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Fountain–Gould Left Orders for Associative Pairs

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Abstract In this paper, we introduce a notion of weak Fountain–Gould left order for associative pairs and give a Goldie-like theory of associative pairs which are weak Fountain–Gould left orders in semiprime pairs coinciding with their socles.

Keywords Fountain–Gould left order, Associtive pair, Goldie theory
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Introduction

In 1990, Fountain and Gould [1] gave a notion of order in a ring not necessarily unital, and obtained in [2] a Goldie-like characterization of two-sided orders in semiprime rings with descending chain condition on principal one-sided ideals, i.e., in semiprime rings coinciding with their socles. Later on, Ánh and Márki extended these results to one-sided orders [3, 4]. In [5], local algebras were used to introduce a notion of order for associative pairs and describe such two-sided orders in semiprime pairs coinciding with their socles.

In our work we will give a pair version of weak Fountain–Gould one-sided orders and study associative pairs which are weak Fountain–Gould one-sided orders in semiprime pairs coinciding with their socles. In this sense, we remove the two-sidedness of the results of [5]. At the present moment some work on the Goldie theory of Jordan systems is being done [6, 7], so that having a suitable Goldie theory for associative systems could be of some help.

We will use envelopes and local algebras to translate pair problems into algebra ones; thus, already known results on Fountain–Gould orders in rings [4, 1, 8] will be very important in our work. We will also study the connections of our notion of order and other related notions given for associative pairs [5, 9].

Our paper is divided into three sections, apart from Section 0 devoted to outlining some preliminary results and definitions. Section 1 deals with the algebra envelopes of associative pairs. These will be the "atmospheres" where pairs embed and calculations are performed; they must be close enough to the pairs we deal with, but, on the other hand, they must be general enough to allow working with a pair and its quotient at the same time. Using algebra envelopes, we introduce in Section 2 the notion of weak Fountain–Gould left order for associative pairs.

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We also show that the quotients obtained via this notion are particular cases of those one-sided quotient pairs studied in [9]. This is used, together with local inverse lifting properties, to show that in most of the cases, our notion does not depend on the algebra envelope where the computations are done, for example, when one is restricted to semiprime pairs. In the last section, we study weak Fountain–Gould left orders in semiprime pairs coinciding with their socles and characterize them in terms of chain conditions on left annihilators and local Goldie dimension.

We finally remark that, though we always work with left orders and left quotients, the results in this paper have their right-side analogues, with obvious changes in the definitions, just reversing products in the proofs or applying the left-side results to the opposite pairs and algebras.

0 Preliminaries

0.1 Throughout this paper Φ will denote a unital commutative associative ring of scalars, and all algebraic systems will be defined on modules over Φ. Recall that an *associative pair* over Φ is a pair (A^+, A^-) of Φ-modules together with Φ-trilinear maps

$$\begin{array}{ccc} A^\sigma \times A^{-\sigma} \times A^\sigma & \to & A^\sigma \\ (x,y,z) & \mapsto & xyz \end{array}$$

satisfying the following identities:

$$uv(xyz) = u(vxy)z = (uvx)yz, \tag{1}$$

for all $u, x, z \in A^\sigma$, $v, y \in A^{-\sigma}$ and $\sigma = \pm$.

The first example of an associative pair is given by $V(\mathscr{A}) := (\mathscr{A}, \mathscr{A})$, for any associative algebra $\mathscr{A}$, where the triple product of x, y, and z is given by the usual product xyz in the algebra $\mathscr{A}$.

0.2 Given an associative pair $A = (A^+, A^-)$, and elements $x, z \in A^\sigma, y \in A^{-\sigma}$, $\sigma = \pm$, recall that *left*, *middle* and *right multiplications* are defined by:

$$\lambda(x,y)z = \mu(x,z)y = \rho(y,z)x = xyz. \tag{1}$$

From (0.1)(1), for any $x, u \in A^\sigma, y, v \in A^{-\sigma}$,

$$\lambda(x,y)\lambda(u,v) = \lambda(xyu, v) = \lambda(x, yuv), \tag{2}$$

and similarly,

$$\rho(u,v)\rho(x,y) = \rho(x, yuv) = \rho(xyu, v). \tag{3}$$

As a consequence of (2) and (3), it is clear that the linear span of all operators $T : A^\sigma \to A^\sigma$ of the form $T = \lambda(x,y)$, for $(x,y) \in A^\sigma \times A^{-\sigma}$, or $T = Id_{A^\sigma}$ is a unital associative algebra; it will be denoted by $\Lambda(A^\sigma, A^{-\sigma})$. Clearly A^σ is a left $\Lambda(A^\sigma, A^{-\sigma})$-module. Similarly, we define $\mathrm{P}(A^{-\sigma}, A^\sigma)$ as the linear span of all the right multiplications and the identity on A^σ, so that A^σ becomes a left $\mathrm{P}(A^{-\sigma}, A^\sigma)$-module. We define the *left ideals* $L \subset A^\sigma$ of A as the $\Lambda(A^\sigma, A^{-\sigma})$-submodules of A^σ, and the *right ideals* $R \subset A^\sigma$ as the $\mathrm{P}(A^{-\sigma}, A^\sigma)$-submodules. A *two-sided ideal* $B \subset A^\sigma$ is both a left and a right ideal. An *ideal* $I = (I^+, I^-)$ of A is a pair of two-sided ideals of A such that $A^\sigma I^{-\sigma} A^\sigma \subseteq I^\sigma, \sigma = \pm$. A Φ-submodule M of A^σ is called an *inner ideal* of A if $xA^{-\sigma}x \subseteq M$, for any $x \in M$.

0.3 An associative pair A is *semiprime* if and only if nonzero ideals of A have a nonzero cube [I ideal of A and $I^\sigma I^{-\sigma} I^\sigma = 0$, $\sigma = \pm$, implies $I = 0$]. We will say that A is *prime* if A is semiprime and two nonzero ideals of A have nonzero intersection. An element a in A^σ generates the ideal $I = (I^+, I^-)$ of A given by $I^\sigma := \Phi a + A^\sigma A^{-\sigma} a + a A^{-\sigma} A^\sigma + A^\sigma A^{-\sigma} a A^{-\sigma} A^\sigma$ and $I^{-\sigma} := A^{-\sigma} a A^{-\sigma}$. This provides elemental characterizations of semiprimeness and primeness (see [10, 1.18]): A is semiprime if and only if A is *nondegenerate* ($aA^\sigma a = 0, a \in A^{-\sigma} \Rightarrow a = 0$, $\sigma = \pm$); A is prime if and only if A is *elementally prime* ($aA^\sigma b = 0, a, b \in A^{-\sigma} \Rightarrow a = 0$ or $b = 0$, $\sigma = \pm$).

0.4 If $\mathscr{A}$ is a subalgebra of a unital associative algebra $\mathscr{Q}$, we say that $\mathscr{A}$ is a *weak classical left order in* $\mathscr{Q}$ (or $\mathscr{Q}$ is a *weak classical left quotient algebra of* $\mathscr{A}$) if for any $q \in \mathscr{Q}$, there exist $a, b \in A$, such that a is invertible in $\mathscr{Q}$ and $q = a^{-1}b$. We say that $\mathscr{A}$ is a *classical left order in* $\mathscr{Q}$ (or $\mathscr{Q}$ is a *classical left quotient algebra of* $\mathscr{A}$) when it is a weak classical left order and every regular element in $\mathscr{A}$ is invertible in $\mathscr{Q}$.

0.5 Given an associative pair $A = (A^+, A^-)$ and $a \in A^{-\sigma}$, the Φ-module $aA^\sigma a$ becomes an associative algebra, denoted by A^σ_a and called the *local algebra of* A *at* a by defining the product

$$axa \cdot aya = axaya, \qquad \text{for any } x, y \in A^\sigma.$$

Let $A^{\sigma(a)}$ be the *a-homotope of* A, i.e., the Φ-module A with product $x \cdot_a y := xay$ for any $x, y \in A^\sigma$. The set $\operatorname{Ker}_A a = \operatorname{Ker} a := \{x \in A^\sigma \mid axa = 0\}$ is an ideal of $A^{\sigma(a)}$ and the quotient $A^{\sigma(a)}/\operatorname{Ker} a$ is isomorphic to A^σ_a (see [5, 11]).

1 Algebra Envelopes of Associative Pairs

1.1 Associative pairs are really "abstract off-diagonal Peirce spaces" of associative algebras: Let $\mathscr{E}$ be a unital associative algebra, and consider the Peirce decomposition $\mathscr{E} = \mathscr{E}_{11} \oplus \mathscr{E}_{12} \oplus \mathscr{E}_{21} \oplus \mathscr{E}_{22}$ of $\mathscr{E}$ with respect to an idempotent $e \in \mathscr{E}$, i.e.,

$$\mathscr{E}_{11} = e\mathscr{E}e, \quad \mathscr{E}_{12} = e\mathscr{E}(1-e), \quad \mathscr{E}_{21} = (1-e)\mathscr{E}e \quad \text{and} \quad \mathscr{E}_{22} = (1-e)\mathscr{E}(1-e).$$

From the Peirce multiplication rules, $(\mathscr{E}_{12}, \mathscr{E}_{21})$ is a subpair of $V(\mathscr{E})$. Conversely, every associative pair $A = (A^+, A^-)$ can be obtained in this way (see [12, 2.3]): Let $\mathscr{C}$ be the Φ-submodule of $\mathscr{B} = \operatorname{End}_\Phi(A^+) \times \operatorname{End}_\Phi(A^-)^{op}$ spanned by $e_1 = (Id_{A^+}, Id_{A^-})$ and all $(\lambda(x, y), \rho(x, y))$, and similarly, let $\mathscr{D}$ be the submodule of $\mathscr{B}^{op}$ spanned by $e_2 = (Id_{A^+}, Id_{A^-})$ and all $(\rho(y, x), \lambda(y, x))$ where $(x, y) \in A^+ \times A^-$. By (0.2), these Φ-linear spans are really subalgebras. Clearly, A^+ is an $(\mathscr{C}, \mathscr{D})$-bimodule if we set $cx = c^+(x)$, $xd = d^+(x)$ for $x \in A^+$ and $c = (c^+, c^-) \in \mathscr{C}$, $d = (d^+, d^-) \in \mathscr{D}$. Similarly, A^- is a $(\mathscr{D}, \mathscr{C})$-bimodule. Now we define bilinear maps on $A^\pm \times A^\mp$ with values in $\mathscr{C}$, respectively, $\mathscr{D}$, by $xy = (\lambda(x, y), \rho(x, y))$, $yx = (\rho(y, x), \lambda(y, x))$. Then it is easy to check that $(\mathscr{C}, A^+, A^-, \mathscr{D})$ is a Morita context which gives rise to a unital associative algebra $\mathscr{E}$ (cf. [12, 2.3]). If we set $e = e_1$, then the pair $A = (A^+, A^-)$ is isomorphic to the associative pair $(\mathscr{E}_{12}, \mathscr{E}_{21})$. Moreover $\mathscr{E}_{11}$ (respectively, $\mathscr{E}_{22}$) is spanned by e and all products $x_{12}y_{21}$ (respectively, $1 - e$ and all products $y_{21}x_{12}$) for $x_{12} \in \mathscr{E}_{12}$, $y_{21} \in \mathscr{E}_{21}$, and has the property that

$$x_{11}\mathscr{E}_{12} = \mathscr{E}_{21}x_{11} = 0 \Longrightarrow x_{11} = 0, \qquad x_{22}\mathscr{E}_{21} = \mathscr{E}_{12}x_{22} = 0 \Longrightarrow x_{22} = 0. \tag{1}$$

1.2 Let $\mathscr{A}$ be the subalgebra of $\mathscr{E}$ generated by $\mathscr{E}_{12} \cup \mathscr{E}_{21}$, i.e., $\mathscr{A} = \mathscr{E}_{12} \oplus \mathscr{E}_{12}\mathscr{E}_{21} \oplus \mathscr{E}_{21}\mathscr{E}_{12} \oplus \mathscr{E}_{21}$. It is immediate that $\mathscr{A}$ is an ideal of $\mathscr{E}$. We will call $\mathscr{A}$ the *standard envelope of* the associative pair A, and write $\tau = (\tau^+, \tau^-)$ for the natural inclusion $\tau^\sigma : A^\sigma \longrightarrow \mathscr{A}$ of A into $\mathscr{A}$.

1.3 Let A be an associative pair, $\mathscr{A}$ be an associative algebra, and $\varphi = (\varphi^+, \varphi^-)$, where $\varphi^\sigma : A^\sigma \longrightarrow \mathscr{A}$ is an injective Φ-linear map, $\sigma = \pm$. We say that A is a *subpair of* $(\mathscr{A}, \varphi)$, if:

(i) $\varphi^+(A^+) \cap \varphi^-(A^-) = 0$;

(ii) $\varphi^+(A^+)\varphi^+(A^+) = \varphi^-(A^-)\varphi^-(A^-) = 0$; and,

(iii) $\varphi : A \longrightarrow V(\mathscr{A})$ is a pair homomorphism (hence monomorphism).

When A is a subpair of $(\mathscr{A}, \varphi)$, $(\mathscr{A}, \varphi)$ is called an *envelope of* A if $\varphi^+(A^+) \cup \varphi^-(A^-)$ generates $\mathscr{A}$ as an algebra, i.e.,

(iv) $\mathscr{A} = \varphi^+(A^+) + \varphi^+(A^+)\varphi^-(A^-) + \varphi^-(A^-)\varphi^+(A^+) + \varphi^-(A^-)$.

An envelope $(\mathscr{A}, \varphi)$ of A is called *tight* if every nonzero ideal of $\mathscr{A}$ hits A $[I \cap (\varphi^+(A^+) \cup \varphi^-(A^-)) \neq 0$ for every nonzero ideal I of $\mathscr{A}]$. We say that $(\mathscr{A}, \varphi)$ and $(\tilde{\mathscr{A}}, \tilde{\varphi})$ are *isomorphic envelopes* of A if there exists an algebra isomorphism $\psi : \mathscr{A} \longrightarrow \tilde{\mathscr{A}}$ such that $\psi \circ \varphi^\sigma = \tilde{\varphi}^\sigma$, $\sigma = \pm$.

Notice that, for an associative pair A, the standard envelope $(\mathscr{A}, \tau)$ of A is an envelope of A in the above sense.

1.4 Let $\mathscr{A}$ be an associative algebra. An element $a \in \mathscr{A}$ is said to be a *left* (respectively, *right*) *total zero divisor* if $a\mathscr{A} = 0$ (respectively, $\mathscr{A}a = 0$). Similarly, an element $a \in A^\sigma$

of an associative pair is called a *left* (respectively, *right*) *total zero divisor* if $aA^{-\sigma}A^{\sigma} = 0$ (respectively, $A^{\sigma}A^{-\sigma}a = 0$). A pair or an algebra not having nonzero left (respectively, right) total zero divisors is said to be *left* (respectively, *right*) *faithful*. It is immediate that a semiprime algebra or pair is both left and right faithful.

Proposition 1.5 *Let A be a left and right faithful (for example, semiprime) associative pair, and $(\mathscr{A}, \varphi)$ be an envelope of A. Then*

$$\begin{aligned} I &= \{x \in \varphi^+(A^+)\varphi^-(A^-) + \varphi^-(A^-)\varphi^+(A^+) \mid x\varphi^\sigma(A^\sigma) = 0 = \varphi^\sigma(A^\sigma)x, \sigma = \pm\} \\ &= \{x \in \varphi^+(A^+)\varphi^-(A^-) \mid x\varphi^+(A^+) = 0 = \varphi^-(A^-)x\} + \\ &\quad + \{x \in \varphi^-(A^-)\varphi^+(A^+) \mid x\varphi^-(A^-) = 0 = \varphi^+(A^+)x\} \end{aligned}$$

is the biggest ideal of $\mathscr{A}$ not hitting $\varphi(A)$, and satisfies $I\mathscr{A} = \mathscr{A}I = 0$. Moreover, if $\phi^\sigma : A^\sigma \longrightarrow \mathscr{A}/I$ is given by $\phi^\sigma(x^\sigma) = \varphi^\sigma(x^\sigma) + I$, $\sigma = \pm$, then $(\mathscr{A}/I, \phi)$ is an envelope of A isomorphic to the standard envelope of A.

Proof From (1.3), it is clear that I is an ideal of $\mathscr{A}$ and that both definitions of I agree. Indeed, $\mathscr{A}x = x\mathscr{A} = 0$ for any $x \in I$. Moreover $(I \cap \varphi^+(A^+), I \cap \varphi^-(A^-))$ is an ideal of $\varphi(A)$ consisting of two-sided total zero divisors, hence it is zero by the faithfulness of $\varphi(A) \cong A$. This means that ϕ^σ is injective for $\sigma = \pm$, and it is straightforward that $(\mathscr{A}/I, \phi)$ satisfies (1.3)(ii)–(iv). To show that $(\mathscr{A}/I, \phi)$ is an envelope of A, we just need to check (1.3)(i). Indeed, an element in $\phi^+(A^+) \cap \phi^-(A^-)$ has the form $\varphi^+(a^+) + I = \varphi^-(a^-) + I$ for some $a^\sigma \in A^\sigma$, $\sigma = \pm$. Thus, there must be $y \in I$ such that $\varphi^+(a^+) = \varphi^-(a^-) + y$, hence

$$\varphi^+(a^+)\varphi^-(A^-)\varphi^+(A^+) = (\varphi^-(a^-) + y)\varphi^-(A^-)\varphi^+(A^+) = 0,$$

because φ satisfies (1.3)(ii). Therefore $\varphi^+(a^+) = 0$, by the left faithfulness of $\varphi(A)$.

Let L be an ideal of $\mathscr{A}$ not hitting $\varphi(A)$. We will show that $L \subseteq I$: if $x \in L$, write $x = x^+ + y + z + x^-$, where $x^\sigma \in \varphi^\sigma(A^\sigma)$, $\sigma = \pm$, $y \in \varphi^+(A^+)\varphi^-(A^-)$, $z \in \varphi^-(A^-)\varphi^+(A^+)$. By (1.3)(ii), $\varphi^+(A^+)x\varphi^+(A^+) = \varphi^+(A^+)x^-\varphi^+(A^+) \in \varphi^+(A^+) \cap L = 0$, hence $x^- = 0$ by the left and right faithfulness of $\varphi(A)$. Similarly $x^+ = 0$. But now, again by (1.3)(ii), $x\varphi^+(A^+) = y\varphi^+(A^+) \in \varphi^+(A^+) \cap L = 0$, and, similarly $x\varphi^-(A^-) = \varphi^+(A^+)x = \varphi^-(A^-)x = 0$, which shows $x \in I$.

Let $(\tilde{\mathscr{A}}, \tau)$ be the standard envelope of A. We can define a linear map $\psi : \mathscr{A} \longrightarrow \tilde{\mathscr{A}}$ given by

$$\begin{aligned} &\psi\Big(\varphi^+(x^+) + \sum_i \varphi^+(y_i^+)\varphi^-(y_i^-) + \sum_j \varphi^-(z_j^-)\varphi^+(z_j^+) + \varphi^-(u^-)\Big) \\ &\qquad = \tau^+(x^+) \oplus \sum_i \tau^+(y_i^+)\tau^-(y_i^-) \oplus \sum_j \tau^-(z_j^-)\tau^+(z_j^+) \oplus \tau^-(u^-), \end{aligned}$$

for any $x^+, y_i^+, z_j^+ \in A^+$, $y_i^-, z_j^-, u^- \in A^-$. Indeed, if

$$a = \varphi^+(x^+) + \sum_i \varphi^+(y_i^+)\varphi^-(y_i^-) + \sum_j \varphi^-(z_j^-)\varphi^+(z_j^+) + \varphi^-(u^-) = 0,$$

then $0 = \varphi^-(A^-)a\varphi^-(A^-) = \varphi^-(A^-)\varphi^+(x^+)\varphi^-(A^-) = \varphi^-(A^-x^+A^-)$ by (1.3), thus $A^-x^+A^- = 0$ by the injectivity of φ^-, which implies $x^+ = 0$ by the left and right faithfulness of A; similarly $u^- = 0$; hence

$$0 = \varphi^+(A^+)a = \varphi^+(A^+)\Big(\sum_j \varphi^-(z_j^-)\varphi^+(z_j^+)\Big) = \sum_j \varphi^+(A^+)\varphi^-(z_j^-)\varphi^+(z_j^+) = \varphi^+\Big(\sum_j A^+z_j^-z_j^+\Big),$$

which implies $\sum_j A^+z_j^-z_j^+ = 0$, and thus

$$0 = \tau^+\Big(\sum_j A^+z_j^-z_j^+\Big) = \sum_j \tau^+(A^+)\tau^-(z_j^-)\tau^+(z_j^+) = \tau^+(A^+)\Big(\sum_j \tau^-(z_j^-)\tau^+(z_j^+)\Big);$$

similarly $\big(\sum_j \tau^-(z_j^-)\tau^+(z_j^+)\big)\tau^+(A^+) = 0$, which implies $\sum_j \tau^-(z_j^-)\tau^+(z_j^+) = 0$ by (1.2) and (1.1)(1); in a similar manner $\sum_i \tau^+(y_i^+)\tau^-(y_i^-) = 0$, and we get that ψ is well defined.

It is clear that ψ is a surjective algebra homomorphism satisfying $\psi \circ \varphi^\sigma = \tau^\sigma$, $\sigma = \pm$. By the very definition of ψ, an element a as above lies in $\operatorname{Ker}\psi$ if and only if $a = \sum_i \varphi^+(y_i^+)\varphi^-(y_i^-) +$

$\sum_j \varphi^-(z_j^-)\varphi^+(z_j^+)$ with $\sum_i \tau^+(y_i^+)\tau^-(y_i^-) \oplus \sum_j \tau^-(z_j^-)\tau^+(z_j^+) = 0$, which is shown to be equivalent to $a\varphi^\sigma(A^\sigma) = \varphi^\sigma(A^\sigma)a = 0$, $\sigma = \pm$, using (1.1)(1). Thus $\operatorname{Ker}\psi = I$, and we can define $\tilde{\psi} : \mathscr{A}/I \longrightarrow \tilde{\mathscr{A}}$ by $\tilde{\psi}(a+I) = \psi(a)$, which turns out to be an algebra isomorphism satisfying $\tilde{\psi} \circ \phi^\sigma = \tau^\sigma$, $\sigma = \pm$.

Corollary 1.6 *Let A be a left and right faithful (for example, semiprime) associative pair, and $(\mathscr{A}, \varphi)$ be an envelope of A. Then the following are equivalent:*

(i) *$(\mathscr{A}, \varphi)$ is tight on A;*

(ii) *$\mathscr{A}$ is left and right faithful;*

(iii) *$(\mathscr{A}, \varphi)$ is isomorphic to the standard envelope of A.*

Proof Apply (1.5) together with the obvious fact that the set of left (respectively, right) total zero divisors of an algebra is an ideal.

Remark 1.7 To simplify notation, from now on, when dealing with a subpair A of $(\mathscr{A}, \varphi)$ we will assume that $A^\sigma \subseteq \mathscr{A}$, the maps φ^σ will be simply the inclusion maps, and we will write $\mathscr{A}$ instead of $(\mathscr{A}, \varphi)$. This will also be applied to the particular case of $(\mathscr{A}, \varphi)$ being an envelope of A.

2 Weak Fountain–Gould Left Orders in Associative Pairs

In this section, we introduce the notion of weak Fountain–Gould left order for associative pairs, inspired by Fountain–Gould's definition of left orders for rings given in [1].

2.1 Let a be an element of an algebra $\mathscr{R}$. We say that b in $\mathscr{R}$ is the *group inverse* of a if $aba = a, bab = b$, and $ab = ba$. It is easy to see that the group inverse is unique when it exists, and that a has a group inverse in $\mathscr{R}$ (say b) if and only if there exists an idempotent e in $\mathscr{R}$ such that a is an invertible element of $e\mathscr{R}e$ (indeed, e is unique, given by $e = ab$). An element a is said to be *locally invertible* if it has a group inverse, which will be denoted by $a^\#$. If a is locally invertible, then any power a^n of a is also locally invertible, and $(a^n)^\# = (a^\#)^n$. It is obvious that an invertible element a in a unital algebra is locally invertible with $a^\# = a^{-1}$.

2.2 A subalgebra $\mathscr{R}$ of an associative algebra $\mathscr{Q}$ is said to be a *weak Fountain–Gould left order in $\mathscr{Q}$* (or $\mathscr{Q}$ is said to be a *weak Fountain–Gould left quotient algebra of $\mathscr{R}$*) if for every element $q \in \mathscr{Q}$ there exist $a, x \in \mathscr{R}$ such that a is locally invertible in $\mathscr{Q}$ and $q = a^\# x$. Clearly, if $\mathscr{R}$ is a weak classical left order in a unital algebra $\mathscr{Q}$, then $\mathscr{R}$ is a weak Fountain–Gould left order in $\mathscr{Q}$. The converse is not true in general, as it was shown by Fountain and Gould in [1, Example 3.1].

A subalgebra $\mathscr{R}$ of an associative algebra $\mathscr{Q}$ is said to be a *Fountain–Gould left order in $\mathscr{Q}$* (or $\mathscr{Q}$ is said to be a *Fountain–Gould left quotient algebra of $\mathscr{R}$*) if $\mathscr{R}$ is a weak Fountain–Gould left order in $\mathscr{Q}$ and every square-cancellable element of $\mathscr{R}$ is locally invertible in $\mathscr{Q}$ (cf. [1]).

2.3 Let A be a subpair of an associative pair Q, which is a subpair of the algebra $\mathscr{Q}$ (in particular A is also a subpair of the algebra $\mathscr{Q}$). We will say that A is a *weak Fountain–Gould left order in Q relative to $\mathscr{Q}$* if for any $q \in Q^\sigma$ there exist $a \in A^\sigma A^{-\sigma}$ which is locally invertible in $Q^\sigma Q^{-\sigma}$, and $b \in A^\sigma$, such that $q = a^\# b$. We will also say that Q is a *weak Fountain–Gould left quotient pair of A relative to $\mathscr{Q}$.*

Remark 2.4 Under the conditions of (2.3), we can always replace $\mathscr{Q}$ by the subalgebra generated by $Q^+ \cup Q^-$, and assume that $\mathscr{Q}$ is an envelope of Q.

Remark 2.5 If A is a weak Fountain–Gould left order in Q relative to $\mathscr{Q}$, for any $q \in Q^\sigma$ we can always find a, b as in (2.3) such that also $aq = b$ [just replace a and b in (2.3) by a^2 and ab, respectively].

Examples 2.6 (i) Let $\mathscr{R}$ be a subalgebra of an associative algebra $\mathscr{Q}$. It is easy to see that $\mathscr{Q}$ is a weak Fountain–Gould left quotient algebra of $\mathscr{R}$ if and only if $V(\mathscr{Q})$ is a weak Fountain–Gould left quotient pair of the associative pair $V(\mathscr{R})$ relative to the algebra of 2×2

matrices over $\mathscr{Q}$ with the natural embedding

$$a \in V(\mathscr{Q})^+ \mapsto \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in \mathscr{M}_{2\times 2}(\mathscr{Q}), \quad b \in V(\mathscr{Q})^- \mapsto \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \in \mathscr{M}_{2\times 2}(\mathscr{Q}).$$

(ii) Let D be a weak classical left order in a division algebra Δ. Then, one can show that, for every pair (m, n) of positive integers, $(\mathscr{M}_{m\times n}(D), \mathscr{M}_{n\times m}(D))$ is a weak Fountain–Gould left order in the associative pair $(\mathscr{M}_{m\times n}(\Delta), \mathscr{M}_{n\times m}(\Delta))$ relative to the algebra $\mathscr{M}_{(m+n)\times(m+n)}(\Delta)$, considering the natural embedding given by

$$a \in \mathscr{M}_{m\times n}(\Delta) \mapsto \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in \mathscr{M}_{(m+n)\times(m+n)}(\Delta),$$

$$b \in \mathscr{M}_{n\times m}(\Delta) \mapsto \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \in \mathscr{M}_{(m+n)\times(m+n)}(\Delta).$$

2.7 The following notion was introduced by Utumi in [13]: Let $\mathscr{R}$ be a subalgebra of an associative algebra $\mathscr{Q}$. We say that $\mathscr{Q}$ is a *(general) left quotient algebra of* $\mathscr{R}$ if for every $x, y \in \mathscr{Q}$ with $x \neq 0$ there is an $a \in \mathscr{R}$ such that $ax \neq 0$ and $ay \in \mathscr{R}$. Notice that an algebra is a left quotient algebra of itself if and only if it is right faithful. If $\mathscr{R}$ is right faithful, then it has a unique maximal left quotient algebra $Q^l_{\max}(\mathscr{R})$, which is unital, called the *Utumi left quotient algebra of* $\mathscr{R}$ [13].

2.8 In [9] a pair version of the above notion is introduced. Let $A = (A^+, A^-)$ be a subpair of an associative pair $Q = (Q^+, Q^-)$. We say that Q is a *left quotient pair of* A if given $p, q \in Q^\sigma$ with $p \neq 0$, $\sigma = \pm$, there exist $a \in A^{-\sigma}$, $b \in A^\sigma$ such that $bap \neq 0$ and $baq \in A^\sigma$.

2.9 Let A be an associative pair and let $X \subset A^\sigma, \sigma = \pm$. The *left annihilator of* X *in* A is defined to be the set $\mathrm{lan}(X) = \mathrm{lan}_A(X) = \{b \in A^{-\sigma} \mid bXA^{-\sigma} = 0, A^\sigma bX = 0\}$. Similarly, the *right annihilator of* X *in* A is defined by $\mathrm{ran}(X) = \mathrm{ran}_A(X) = \{b \in A^{-\sigma} \mid XbA^\sigma = 0, A^{-\sigma}Xb = 0\}$.

Clearly, $\mathrm{lan}(X)$ is a left ideal of A and $\mathrm{ran}(X)$ is a right ideal of A.

Let A be an associative pair which is a subpair of the algebra $\mathscr{A}$. For $\sigma = \pm$ and an element $a \in A^\sigma A^{-\sigma}$ we can also consider $\mathrm{lan}_A(a) = \{x \in A^{-\sigma} \mid xa = 0\}$, (respectively, $\mathrm{ran}_A(a) = \{x \in A^\sigma \mid ax = 0\}$) which is a left (respectively, right) ideal of A.

Lemma 2.10 *Let A be a subpair of an associative pair Q, which is a subpair of the algebra $\mathscr{Q}$, and let $a \in A^\sigma A^{-\sigma}$ satisfy that there exists $a^\# \in Q^\sigma Q^{-\sigma}$. Then*

(i) $Q^{-\sigma} = Q^{-\sigma}a + \mathrm{lan}_Q(a)$.

If A is a weak Fountain–Gould left order in Q relative to $\mathscr{Q}$, then

(ii) $\mathrm{lan}_Q(a) = Q^{-\sigma}Q^\sigma\,\mathrm{lan}_A(a)$.

(iii) $Q^{-\sigma} = Q^{-\sigma}Q^\sigma(\mathrm{lan}_A(a) + A^{-\sigma}a)$.

Proof (i) If $q \in Q^{-\sigma}$, then $q = qa^\#a + (q - qa^\#a)$, where $qa^\#a \in Q^{-\sigma}a$ and $(q - qa^\#a) \in \mathrm{lan}_Q(a)$, and we have shown $Q^{-\sigma} \subseteq Q^{-\sigma}a + \mathrm{lan}_Q(a)$. The converse is obvious.

(ii) Let $q \in \mathrm{lan}_Q(a)$ and write $q = u^\#v$, with $u \in A^{-\sigma}A^\sigma$ ($u^\# \in Q^{-\sigma}Q^\sigma$), $v \in A^{-\sigma}$, and $uq = v$. Then $v \in \mathrm{lan}_Q(a) \cap A^{-\sigma} = \mathrm{lan}_A(a)$ and $q = u^\#v \in Q^{-\sigma}Q^\sigma\,\mathrm{lan}_A(a)$. The converse is obvious.

(iii) follows from (i), (ii), and the fact that $Q^{-\sigma}Q^\sigma A^{-\sigma} = Q^{-\sigma}$.

Proposition 2.11 *If A is a weak Fountain–Gould left order in an associative pair Q relative to an algebra $\mathscr{Q}$, then Q is a left quotient pair of A.*

Proof Let $p, q \in Q^\sigma$ with $p \neq 0$. We can write $q = a^\#b$, $p = c^\#d$ where $a, c \in A^\sigma A^{-\sigma}$ ($a^\#, c^\# \in Q^\sigma Q^{-\sigma}$) and $b, d \in A^\sigma$. By (2.5), we can also assume that $aq = b$. Notice that $0 \neq p = c^\#d = c^\#cc^\#d = c^\#cp \in Q^\sigma Q^{-\sigma}p$. Hence $0 \neq Q^\sigma Q^{-\sigma}p = Q^\sigma Q^{-\sigma}Q^\sigma(\mathrm{lan}_A(a) + A^{-\sigma}a)p$ by (2.10)(iii), and there exist $t \in Q^\sigma$, $x \in \mathrm{lan}_A(a)$ and $y \in A^{-\sigma}$ such that $0 \neq t(x + ya)p$. If we write $t = u^\#v$, with $u \in A^\sigma A^{-\sigma}$ ($u^\# \in Q^\sigma Q^{-\sigma}$) and $v \in A^\sigma$, then $v(x + ya)p \neq 0$ and

$$v(x + ya)q = vxq + vyaq = vxa^\#b + vyaq = vxaa^\#a^\#b + vyaq$$

$$= vyaq \ (xa = 0 \text{ since } x \in \mathrm{lan}_A(a)) \ = vyb \in A^\sigma.$$

Next we will show that, for pairs without total zero divisors, the notion of weak Fountain–Gould left order is intrinsic, independent of the "atmosphere" algebra $\mathscr{Q}$ we deal with.

Lemma 2.12 *Let Q be a left quotient pair of an associative pair A. Then A and Q are both right faithful. Moreover,*

(i) *If A is left faithful, then Q is left faithful*;

(ii) *If A is semiprime, then Q is semiprime.*

Proof Let $0 \neq q \in Q^\sigma$. By the very definition of left quotient pair, we can find $a \in A^\sigma, b \in A^{-\sigma}$ such that $0 \neq abq \in A^\sigma$. But $abq \in A^\sigma A^{-\sigma} q \subseteq Q^\sigma Q^{-\sigma} q$, which shows the first assertion.

(i) If $qQ^{-\sigma}Q^\sigma = 0$, then $abqA^\sigma A^{-\sigma} \subseteq abqQ^{-\sigma}Q^\sigma = 0$, which is a contradiction if A is left faithful.

(ii) is [9, 2.13(ii)].

Lemma 2.13 *Let $\mathscr{R}$ be an associative algebra, and I be an ideal of $\mathscr{R}$ such that $\mathscr{R}I = I\mathscr{R} = 0$. If $x \in \mathscr{R}/I$ is locally invertible in $\mathscr{R}/I$ with group inverse $x^\# \in \mathscr{R}/I$, then there exists a locally invertible element $a \in \mathscr{R}$ such that $x = a + I$ and $x^\# = a^\# + I$. Indeed, if $x = u + I$, and $x^\# = v + I$, we can set $a = uvu$, $a^\# = vuv$.*

Proof Let $x = u+I$, $x^\# = v+I$, where $u, v \in \mathscr{R}$. By the definition of group inverse, we obtain

$$uv - vu \in I, \qquad uvu - u \in I, \qquad vuv - v \in I. \tag{1}$$

Let $a = uvu$, $a' = vuv$. By (1), $x = u + I = uvu + I = a + I$, $x^\# = v + I = vuv + I = a' + I$. Moreover,

$$\begin{aligned} a'a &= vuvuvu = vu(vuv - v)u + vuvu = vuvu \ [\text{since } vu(vuv - v)u \in \mathscr{R}I\mathscr{R} = 0] \\ &= vu(vu - uv) + vuuv = vuuv \ [\text{since } vu(vu - uv) \in \mathscr{R}I = 0] \\ &= (vu - uv)uv + uvuv = uvuv \ [\text{since } (vu - uv)uv \in I\mathscr{R} = 0] \\ &= (u - uvu)vuv + uvuvuv = uvuvuv \ [\text{since } (u - uvu)vuv \in I\mathscr{R} = 0] \ = aa'. \end{aligned}$$

Similarly, $a'aa' = a'$ and $aa'a = a$, which shows a' is the group inverse of a.

Proposition 2.14 *Let A be a subpair of an associative pair Q such that either A or Q is left faithful (for example, if either A or Q is semiprime), and let $\mathscr{Q}$ be an associative algebra such that Q is a subpair of $\mathscr{Q}$. Then A is a weak Fountain–Gould left order in Q relative to $\mathscr{Q}$ if and only if A is a weak Fountain–Gould left order in Q relative to the standard envelope $\tilde{\mathscr{Q}}$ of Q.*

Proof Notice that (2.11) and (2.12) imply that Q is left and right faithful in any case. By (2.4), we can assume that $\mathscr{Q}$ is an envelope of Q, and, by (1.5) there exists an ideal I of $\mathscr{Q}$ not hitting Q such that $\mathscr{Q}/I$ and $\tilde{\mathscr{Q}}$ are isomorphic envelopes of Q, and $\mathscr{Q}I = I\mathscr{Q} = 0$.

If A is a weak Fountain–Gould left order in Q relative to $\mathscr{Q}$, then it is straightforward that A is a weak Fountain–Gould left order in Q relative to $\mathscr{Q}/I$, i.e., A is a weak Fountain–Gould left order in Q relative to $\tilde{\mathscr{Q}}$.

Conversely, assume that A is a weak Fountain–Gould left order in Q relative to $\tilde{\mathscr{Q}}$, i.e., A is a weak Fountain–Gould left order in Q relative to $\mathscr{Q}/I$. Thus, given $q \in Q^\sigma$, there exist $u \in A^\sigma A^{-\sigma}$, $b \in A^\sigma$ such that $u+I$ is locally invertible, $(u+I)^\# = v+I$ with $v \in Q^\sigma Q^{-\sigma}$, and $q + I = (u+I)^\#(b+I)$. By (2.13), $a = uvu \in Q^\sigma Q^{-\sigma}$ is locally invertible with $a^\# = vuv \in Q^\sigma Q^{-\sigma}$, and $u + I = a + I$, $(u+I)^\# = v + I = a^\# + I$. Thus $q - a^\# b \in I \cap Q^\sigma = 0$, i.e., $q = a^\# b$. The problem is that we do not know whether a is in $A^\sigma A^{-\sigma}$. Anyway,

$$\begin{aligned} a^2 &= au + a(a-u) = au \ [\text{since } a(a-u) \in \mathscr{Q}I = 0] \\ &= u^2 + (a-u)u = u^2 \ [\text{since } (a-u)u \in I\mathscr{Q} = 0] \ \in A^\sigma A^{-\sigma}, \\ ab &= ub + (a-u)b = ub \ [\text{since } (a-u)b \in I\mathscr{Q} = 0] \ \in A^\sigma A^{-\sigma} A^\sigma \subseteq A^\sigma, \end{aligned}$$

and $q = a^\# b = (a^\#)^2 ab$, where $(a^2)^\# = (a^\#)^2 \in Q^\sigma Q^{-\sigma}$.

Remark 2.15 Under the conditions of (2.14), there is no need to specify the algebra $\mathscr{Q}$ where the calculations are made, and we will simply say that A is a weak Fountain–Gould left order in Q.

3 Weak Fountain–Gould Left Orders in Semiprime Pairs Coinciding with their Socles

3.1 For an associative pair, different reasonable notions of socle can be considered (see [14]). However, we will restrict ourselves to semiprime pairs, where these notions coincide [14, 3.7]: Given a semiprime associative pair A the *socle* $\mathrm{Soc}(A)$ of A is the pair $(\mathrm{Soc}(A)^+, \mathrm{Soc}(A)^-)$, where $\mathrm{Soc}(A)^\sigma$ is the sum of all minimal left ideals of A contained in A^σ ($\mathrm{Soc}(A)^\sigma = 0$ if A does not have minimal left ideals contained in A^σ), and $\mathrm{Soc}(A)^\sigma$ coincides with the sum of all minimal right or inner ideals of A contained in A^σ.

Lemma 3.2 *Let Q be an associative pair, and $\mathscr{Q}$ be its standard envelope. Then Q is semiprime if and only if $\mathscr{Q}$ is semiprime.*

Proof Let $\mathscr{E}$ be the associative algebra built out of Q, as in (1.1), so that $\mathscr{Q}$ is an ideal of $\mathscr{E}$ (1.2).

If Q is semiprime, then $\mathscr{E}$ is semiprime by [5, 4.2] and $\mathscr{Q}$ is semiprime because it is an ideal of $\mathscr{E}$. Conversely, if $\mathscr{Q}$ is semiprime, we will show that Q is so. Indeed, let us show that Q is nondegenerate (0.3): If $q \in Q^\sigma$ satisfies $0 = qQ^{-\sigma}q$, then $q\mathscr{Q}q = q(Q^\sigma + Q^{-\sigma} + Q^\sigma Q^{-\sigma} + Q^{-\sigma}Q^\sigma)q = qQ^{-\sigma}q = 0$ since $qQ^\sigma, Q^\sigma q \subseteq Q^\sigma Q^\sigma = 0$; thus $q = 0$ by the semiprimeness (nondegeneracy) of $\mathscr{Q}$.

Lemma 3.3 *Let Q be a semiprime associative pair, $\mathscr{Q}$ its standard envelope. Then* $\mathrm{Soc}(Q)^\sigma =$ $\mathrm{Soc}(\mathscr{Q}) \cap Q^\sigma$*. Moreover, $Q = Soc(Q)$ if and only if $\mathscr{Q} = \mathrm{Soc}(\mathscr{Q})$, and, in this case, $Q^\sigma Q^{-\sigma}$ ($\sigma = \pm$) is a semiprime algebra which coincides with its socle.*

Proof Let $\mathscr{E}$ be the associative algebra built out of Q, as in (1.1), so that $\mathscr{Q}$ is an ideal of $\mathscr{E}$ (1.2). By [5, 4.2], $\mathscr{E}$ is semiprime, hence $\mathrm{Soc}(\mathscr{Q}) = \mathrm{Soc}(\mathscr{E}) \cap \mathscr{Q}$ using (3.1), and [15, Prop. 3] applied to $V(\mathscr{E})^{(+)}$, together with [14, 2.1, 4.2]. Thus $\mathrm{Soc}(\mathscr{Q}) \cap Q^\sigma = \mathrm{Soc}(\mathscr{E}) \cap \mathscr{Q} \cap Q^\sigma = \mathrm{Soc}(\mathscr{E}) \cap Q^\sigma = \mathrm{Soc}(Q)^\sigma$ by [5, 4.7(i)].

Obviously, the above equality shows $Q = \mathrm{Soc}(Q)$ when $\mathscr{Q} = \mathrm{Soc}(\mathscr{Q})$. If, conversely, $Q = \mathrm{Soc}(Q)$, then $\mathrm{Soc}(\mathscr{Q})$ is an ideal of $\mathscr{Q}$, containing $Q^+ \cup Q^-$, hence $\mathscr{Q} = \mathrm{Soc}(\mathscr{Q})$ since $Q^+ \cup Q^-$ generates $\mathscr{Q}$ as an algebra.

Assume now that $\mathscr{Q} = \mathrm{Soc}(\mathscr{Q})$, and let e be the idempotent of $\mathscr{E}$ given in (1.1). Now $e\mathscr{E}e$, $(1-e)\mathscr{E}(1-e)$ are semiprime algebras (using (0.3)), and $Q^+Q^- = \mathscr{Q} \cap e\mathscr{E}e$, $Q^-Q^+ = \mathscr{Q} \cap (1-e)\mathscr{E}(1-e)$ are ideals of them, hence semiprime. Moreover, again by (3.1) and [15, Prop. 3] we get

$$\mathrm{Soc}(Q^+Q^-) = \mathrm{Soc}(e\mathscr{E}e) \cap Q^+Q^-,\ \mathrm{Soc}(Q^-Q^+) = \mathrm{Soc}((1-e)\mathscr{E}(1-e)) \cap Q^-Q^+. \tag{1}$$

On the other hand, [15, Prop. 3] also applies to full subpairs, which, again together with (3.1), yields

$$\mathrm{Soc}(e\mathscr{E}e) = \mathrm{Soc}(\mathscr{E}) \cap e\mathscr{E}e,\ \mathrm{Soc}((1-e)\mathscr{E}(1-e)) = \mathrm{Soc}(\mathscr{E}) \cap (1-e)\mathscr{E}(1-e). \tag{2}$$

Finally, using (1) and (2),

$$\begin{aligned}\mathrm{Soc}(Q^+Q^-) &= \mathrm{Soc}(e\mathscr{E}e) \cap Q^+Q^- = \mathrm{Soc}(\mathscr{E}) \cap e\mathscr{E}e \cap Q^+Q^- \\ &= \mathrm{Soc}(\mathscr{E}) \cap Q^+Q^- \supseteq \mathrm{Soc}(\mathscr{Q}) \cap Q^+Q^- = \mathscr{Q} \cap Q^+Q^- = Q^+Q^-,\end{aligned}$$

hence $\mathrm{Soc}(Q^+Q^-) = Q^+Q^-$, and, similarly, $\mathrm{Soc}(Q^-Q^+) = Q^-Q^+$.

Theorem 3.4 *Let A be a weak Fountain–Gould left order in a simple associative pair Q coinciding with its socle. Then A is a prime associative pair.*

Proof By (2.14), we may assume that A is a weak Fountain–Gould left order in Q relative to the standard envelope $\mathscr{Q}$ of Q.

Let a, c be nonzero elements of A^σ. Since Q is simple, it is prime. Hence there is an element $q \in Q^{-\sigma}$ satisfying $aqc \neq 0$ (0.3). On the other hand, $q = u^\# v$, where $u \in A^{-\sigma}A^\sigma$ ($u^\# \in Q^{-\sigma}Q^\sigma$), $v \in A^{-\sigma}$, and $uu^\# v = v$ (2.5). Then $0 \neq aqc = au^\# vc$. By the semiprimeness of Q (0.3), $au^\# vcQ^{-\sigma} \neq 0$ and, since $Q = \mathrm{Soc}(Q)$, there exists an element t in a minimal right ideal I of Q contained in $Q^{-\sigma}$, satisfying $0 \neq au^\# vct \in Q^{-\sigma}Q^\sigma$.

Let us consider the set $\Gamma = \{u^n u^\# vctQ^\sigma \mid n = 1, 2, \dots\}$ of right ideals of $V := V(Q^{-\sigma}Q^\sigma)$ contained in $uV^{-\sigma}V^\sigma = uQ^{-\sigma}Q^\sigma Q^{-\sigma}Q^\sigma$. Notice that all the right ideals in Γ are nonzero ($0 \neq u^\# vct = (u^\#)^n u^n u^\# vct \Longrightarrow u^n u^\# vct \neq 0 \Longrightarrow u^n u^\# vctQ^\sigma \neq 0$ by the semiprimeness of Q). On the other hand, $Q^{-\sigma}Q^\sigma$ is a semiprime algebra coinciding with its socle by (3.3), so that V is a semiprime pair coinciding with its socle [14, 2.1, 4.2]. Hence, by [5, 5.2(vi)], the sum of the elements of Γ cannot be direct, and so there exist $\alpha_1, \dots, \alpha_m \in Q^\sigma$ such that

$$\sum_{i=1}^{m} u^i u^\# vct\alpha_i = 0, \tag{1}$$

and at least one summand is nonzero. Let $u^k u^\# vct\alpha_k$ be the first nonzero summand. By multiplying (1) by $(u^\#)^k$ on the left and renumbering the α's, we obtain

$$u^\# vct\alpha_1 + uu^\# vct\alpha_2 + \cdots + u^{r-1}u^\# vct\alpha_r = 0, \tag{2}$$

with $u^\# vct\alpha_1 \neq 0$. On the other hand, $t\alpha_1$ is a nonzero element in the semiprime algebra $Q^{-\sigma}Q^\sigma$, hence $t\alpha_1 Q^{-\sigma}Q^\sigma \neq 0$, which implies that $t\alpha_1 Q^{-\sigma}$ is a nonzero right ideal of Q contained in I. By the minimality, $I = t\alpha_1 Q^{-\sigma}$, hence $t = t\alpha_1\alpha$ for some $\alpha \in Q^\sigma$. Multiplying (2) by a on the left and by α on the right we obtain

$$au^\# vct\alpha_1\alpha + auu^\# vct\alpha_2\alpha + \cdots + au^{r-1}u^\# vct\alpha_r\alpha = 0,$$

i.e.,

$$au^\# vct + auu^\# vct\alpha_2\alpha + \cdots + au^{r-1}u^\# vct\alpha_r\alpha = 0. \tag{3}$$

Then

$$\begin{aligned} 0 \neq -au^\# vct &= auu^\# vct\alpha_2\alpha + \cdots + au^{r-1}u^\# vct\alpha_r\alpha \\ &= avct\alpha_2\alpha + \cdots + au^{r-2}vct\alpha_r\alpha \end{aligned} \tag{4}$$

since $uu^\# v = v$. Thus $au^k vct\alpha_{k+2}\alpha \neq 0$ for some $k \in \{0, \dots, r-2\}$ and, consequently, $0 \neq au^k vc \in aA^{-\sigma}c$, which proves that A is prime (0.3).

Corollary 3.5 *Let A be a weak Fountain–Gould left order in a semiprime associative pair Q coinciding with its socle. Then A is semiprime.*

Proof By [16, Theorem 1], Q is a direct sum of ideals Q_i ($i \in I$), where Q_i is simple and coincides with its socle. For every $i \in I$, let $\pi_i = (\pi_i^+, \pi_i^-) : Q \to Q_i$ be the canonical projection. It is easy to see that $\pi_i(A)$ is a weak Fountain–Gould left order in Q_i and, by (3.4), we have that $\pi_i(A)$ is a prime algebra, for any $i \in I$.

We will show that A is nondegenerate (0.3): Given $0 \neq x \in A^\sigma$, there exists $j \in I$ such that $\pi_j^\sigma(x) \neq 0$. By the primeness of $\pi_j(A)$, $0 \neq \pi_j^\sigma(x)(\pi_j(A))^{-\sigma}\pi_j^\sigma(x) = \pi_j^\sigma(x)\pi_j^{-\sigma}(A^{-\sigma})\pi_j^\sigma(x) = \pi_j^\sigma(xA^{-\sigma}x)$, which implies $xA^{-\sigma}x \neq 0$.

In the next result we express the fact of A being a weak Fountain–Gould left order in a semiprime pair Q coinciding with its socle in terms of the algebras naturally attached to these pairs.

Remark 3.6 We claim that [5, 8.7, 8.8] remain true when replacing "A being a left triple product order in Q" by "Q being a left quotient pair of A". Indeed, the proofs given in [5] remain valid under this new condition, which can be readily checked (also cf. [9, 2.14]).

Theorem 3.7 *Let A be a subpair of a semiprime associative pair Q which coincides with its socle, and $\mathscr{Q}$ be the standard envelope of Q. Let $\mathscr{A}$ be the subalgebra of $\mathscr{Q}$ generated by $A^+ \cup A^-$, i.e., $\mathscr{A} = A^+ + A^+A^- + A^-A^+ + A^-$. The following are equivalent conditions:*

(i) *A is a weak Fountain–Gould left order in Q;*

(ii) *A is semiprime, $Q^\sigma = A^\sigma Q^{-\sigma}A^\sigma$, $\sigma = \pm$, and Q is a left quotient pair of A;*

(iii) *A is semiprime, $Q^\sigma = A^\sigma Q^{-\sigma}A^\sigma$ and, for every $a \in A^{-\sigma}$, A_a^σ is a classical left order in Q_a^σ, $\sigma = \pm$;*

(iv) *$\mathscr{A}$ is a Fountain–Gould left order in $\mathscr{Q}$;*

(v) *$A^\sigma A^{-\sigma}$ is a Fountain–Gould left order in $Q^\sigma Q^{-\sigma}$, for $\sigma = \pm$.*

Moreover, under any of the above conditions, $Q^\sigma = Q^\sigma A^{-\sigma}A^\sigma = A^\sigma A^{-\sigma}Q^\sigma, \sigma = \pm$, and $\mathscr{A}$ is isomorphic to the standard envelope of A.

Proof (i)⇒(ii) By (3.5), A is a semiprime associative pair. For any $q \in Q^\sigma$, there exist $a \in A^\sigma A^{-\sigma}$, $b \in A^\sigma$, with $a^\# \in Q^\sigma Q^{-\sigma}$ such that $q = a^\# b = a(a^\#)^2 b \in A^\sigma A^{-\sigma} Q^\sigma Q^{-\sigma} A^\sigma = A^\sigma (A^{-\sigma} Q^\sigma Q^{-\sigma}) A^\sigma \subseteq A^\sigma Q^{-\sigma} A^\sigma$, and Q is a left quotient pair of A by (2.11). Moreover, $q = a^\# b = (a^\#)^2 ab \in Q^\sigma Q^{-\sigma} A^\sigma A^{-\sigma} A^\sigma = (Q^\sigma Q^{-\sigma} A^\sigma) A^{-\sigma} A^\sigma \subseteq Q^\sigma A^{-\sigma} A^\sigma$, and $q = a^\# b = a(a^\#)^2 b \in A^\sigma A^{-\sigma} Q^\sigma Q^{-\sigma} A^\sigma = A^\sigma A^{-\sigma}(Q^\sigma Q^{-\sigma} A^\sigma) \subseteq A^\sigma A^{-\sigma} Q^\sigma$.

(ii)⇒(iii) follows from [5, 8.8(ii)] and (3.6).

(iii)⇒(ii) First we claim that

(1) If $0 \neq aqc$, where $a, c \in A^{-\sigma}$, $q \in Q^\sigma$, then the set $\Lambda_{aqc} = \{aba \in A^\sigma_a \mid abaqc \in A^{-\sigma}\}$ is an essential left ideal of A^σ_a.

Indeed Λ_{aqc} is clearly a left ideal of A^σ_a. Any nonzero left ideal of A^σ_a has the form aIa, where $I \subseteq A^\sigma$. If $aIaqc = 0$, then $aIa \subseteq \Lambda_{aqc}$, hence aIa hits Λ_{aqc}. Assume $y \in I$ satisfies $ayaqc \neq 0$. By the semiprimeness of Q (0.3), $ayaqcpayaqc \neq 0$ for some $p \in Q^\sigma$, hence $cpayaqc \neq 0$. Since A^σ_c is a classical left order in Q^σ_c, $cpayaqc = c\bar{u}cvc$, for some elements $cuc, cvc \in A^\sigma_c$, such that cuc is regular in A^σ_c, and $c\bar{u}c$ is the inverse of cuc in Q^σ_c. Then $0 \neq cucpayaqc = cvc \in A^\sigma_c$. By [16, Theorem 1], Q is a direct sum of ideals Q_α ($\alpha \in J$), where the Q_α's are simple and coincide with their socles. Let $\pi_\alpha = (\pi^+_\alpha, \pi^-_\alpha) : Q \to Q_\alpha$ be the canonical projection, $\alpha \in J$. For every $w \in A^{-\sigma}$, $\pi^{-\sigma}_\alpha$ induces $\mu_\alpha : Q^\sigma_w \longrightarrow Q^\sigma_{\pi^{-\sigma}_\alpha(w)} = (Q_\alpha)^\sigma_{\pi^{-\sigma}_\alpha(w)}$, an algebra epimorphism . Now, [8, 4.6] is applied to obtain that $\mu_\alpha(A^\sigma_w) = \pi^\sigma_\alpha(A^\sigma)_{\pi^{-\sigma}_\alpha(w)}$ is a classical left order in $(Q_\alpha)^\sigma_{\pi^{-\sigma}_\alpha(w)}$, whenever $\pi^{-\sigma}_\alpha(w) \neq 0$. In particular $\pi^\sigma_\alpha(A^\sigma)_{\pi^{-\sigma}_\alpha(w)} \neq 0$ if $\pi^{-\sigma}_\alpha(w) \neq 0$, which implies that $\pi_\alpha(A)$ is semiprime (0.3). Since Q_α is simple for any $\alpha \in J$, by [5, 5.2(iii)(v)], $(Q_\alpha)^\sigma_{\pi^{-\sigma}_\alpha(w)}$ is a simple artinian algebra, and, by Goldie's First Theorem [17, 3.2.16], $\pi^\sigma_\alpha(A^\sigma)_{\pi^{-\sigma}_\alpha(w)}$ is a prime algebra, whenever $0 \neq \pi^{-\sigma}_\alpha(w)$, which implies that $\pi_\alpha(A)$ is prime, using [5, 5.2(ii)]. Choose $\beta \in J$, such that $0 \neq \pi^{-\sigma}_\beta(cvc) = \pi^{-\sigma}_\beta(cucpayaqc)$. Then $\pi^{-\sigma}_\beta(a) \neq 0$, and, by the primeness (0.3) of $\pi_\beta(A)$, there exists $x \in A^\sigma$ such that $0 \neq \pi^{-\sigma}_\beta(a)\pi^\sigma_\beta(x)\pi^{-\sigma}_\beta(cvc) = \pi^{-\sigma}_\beta(axcvc)$. In particular,

$$axcvc = axcucpayaqc \neq 0. \tag{2}$$

Take $ara, asa \in A^\sigma_a$, such that ara is regular in A^σ_a and $axcucpa = a\bar{r}asa$, where $a\bar{r}a$ denotes the inverse of ara in Q^σ_a. Then $araxcucpa = asa$, and (2) yields

$$0 \neq araxcucpayaqc = araxcvc \in araxcA^\sigma c \subseteq A^{-\sigma}$$

and $araxcucpaya = asaya$ is a nonzero element in $\Lambda_{aqc} \cap aIa$, which proves (1).

Now we can prove that Q is a left quotient pair of A: Let p, q be in Q^σ, with $p \neq 0$. Since Q is semiprime $Q^\sigma Q^{-\sigma} p \neq 0$ and since $Q^\sigma = A^\sigma Q^{-\sigma} A^\sigma$, $\sigma = \pm$, then $0 \neq Q^\sigma Q^{-\sigma} p = Q^\sigma A^{-\sigma} Q^\sigma A^{-\sigma} p = Q^\sigma A^{-\sigma} A^\sigma Q^{-\sigma} A^\sigma A^{-\sigma} p$. Hence there exist $a \in A^\sigma$, $b \in A^{-\sigma}$ such that $abp \neq 0$. If $abq = 0$, we are done. Otherwise, we can use the fact that $q \in Q^\sigma = A^\sigma Q^{-\sigma} A^\sigma$ to find $q_j \in Q^{-\sigma}$, $d_j \in A^\sigma$ such that $abq = \sum_{j=1}^n aq_jd_j$, and $aq_jd_j \neq 0$ for any $j = 1, \dots, n$. By (1), the $\Lambda_{aq_jd_j}$'s are essential left ideals of $A^{-\sigma}_a$ and, consequently, $\Lambda = (\cap_{j=1}^n \Lambda_{aq_jd_j})$ is an essential left ideal of $A^{-\sigma}_a$. Applying Goldie's Second Theorem [17, 3.2.14], there exists $aua \in \Lambda$ such that aua is regular in $A^{-\sigma}_a$. Then $auabq = \sum_{j=1}^n auaq_jd_j \in A^\sigma$ because $aua \in \Lambda_{aq_jd_j}$ for every j, and $auabp \neq 0$ ($0 \neq abp \Longrightarrow 0 \neq abpQ^{-\sigma}abp$ by the semiprimeness, but $abpQ^{-\sigma}abp = a\bar{u}auabpQ^{-\sigma}abp$, because $a\bar{u}a$ is the inverse of aua in $Q^{-\sigma}_a$).

(ii)⇒(iv) By (3.2), the algebra $\mathscr{Q}$ is semiprime, and by (3.3), it coincides with its socle. Since A is semiprime, by the proof of [9, 2.5(i)(iii)] $\mathscr{Q}$ is a left quotient algebra of $\mathscr{A}$, and the latter is isomorphic to the standard envelope of A, which is semiprime using (3.2). The equality $Q^\sigma = A^\sigma Q^{-\sigma} A^\sigma$ readily implies $\mathscr{Q} = \mathscr{A}\mathscr{Q}$, hence $\mathscr{A}$ is a Fountain–Gould left order in $\mathscr{Q}$ by [8, 4.11].

(iv)⇒(v) By (3.3), $Q^\sigma Q^{-\sigma}$ is semiprime and coincides with its socle. Moreover, by $\mathscr{A}$ being a Fountain–Gould left order in $\mathscr{Q}$, we have

$$\mathscr{Q} = \mathscr{A}\mathscr{Q}, \tag{1}$$

which yields

$$Q^\sigma = A^\sigma Q^{-\sigma} Q^\sigma + A^\sigma A^{-\sigma} Q^\sigma \subseteq A^\sigma Q^{-\sigma} Q^\sigma. \tag{2}$$

Hence

$$Q^\sigma Q^{-\sigma} = A^\sigma Q^{-\sigma} + A^\sigma A^{-\sigma} Q^\sigma Q^{-\sigma} \text{ (by (1)) } \subseteq A^\sigma A^{-\sigma} Q^\sigma Q^{-\sigma} \subseteq Q^\sigma Q^{-\sigma} \tag{3}$$

by (2). Similarly, $\mathscr{Q} = \mathscr{Q}\mathscr{A}$ implies $Q^\sigma Q^{-\sigma} = Q^\sigma Q^{-\sigma} A^\sigma A^{-\sigma}$, which implies, using (3), that

$$Q^\sigma Q^{-\sigma} = A^\sigma A^{-\sigma} Q^\sigma Q^{-\sigma} A^\sigma A^{-\sigma}.$$

For every $0 \neq a \in A^\sigma A^{-\sigma}$, $(A^\sigma A^{-\sigma})_a = \mathscr{A}_a$ is a classical left order in $\mathscr{Q}_a = (Q^\sigma Q^{-\sigma})_a$ [8, 4.7], hence $A^\sigma A^{-\sigma}$ is a Fountain–Gould left order in $Q^\sigma Q^{-\sigma}$ by [8, 4.11].

(v)⇒(i) Let $p \in Q^\sigma$. By (3.3), $\mathscr{Q} = \mathrm{Soc}(\mathscr{Q})$. In particular $\mathscr{Q}$ is von Neumann regular [15, Th. 1], hence there exists $q \in \mathscr{Q}$ such that $p = pqp$. Notice that this equality also holds if we replace q by its component in $Q^{-\sigma}$, thus we can assume that $q \in Q^{-\sigma}$. Now, $qp = a^\# b$ for some $a, b \in A^{-\sigma} A^\sigma$ such that $a^\# \in Q^{-\sigma} Q^\sigma$. Let $b_i \in A^{-\sigma}$ and $c_i \in A^\sigma$ satisfy $b = \sum_{i=1}^n b_i c_i$. Then $p = pqp = pa^\# b = \sum_{i=1}^n pa^\# b_i c_i$. Using [4, Theorem 5], $pa^\# b_i = u^\# v_i$, $i = 1, \ldots, n$, for some $u, v_i \in A^\sigma A^{-\sigma}$ such that $u^\# \in Q^\sigma Q^{-\sigma}$. Then $p = u^\# (\sum_{i=1}^n v_i c_i)$, which proves (i).

Now we establish an analogue of [4, Theorem 5] for pairs. The way we state the result is due to the fact that, unlike the algebra case, having a common denominator for two elements does not seem to imply the property for an arbitrary finite family.

Corollary 3.8 (Common Denominator Property) *Let A be a weak Fountain–Gould left order in a semiprime associative pair Q which coincides with its socle. Then, given any finite number of elements $p_1, \ldots, p_n \in Q^\sigma$, there exist $a \in A^\sigma A^{-\sigma}$, $b_1, \ldots, b_n \in A^\sigma$ such that a is locally invertible in $Q^\sigma Q^{-\sigma}$, and $p_i = a^\# b_i$, for $i = 1, \ldots, n$.*

Proof We can write $p_i = (u_i)^\# v_i$, with $u_i \in A^\sigma A^{-\sigma}$ and $v_i \in A^\sigma$. By (3.7), $A^\sigma A^{-\sigma}$ is a Fountain–Gould left order in $Q^\sigma Q^{-\sigma}$, and by [4, Theorem 5], given $(u_1)^\#, \ldots, (u_n)^\#$ there exist $a, c_1, \ldots, c_n \in A^\sigma A^{-\sigma}$ such that $(u_i)^\# = a^\# c_i$, $i = 1, \ldots, n$. Hence, if we define $b_i := c_i v_i$, we have $p_i = a^\# b_i$, for $i = 1, \ldots, n$.

We finally express the fact of A being a weak Fountain–Gould left order in a semiprime pair Q coinciding with its socle in intrinsic pair terms and Goldie theory notions [5].

Theorem 3.9 *For an associative pair A the following conditions are equivalent*:

(i) *A is a weak Fountain–Gould left order in a semiprime associative pair Q coinciding with its socle*;

(ii) *A is semiprime, satisfies the ascending chain condition on $\mathrm{lan}_A(x)$, with $x \in A^\sigma$, $\sigma = \pm$, and has finite left local Goldie dimension*;

(iii) *A is a semiprime left local Goldie associative pair*;

(iv) *A is semiprime and all its local algebras at nonzero elements are left Goldie*;

(v) *A is semiprime and there is a semiprime associative pair coinciding with its socle which is a left quotient pair of A.*

In this case : (1) *A is prime if and only if Q is simple*; *and,*

(2) *A is left Goldie if and only if Q is artinian.*

Proof (i)⇒(ii) By (3.5) A is semiprime. Moreover since Q is a left quotient pair by (2.11), it follows from [5, 8.8(i)] and (3.6) that A is left local Goldie, hence it has finite left local Goldie dimension. By [5, 8.7(ii), 2.5(1)(iii)] and (3.6), A satisfies the ascending chain condition on the left annihilators of a single element.

(ii)⇒(iii) is a consequence of [5, 3.6].

(iii)⇒(i) Let $\mathscr{A}$ be the standard envelope of A.

We will first show that $\mathscr{A}$ is a semiprime left local Goldie associative algebra: By (3.2) and [9, 1.9], the standard envelope $\mathscr{A}$ of A is semiprime and left nonsingular. On the other hand, for every $a \in A^\sigma$, the Goldie (or uniform) dimension $\text{u-dim}_{\mathscr{A}}(a)$ of a in $\mathscr{A}$ equals the uniform dimension $\text{u-dim}(\mathscr{A}_a)$ of $\mathscr{A}_a$ by [8, 2.1(iv)], and we remark $A_a^{-\sigma} = \mathscr{A}_a$. Moreover, by [5, 5.2(iv)], $\text{u-dim}(A_a^{-\sigma}) = \text{u-dim}_A(a) < \infty$ since A is left local Goldie. Thus we have shown that $\text{u-dim}_{\mathscr{A}}(a) < \infty$, for any $a \in A^\sigma$, i.e., $A^+ \cup A^- \subseteq I(\mathscr{A})$, where $I(\mathscr{A})$ denotes the set of elements of $\mathscr{A}$ of finite uniform dimension. But $I(\mathscr{A})$ is an ideal of $\mathscr{A}$ by [18, Prop. 1; 9, 3.2(iii)], and $A^+ \cup A^-$ generate $\mathscr{A}$, hence the whole algebra $\mathscr{A}$ is contained in $I(\mathscr{A})$, i.e., $\mathscr{A} = I(\mathscr{A})$.

Let $\mathscr{Q} = Q^l_{\max}(\mathscr{A})$. By [8, 4.9(i)] $\mathscr{A}$ is a Fountain–Gould left order in $\mathscr{A}\mathscr{Q}$, which is a semiprime algebra coinciding with its socle.

Let $\mathscr{E}$ be the algebra built out of A, as in (1.1), having a unit element 1 and an idempotent e such that $(A^+, A^-) = (e\mathscr{E}(1-e), (1-e)\mathscr{E}e)$. By [9, 2.9(i)], $\mathscr{Q} = Q^l_{\max}(\mathscr{E})$, and $e, 1-e \in \mathscr{Q}$. Let $Q := (e\mathscr{A}\mathscr{Q}(1-e), (1-e)\mathscr{A}\mathscr{Q}e)$. Notice that $A \subseteq Q$, and $Q^+ \cup Q^- \subseteq \mathscr{A}\mathscr{Q}$ since $e\mathscr{A}, (1-e)\mathscr{A} \subseteq \mathscr{A}$.

We claim that

(a) Q is semiprime and $Q = \mathrm{Soc}(Q)$.

Let x be a nonzero element in Q^σ, $\sigma \in \{+,-\}$. Since $\mathscr{A}\mathscr{Q}$ is semiprime and coincides with its socle, $(\mathscr{A}\mathscr{Q})_x$ is a (nonzero) semiprime artinian algebra by [8, 2.1(i)(v)]. But $Q_x^{-\sigma} = (\mathscr{A}\mathscr{Q})_x$, hence (a) follows by [5, 5.2(v)].

(b) A is a weak Fountain–Gould left order in Q.

By [9, 2.9, 2.11], $Q^l_{max}(A) = (e\mathscr{Q}(1-e), (1-e)\mathscr{Q}e)$. Thus $A \subseteq Q \subseteq Q^l_{max}(A)$, which obviously implies that Q is a left quotient pair of A. Now, $\mathscr{A}\mathscr{Q} = \mathscr{A}\mathscr{A}\mathscr{Q}\mathscr{A}$, because $\mathscr{A}$ is a Fountain–Gould left order in $\mathscr{A}\mathscr{Q}$ [for any $x \in \mathscr{A}\mathscr{Q}$, $x = a^\# b$, where $a \in \mathscr{A}$, $a^\# \in \mathscr{A}\mathscr{Q}$, $b \in \mathscr{A}$, hence $x = a(a^\#)^2 b \in \mathscr{A}\mathscr{A}\mathscr{Q}\mathscr{A}$], hence, $Q^\sigma = A^\sigma Q^{-\sigma} A^\sigma$ [for example $Q^+ = e\mathscr{A}\mathscr{Q}(1-e) = e\mathscr{A}\mathscr{A}\mathscr{Q}\mathscr{A}(1-e) = e(A^+ + A^+A^-)\mathscr{A}\mathscr{Q}(A^-A^+ + A^+)(1-e) \subseteq eA^+\mathscr{A}\mathscr{Q}A^+(1-e) = A^+\mathscr{A}\mathscr{Q}A^+ = A^+(1-e)\mathscr{A}\mathscr{Q}eA^+ = A^+Q^-A^+$]. Now, (b) follows from (3.7).

(iii)⇔(iv) follows from [5, 5.2(iv), 5.3].

(v)⇒(iii) is just [5, 8.8(i)] together with (3.6).

(i)⇒(v) follows from (2.11) and (3.5).

Finally (1) and (2) are consequences of [5, 8.8], (2.11), and (3.6), since A is left local Goldie.

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