \mu a_i\big)f_q\big)f_{q'} + \sum x_i((\mu a_i)f_q)f_{q'}\\
&\quad = -\big(\big(\sum x_i[\mu, a_i]\big)f_q\big)f_{q'} - \big(\big(\sum x_i a_i \mu\big)f_q\big)f'_{q'}\\
&\qquad + \sum x_i(([\mu, a_i])f_q)f_{q'} + \sum x_i((a_i\mu)f_q)f_{q'}\\
&\quad =^{(2)} -\big(\big(\sum x_i[\mu, a_i]\big)f_q\big)f_{q'} + \big(\big(\sum x_i[\mu, a_i]\big)f_q\big)f_{q'} = 0
\end{aligned}$$

(1) is a consequence of μ and $[\mu, x_i]$ belonging to $N(R)$ and f_q and $f_{q'}$ being homomorphisms of left $N(R)$-modules, and (2) uses the previous facts and $([\mu, a_i])f_q \in N(R^1 A_{q'})$, since $([\mu, a_i])f_q$ belongs to $N(R)$ and also belongs to $R^1 A_{q'}$ (because $[\mu, a_i] \in A_{qq'}$), hence $((\sum x_i[\mu, a_i])f_q)f_{q'} = \sum x_i(([\mu, a_i])f_q)f_{q'}$ by definition of $Hom^*_{N(R)}(I_{q'}, R)$.

2.11. Theorem. *Let Q be as above. Then*

(i) *R is a subring of Q. Moreover, R is a dense left $N(R)$-submodule of Q.*
(ii) *$N(R) \subset N(Q)$.*
(iii) *For every $q \in Q$ and $\lambda \in N(R)$, $[\lambda, q] \in N(Q)$.*
(iv) *The associator is a skew-symmetric function on Q.*
(v) *If $D(R)$ is 2-torsion free or semiprime, Q is an alternative ring.*

Proof. (i). The map $\Psi : R \to Q$ given by $\Psi(r) = [R^1 N(R), \rho_r]$ defines a monomorphism of rings: it is clear that Ψ is a monomorphism of $N(R)$-modules, since ρ_r cannot vanish on a dense left ideal of R. Moreover, $\Psi(r)\Psi(r') = [R^1 N(R), \rho_r][R^1 N(R), \rho_{r'}] = [R^1 N(R), f_{rr'}]$, where for every $x \in R$ and $\lambda \in N(R)$, $(x\lambda)(f_{rr'}) = x((\lambda)\rho_r\rho_{r'}) = x(\lambda rr') = (x\lambda)\rho_{rr'}$, so $\Psi(r)\Psi(r') = \Psi(rr')$. Now, given $q, q' \in Q$ with $q \neq 0$, by construction there exists $r \in A_q \cap A_{q'}$ such that $(r)f_q \neq 0$ (since $I_q \cap I_{q'}$ is a dense left ideal of R). Hence $[R^1 N(R), \rho_r][I_q, f_q] = [R^1 N(R), \rho_{(r)f_q}] \neq 0$ and $[R^1 N(R), \rho_r][I_{q'}, f_{q'}] = [R^1 N(R), \rho_{(r)f_{q'}}] \in R$.

(ii). Let us consider $q_j \in Q$, for $j = 1, 2, 3$, and take $a \in A_{(q_1 q_2)q_3} \cap A_{q_1(q_2 q_3)}$. By definition of $A_{q_1(q_2 q_3)}$ we have $(a)f_{q_1} \in I_{q_2 q_3}$; therefore there exist $y_i \in R^1$ and $a_i \in A_{q_2 q_3}$ such that $(a)f_{q_1} = \sum y_i a_i$. Then for every $x \in R^1$,

$$\begin{aligned}
(xa)f_{(q_1 q_2)q_3} &= x((a)f_{q_1 q_2})f_{q_3} = x(((a)f_{q_1})f_{q_2})f_{q_3}\\
&= \sum x((y_i a_i)f_{q_2})f_{q_3} = \sum x((y_i)((a_i)f_{q_2}))f_{q_3}\\
(xa)f_{q_1(q_2 q_3)} &= x((a)f_{q_1})f_{q_2 q_3} = x\big(\sum y_i a_i\big)f_{q_2 q_3}\\
&= \sum x(y_i((a_i)f_{q_2})f_{q_3}).
\end{aligned}$$

So, if $[I_{q_1}, f_{q_1}] \in N(R)$ (i.e., $f_{q_1} = \rho_\lambda$ with $\lambda \in N(R)$), then $(a)f_{q_1} = a\lambda \in N(I_{q_2q_3})$ and therefore $(xa)f_{(q_1q_2)q_3} = x((a\lambda)f_{q_2})f_{q_3} = (xa)f_{q_1(q_2q_3)}$; if $[I_{q_2}, f_{q_2}] \in N(R)$, then $(y_i(a_i)f_{q_2})f_{q_3} = y_i((a_i)f_{q_2})f_{q_3}$, and if $[I_{q_3}, f_{q_3}] \in N(R)$, then $y_i((a_i)f_{q_2})f_{q_3} = (y_i((a_i)f_{q_2}))f_{q_3}$. So, in any case, $(xa)f_{(q_1q_2)q_3} = (xa)f_{q_1(q_2q_3)}$, which implies that $N(R) \subset N(Q)$.

(iii). For every $r \in R$ and $q \in Q$, we have

$$(q, r, r) = (r, q, r) = (r, r, q) = 0. \qquad (*)$$

Otherwise, if there exists $q \in Q$ such that $(r, q, r) \neq 0$, let us consider $\lambda \in N(R)$ such that $\lambda q \in R$ and $0 \neq \lambda(r, q, r) = (\lambda r, q, r) = ([\lambda, r], q, r) + (r\lambda, q, r) = (r, \lambda q, r) = 0$ (by (ii) since $[\lambda, r] \in N(R)$), a contradiction. In a similar way, (q, r, r) and (r, r, q) are zero.

Now, given $\lambda \in N(R)$ and $q \in Q$, there exists $\mu \in N(R)$ such that $\mu[\lambda, q]$ belongs to R. So for every $r, s \in R$ we have

$$\begin{aligned}(r, \mu\lambda q, s) &=^{(1)} (r\mu\lambda, q, s) =^{(2)} -(s, q, r\mu\lambda) = -(s, q, [r\mu, \lambda] + \lambda r\mu)\\ &=^{(3)} -(s, q\lambda, r\mu) =^{(4)} (r\mu, q\lambda, s) =^{(5)} (r, \mu q\lambda, s)\end{aligned}$$

(1), (3), and (5) follow from (ii) (since $[r\mu, \lambda] \in N(R)$), (2), and (4) follow from $(*)$. So we have $\mu[\lambda, q] \in N(R)$ for every $q \in Q$, $\lambda \in N(R)$ and $\mu \in A_{[\lambda,q]}$.

Now, if there exist $p, p' \in Q$ such that $([\lambda, q], p, p') \neq 0$, then there exists $\mu \in N(R)$ such that $\mu[\lambda, q] \in R$ and $0 \neq \mu([\lambda, q], p, p') = (\mu[\lambda, q], p, p') = 0$, by (ii) and $(*)$, a contradiction. Moreover, if $(p, [\lambda, q], p') \neq 0$, there exists $\mu' \in N(R)$ such that $\mu' p \in R$ and $0 \neq \mu'(p, [\lambda, q], p') = (\mu' p, [\lambda, q], p')$ and there exists $\mu \in N(R)$ such that $\mu[\lambda, q] \in R$ and $0 \neq \mu(\mu' p, [\lambda, q], p') = (\mu(\mu' p), [\lambda, q], p') = ([\mu, \mu' p], [\lambda, q], p') + ((\mu' p)\mu, [\lambda, q], p') = (\mu' p, \mu[\lambda, q], p') = 0$, a contradiction. In a similar way, it can be proved that $(p, p', [\lambda, q]) = 0$, which implies that $[\lambda, q] \in N(Q)$.

(iv). Let $p_1, p_2, p_3 \in Q$ and $p = (p_1, p_2, p_3) + (p_2, p_1, p_3) \neq 0$. In view of (ii) and (iii), for any $\lambda \in N(R)$ we have

$$\begin{aligned}\lambda(p_1, p_2, p_3) &= (\lambda p_1, p_2, p_3) = (p_1\lambda, p_2, p_3) = (p_1, \lambda p_2, p_3)\\ &= (p_1, p_2\lambda, p_3) = (p_1, p_2, \lambda p_3).\end{aligned}$$

By (i) and (ii), there exists $\lambda_3 \in N(R) \subseteq N(Q)$, such that $\lambda_3 p \neq 0$ and $\lambda_3 p_3 \in R$. Similarly, there exists $\lambda_2 \in N(R) \subseteq N(Q)$, such that $\lambda_2\lambda_3 p \neq 0$ and $\lambda_2 p_2 \in R$. Finally, there exists $\lambda_1 \in N(R) \subseteq N(Q)$, such that $\lambda_1\lambda_2\lambda_3 p \neq 0$ and $\lambda_1 p_1 \in R$. Therefore

$$\lambda_1\lambda_2\lambda_3 p = (\lambda_1 p_1, \lambda_2 p_2, \lambda_3 p_3) + (\lambda_2 p_2, \lambda_1 p_1, \lambda_3 p_3) = 0,$$

since R is alternative. The contradiction proves that $p = 0$.

(v). Suppose that there exist $p, q \in Q$ such that $(p, p, q) \neq 0$. Then for every $\lambda \in A_p$, $\lambda^2(p, p, q) = (\lambda p, \lambda p, q) = 0$, by $(*)$.

Suppose first that $D(R)$ is semiprime. We know that there exist $\alpha, \beta, \gamma \in N(R)$ such that αp, βp, and γq belong to R and $(\alpha p, \beta p, \gamma q) \neq 0$. Now, there exists $\lambda \in A_p$ such that $\lambda(\alpha p, \beta p, \gamma q) \neq 0$ and therefore, since $D(R)$ is semiprime, $\lambda^2(\alpha p, \beta p, \gamma q) \neq 0$ (otherwise the ideal generated by $\lambda(\alpha p, \beta p, \gamma q)$ will be nilpotent—see Gómez Lozano and Siles Molina (Preprint), (1.5)—a contradiction.

Suppose now that $D(R)$ is 2-torsion free. If $\lambda, \mu \in A_p$,

$$0 = (\lambda + \mu)^2(p, p, q) = \lambda^2(p, p, q) + 2\lambda\mu(p, p, q) + \mu^2(p, p, q) = 2\lambda\mu(p, p, q).$$

Note that $\lambda\mu(p, p, q) = \mu\lambda(p, p, q)$, so $\lambda\mu(p, p, q) = 0$, a contradiction.

In a similar way we can prove that (q, p, p) is zero for every $p, q \in Q$. □

2.12. Lemma. *Let S be a left quotient ring of R and consider $q \in S$. Then $(N(R) : q) := \{\lambda \in N(R) \mid \lambda q \in R\} \in \mathcal{F}^*$.*

Proof. Given $0 \neq x \in R$ and $\mu \in N(R)$, there exists $\gamma \in N(R)$ such that $\gamma x \neq 0$ and $\gamma(\mu q) \in R$; hence $\gamma\mu \in (N(R) : q)$. □

2.13. Theorem. *Let R be an alternative ring such that $D(R)$ is 2-torsion free or semiprime. Then R is a left quotient ring of itself if and only if the maximal left quotient ring of R exists.*

Proof. Let S be a left quotient ring of R. Given an element $q \in S$, by (2.12) and (2.6), $I_q := R^1(N(R) : q)$ is a dense left ideal of R. Now, following the proof of (2.11) (i), the map $\Phi : S \to Q$ defined by $\Phi(q) := [I_q, \rho_q]$ is a monomorphism of alternative rings. □

2.14. Remark. By construction, the maximal left quotient ring of an alternative ring is unital with unit element $[R^1 N(R), Id_R]$.

Some examples of maximal left quotient rings are the following ones:

2.15. Examples. (1). It is clear that the maximal left quotient ring of an associative ring is its maximal left quotient ring as an alternative ring.

(2). Let Q be a Cayley–Dickson algebra over its center. Then $Q^l_{max}(Q) = Q$: Let S be a left quotient ring of Q, and take $s \in S$. By hypothesis there exists $n \in Z(Q)$ (which is a field) such that $ns \in Q$. So $s = n^{-1}(as) \in Q$.

(3). If R is a Cayley–Dickson ring, its maximal left quotient ring is a Cayley–Dickson algebra: by definition R is a central order in a Cayley–Dickson algebra, denoted by Q. So Q is a left quotient ring of R, which implies that $Q^l_{max}(R) = Q^l_{max}(Q) = Q$, by (2).

(4). Let us consider a family $\{R_\alpha\}$ of alternative rings such that for every α there exists the maximal left quotient ring of R_α, which we denote by Q_α. Then $Q^l_{max}(\bigoplus R_\alpha)$ exists and is equal to ΠQ_α, the direct product of Q_α: the proof is analogous to Utumi (1956), (2.1).

(5). Let R be a nondegenerate and purely alternative ring. Then the nearly classical localization of R, given in Beidar and Mikhalev (1989), Section 1, is the maximal left quotient ring of R:

(i) By Zhevlakov et al. (1982), Theorem 8.11, $N(R) = Z(R)$.
(ii) Every dense left ideal of $Z(R)$ is contained in $\mathcal{F}^*$. Let A be a dense left ideal of $Z(R)$. Given $0 \neq x \in R$ and $\lambda \in Z(R)$, by Zhevlakov et al. Lemma 9.2.7 and Theorem 9.2.7, there exists $y \in {}_R(x)_R$ (the ideal of R generated by x) contained in $Z(R)$. Now, since A is a dense left ideal of $Z(R)$, let $\mu \in Z(R)$ such that $\mu y \neq 0$, which implies that $\mu x \neq 0$, and $\mu\lambda \in A$.
(iii) So R satisfies Beidar and Mikhalev (1989), (1.4), and by Beidar and Mikhalev (1989), (1.5) (1) $R_{\mathcal{F}}$, the nearly classical localization of R, is an alternative ring.
(iv) $A_{\mathcal{F}}$ is the maximal left quotient ring of R. If S is a left quotient ring of R, by (2.12), for every $s \in S$, $(Z(R) : s) = \{a \in Z(R) | as \in R\} \in \mathcal{F}^*$ and the map $\Psi : S \to R_{\mathcal{F}}$ defined by $\Psi(s) = ((Z(R) : s), \lambda_s)$ is a monomorphism of alternative rings.

3. CLASSICAL LEFT QUOTIENT RINGS

The next proposition, which is in Gómez Lozano and Siles Molina (Preprint), (5.7) and (6.7) (i), shows that the maximal ring of quotients gives us an appropriate framework in which to settle the different left quotient rings that have been investigated (Fountain–Gould and classical); see Essannouni and Kaidi (1994) and Gómez Lozano and Siles Molina (Preprint) for definitions. This fact was used by Áhn and Márki to give a general theory of Fountain and Gould left order in the setting of associative rings.

3.1. Proposition. *Let R be an alternative ring. If R is a classical* (*Fountain and Gould*) *left order in an alternative ring S, then S is a left quotient ring of R. So S is a subring of $Q^l_{max}(R)$.*

Let us construct the classical left order of a left Ore alternative ring R as the subring of $Q^l_{max}(R)$ generated by R and the set $\{a^{-1} | a \in Reg(R) \cap N(R)\}$.

Let R be an alternative ring. We recall that R satisfies the left Ore condition relative to a nonempty set S if for every $a \in S$ and $x \in R$ there exist $b \in S$ and $y \in R$ such that $bx = ya$. We will say that R is left Ore if it verifies the left Ore condition relative to $Reg(R) \cap N(R) \neq \emptyset$, where $Reg(R)$ denotes the set of all regular elements of R.

Note that $Reg(R) \cap N(R) \neq 0$ implies that R is a left quotient ring of itself. So there exists the maximal left quotient ring of R, denoted by Q.

3.2. Lemma. *Every element $a \in Reg(R) \cap N(R)$ is invertible in Q.*

Proof. It is easy to prove that Ra is a dense left ideal of R. Moreover, the map $h : Ra \to R$, defined by $(xa)h = x$ for every $xa \in Ra$, belongs to $Hom^*_{N(R)}(Ra, R)$. Now $[Ra, h]$ is the inverse of a in Q. Furthermore, $[Ra, h] \in N(Q)$. □

3.3. Lemma (common denominator theorem). *For every elements $a, b \in Reg(R) \cap N(R)$ there exist $c, d \in Reg(R) \cap N(R)$ such that $cb = da$.*

3.4. Theorem. *Let R be a ring that satisfies the left Ore condition. Then* $T = \{a^{-1}x \mid a \in Reg(R) \cap N(R),\ x \in R\}$ *is a subring of Q such that R is a classical left order in T.*

Proof. Given $a^{-1}x$, $b^{-1}y$, where $a, b \in Reg(R) \cap N(R)$ and $x, y \in R$, by (3.3) there exist $c, d \in Reg(R) \cap N(R)$ such that $cb = da$. So $a^{-1}x + b^{-1}y = a^{-1}d^{-1}dx = b^{-1}c^{-1}cy = (da)^{-1}(dx + cy) \in T$. It is straightforward that $a^{-1}xb^{-1}y \in T$.

Let us show that T is an alternative ring. Given $p, q \in T$, there exist $a, b \in Reg(R) \cap N(R)$ such that $ap, bq \in R$, so $a^2b(p, p, q) = 0$ and $a^2b(q, p, p) = 0$, which implies that $(p, p, q) = 0 = (q, p, p)$.

Now it is trivial that R is a classical left order in T. □

3.5. Proposition. *Let R be an alternative ring.*

(i) *If R is nondegenerate and artinian, then* $Q^l_{max}(R) = R$.
(ii) *If R is nondegenerate and left Goldie,* $Q^l_{max}(R) = Q^l_{cl}(R)$, *where* $Q^l_{cl}(R)$ *denotes the classical left quotient ring of R.*

Proof. (i). This follows from Zhevlakov et al. (1982), Theorem 12.2.3 and (2.15) (1), (2) and (4).

(ii). This follows from (i), Goldie's theorems for alternative rings; see Gómez Lozano and Siles Molina (Preprint), (7.1) and (1.12). □

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