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Jordan centers and Martindale-like covers

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Abstract

In this paper we show that the scalar center of a nondegenerate quadratic Jordan algebra is contained in the scalar center of any of its Martindale-like covers.

Keywords: Jordan algebra; Scalar center; Martindale-like cover

Introduction

The notion of (weak) scalar center, introduced by Fulgham in [3], has revealed a central tool in the study of Martindale-like quotients [1,4] of linear Jordan algebras mainly due to two facts:

(i) any nonzero ideal of a nondegenerate PI Jordan algebra contains nonzero central elements [2, 3.6], and

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(ii) the scalar center of a nondegenerate linear Jordan algebra is contained in the scalar center of any of its Martindale-like algebras of quotients [1, 4.1].

Our aim in this paper is showing that the general (quadratic) version of (ii) holds. Indeed we will work at the slightly more general setting of what we call "Martindale-like covers," defined in terms of natural "ideal absorption properties." This result is basic in our forthcoming paper on polynomial identities and speciality of quadratic Martindale-like quotients, as well as we expect it to be useful in the description of Martindale-like quotients of strongly prime quadratic Jordan algebras satisfying a polynomial identity.

The proof of our main result is purely combinatorial, based on the fact that $2J + \operatorname{Ker} 2\operatorname{Id}_J$ is an essential ideal of any nondegenerate Jordan algebra J, which, with the use of annihilators, allows to split the problem into the 2-torsion free and the characteristic 2 cases.

The paper is divided into four sections. Section 0 is devoted to recalling basic facts and notions, including the essentialness of $2J + \operatorname{Ker} 2\operatorname{Id}_J$, mentioned above, and the definition of the scalar center. In Section 1 we study characteristic 2 phenomena needed in the sequel, and their natural extensions to arbitrary Jordan algebras in terms of the annihilator $\operatorname{Ann}_J(\operatorname{Ker} 2\operatorname{Id}_J)$ of $\operatorname{Ker} 2\operatorname{Id}_J$. In the next section we establish the fundamental properties of Martindale-like covers. Finally, in Section 3, we prove our main theorem asserting the inheritance of the scalar center by Martindale-like covers of nondegenerate Jordan algebras. It turns out that for a central element z of J, and a cover Q of J, V_z is in the centroid of Q as soon as Q satisfies the natural outer ideal absorption properties, while for the fact that z is indeed central in Q, the inner ideal absorption property must be assumed too.

0. Preliminaries

0.1. We will deal with Jordan algebras over a ring of scalars Φ. The reader is referred to [5,7,11] for definitions and basic properties not explicitly mentioned or proved in this section. Given a Jordan algebra J, its products will be denoted x^2, $U_x y$, for $x, y \in J$. They are quadratic in x and linear in y and have linearizations denoted $V_x y = x \circ y$, $U_{x,z} y = \{x, y, z\} = V_{x,y} z$, respectively. A Jordan algebra J is said to be *unital* if there is an element $1 \in J$ satisfying $U_1 = \operatorname{Id}_J$ and $U_x 1 = x^2$, for any $x \in J$ (such an element can be shown to be unique and it is called the *unit* of J).

Every Jordan algebra J embeds in a unital Jordan algebra $\hat{J} = J \oplus \Phi 1$ called its (*free*) *unitization* [11, 0.6].

A Jordan algebra J is said to be *nondegenerate* if zero is the only *absolute zero divisor*, i.e., zero is the only $x \in J$ such that $U_x = 0$.

0.2. We will need the following identities valid for arbitrary Jordan algebras.

(i) $(x \circ y) \circ z = \{x, y, z\} + \{y, x, z\}$,
(ii) $z \circ U_x y = \{z, x, y\} \circ x - y \circ U_x z$,
(iii) $U_{U_x y} = U_x U_y U_x$, $U_{x^2} = (U_x)^2$,
(iv) $\{x, U_z x, y\} = \{U_x z, z, y\}$,
(v) $2U_x y = x \circ (x \circ y) - x^2 \circ y$,
(vi) $\{z, x, U_{y_1} y_2\} = \{z, \{x, y_1, y_2\}, y_1\} - \{z, y_2, U_{y_1} x\}$,
(vii) $U_x(y \circ z) = \{x \circ y, z, x\} - y \circ U_x z$,
(viii) $U_{z \circ x} y = U_z U_x y + U_x U_z y + z \circ U_x(y \circ z) - \{U_z x, y, x\}$,

(ix) $(U_x y)^2 = U_x U_y x^2$,
(x) $(x \circ y)^2 = U_x y^2 + U_y x^2 + x \circ U_y x$,
(xi) $U_{\{a,x,b\}} z = U_a U_x U_b z + U_b U_x U_a z + \{a, x, U_b\{x, a, z\}\} - \{U_a U_x b, z, b\}$,
(xii) $\{a, z, U_a U_x a\} = \{U_a z, x, U_a x\}$,
(xiii) $2U_a U_x U_a z = \{a, x, U_a\{x, a, z\}\} - \{U_a x, x, U_a z\}$,
(xiv) $U_x\{a, b, c\} = \{x, a, \{b, c, x\}\} - \{U_x a, c, b\}$.

Indeed, (vi) is [7, JP10], (xi) is [7, JP21], (xiv) is [7, JP12], and the rest of them follow from Macdonald's theorem [6].

0.3. We recall that an *ideal* I of a Jordan algebra J is just a Φ-submodule of J satisfying $U_I J + I^2 + U_J I + I \circ J \subseteq I$, equivalently, $U_I \hat{J} + U_{\hat{J}} I \subseteq I$, which implies $\{I, J, J\} \subseteq I$ using (0.2)(i). An ideal I of J is said to be *essential* if it hits every nonzero ideal of J, i.e., $I \cap L \neq 0$ for any nonzero ideal L of J.

0.4. In a Jordan algebra J, the *annihilator* $\mathrm{Ann}_J(I)$ of an ideal I of J is an ideal of J which, when J is nondegenerate, is given by

$$\mathrm{Ann}_J(I) = \{x \in J \mid U_x I = 0\} = \{x \in J \mid U_I x = 0\}$$

[8, 1.3, 1.7], [12, 1.3]. An ideal I of J will be said *sturdy* if $\mathrm{Ann}_J(I) = 0$. It is easy to prove that essential ideals coincide with sturdy ideals in any semiprime Jordan algebra.

0.5. The *centroid* $\Gamma(J)$ of a Jordan algebra J is the set of linear maps acting "scalarly" in Jordan products [10]:

$$\Gamma(J) = \{T \in \mathrm{End}_\Phi(J) \mid TU_x = U_x T,\ TV_x = V_x T,\\ T^2(x^2) = (T(x))^2,\ T^2 U_x = U_{T(x)},\ \text{for any } x \in J\}.$$

It is immediate that $TV_{x,y} = V_{x,y}T$, $TU_{x,y} = U_{x,y}T$ for any $T \in \Gamma(J)$, and any $x, y \in J$. Clearly, $\Phi\,\mathrm{Id}_J \subseteq \Gamma(J)$. By [10, 2.5], when J has no nonzero extreme elements (for example, when J is nondegenerate), $\Gamma(J)$ is a unital associative commutative Φ-algebra and J is a Jordan algebra over $\Gamma(J)$.

0.6. Lemma. *If J is a nondegenerate Jordan algebra and $T \in \Gamma(J)$, then*

(i) *the sum $T(J) + \mathrm{Ker}\,T$ is direct and, indeed, $\mathrm{Ker}\,T = \mathrm{Ker}\,T^n$ for any positive integer n,*
(ii) *$T(J)$ and $\mathrm{Ker}\,T$ are ideals of J,*
(iii) *$T(J) + \mathrm{Ker}\,T$ is an essential ideal of J.*

Proof. (i) Let $x \in J$ such that $T^2(x) = 0$. Then, for any $y \in J$, we have

$$U_{U_{T(x)}y} = U_{T(x)} U_y U_{T(x)} = U_{T(x)} U_y T^2 U_x = T^2 U_{T(x)} U_y U_x = U_{T^2(x)} U_y U_x = 0,$$

hence $U_{T(x)} y = 0$ by nondegeneracy. This shows $U_{T(x)} = 0$, hence $T(x) = 0$ again by nondegeneracy. We have proved $\mathrm{Ker}\,T = \mathrm{Ker}\,T^2$, which readily implies our assertion.

(ii) By [10, 2.6] we already know that $T(J)$ is an ideal of J and $\operatorname{Ker} T$ is an outer ideal of J. But, under nondegeneracy, $\operatorname{Ker} T$ is also an inner ideal of J: for any $x \in \operatorname{Ker} T$, $y \in \hat{J}$, $T^2(U_x y) = U_{T(x)} y = 0$, hence $U_x y \in \operatorname{Ker} T^2 = \operatorname{Ker} T$ by (i).

(iii) Given a nonzero ideal L of J, if $T(L) = 0$, then $0 \neq L \subseteq L \cap \operatorname{Ker} T \subseteq L \cap (T(J) + \operatorname{Ker} T)$. Otherwise, there exists $x \in L$ such that $T(x) \neq 0$. By nondegeneracy, $0 \neq U_{T(x)} J = T^2 U_x J = U_x T^2(J) \subseteq U_L J \cap T(J) \subseteq L \cap (T(J) + \operatorname{Ker} T)$. □

0.7. Following [3], the (*weak*) *center* of J is the set $C(J)$ of all elements $z \in J$ such that $U_z, V_z \in \Gamma(J)$, which is a subalgebra of J when J is nondegenerate [3, Theorems 1, 2]. More explicitly, $z \in J$ lies in $C(J)$ if and only if

$$c_i(z, J, J) = 0, \quad \text{for } i = 1, 2, 3, 5, 6, \quad \text{and} \quad c_i(z, J) = 0 \quad \text{for } i = 4, 7,$$

where

$$\begin{aligned}
c_1(z, x, y) &= V_z U_x y - U_x V_z y = z \circ U_x y - U_x(z \circ y), \\
c_2(z, x, y) &= V_z V_x y - V_x V_z y = z \circ (x \circ y) - x \circ (z \circ y), \\
c_3(z, x, y) &= U_{V_z x} y - V_z^2 U_x y = U_{z \circ x} y - z \circ (z \circ U_x y), \\
c_4(z, x) &= (V_z x)^2 - V_z^2 x^2 = (z \circ x)^2 - z \circ \big(z \circ x^2\big), \\
c_5(z, x, y) &= U_z U_x y - U_x U_z y, \\
c_6(z, x, y) &= U_z V_x y - V_x U_z y = U_z(x \circ y) - x \circ U_z y, \\
c_7(z, x) &= (U_z x)^2 - U_z^2 x^2,
\end{aligned}$$

since $c_5(z, J, J) = 0$ and (0.2)(iii) imply $U_{U_z x} = U_z U_x U_z = U_z^2 U_x$, for any $x \in J$.

We claim that $z \in C(J)$ also satisfies $c_8(z, J, J) = 0$, where $c_8(z, x, y) = \{U_z x, y, x\} - 2U_z U_x y$, which readily follows from the fact that $U_z \in \Gamma(J)$. If J is nondegenerate then also $c_9(z, J) = 0$ for $c_9(z, x) = U_z x^2 - U_x z^2$, since $c_9(z, x) = c_5(z, x, 1)$ and $C(J) \subseteq C(\hat{J})$ by [3, Corollary 1].

1. Characteristic 2 phenomena

1.1. We remark that, by applying (0.6) to $T = 2\operatorname{Id}_J$ in a nondegenerate Jordan algebra J, $2x = 0$ if and only if $2^n x = 0$ for a positive integer n.

On the other hand, if $2x = 0$ and $x \in \operatorname{Ann}(\operatorname{Ker} 2\operatorname{Id}_J)$, then $x = 0$: $x \in \operatorname{Ker} 2\operatorname{Id}_J \cap \operatorname{Ann}(\operatorname{Ker} 2\operatorname{Id}_J) = 0$ since J is semiprime and $\operatorname{Ker} 2\operatorname{Id}_J$ is an ideal of J by (0.6)(ii).

1.2. Remark. In a nondegenerate Jordan algebra J, $U_y x = U_y(-x)$, i.e., $U_y 2x = 0$, for any $x \in J$, $y \in \operatorname{Ker} 2\operatorname{Id}_J$: $U_y 2x = 2U_y x = 0$ since $U_y x \in \operatorname{Ker} 2\operatorname{Id}_J$ by (0.6)(ii).

1.3. Lemma. *Let J be a nondegenerate Jordan algebra, and let $a, b \in J$. If $(U_a - U_b)J \subseteq \operatorname{Ann}_J(\operatorname{Ker} 2\operatorname{Id}_J)$, then $a - b \in \operatorname{Ann}_J(\operatorname{Ker} 2\operatorname{Id}_J)$.*

Proof. (I) $U_{\{a,J,b\}} J \subseteq \operatorname{Ann}_J(\operatorname{Ker} 2\operatorname{Id}_J)$: for any $y \in \operatorname{Ker} 2\operatorname{Id}_J$, $x, z \in J$, using (0.2)(xi),

$$\begin{aligned}
U_yU_{\{a,x,b\}}z &= U_y\big[U_aU_xU_bz + U_bU_xU_az + \big\{a,x,U_b\{x,a,z\}\big\} - \{U_aU_xb,z,b\}\big]\\
&= U_y\big[U_aU_xU_az + U_aU_xU_az + \big\{a,x,U_a\{x,a,z\}\big\} - \{U_bU_xb,z,b\}\big]\\
&\quad \big(\text{for } t\in J,\ U_at - U_bt \in \mathrm{Ann}_J(\mathrm{Ker}\,2\,\mathrm{Id}_J),\ \text{which is an ideal of } J\big)\\
&= U_y\big[2U_aU_xU_az + \big\{a,x,U_a\{x,a,z\}\big\} - \{U_bx,x,U_bz\}\big] \quad \big(\text{by (0.2)(xii)}\big)\\
&= U_y\big[2U_aU_xU_az + \big\{a,x,U_a\{x,a,z\}\big\} - \{U_ax,x,U_az\}\big]\\
&\quad \big(\text{for } t\in J,\ U_at - U_bt \in \mathrm{Ann}_J(\mathrm{Ker}\,2\,\mathrm{Id}_J),\ \text{which is an ideal of } J\big)\\
&= U_y[4U_aU_xU_az] \quad \big(\text{by (0.2)(xiii)}\big)\\
&= 0
\end{aligned}$$

by (1.2).

(II) $\{a,J,b\}\subseteq \mathrm{Ann}_J(\mathrm{Ker}\,2\,\mathrm{Id}_J)$: using (0.2)(iii), for any $y\in \mathrm{Ker}\,2\,\mathrm{Id}_J$, $x\in J$, $U_{U_y\{a,x,b\}} = U_yU_{\{a,x,b\}}U_y = 0$ by (I), hence $U_y\{a,x,b\}=0$ by nondegeneracy of J.

(III) $U_{a-b}J\subseteq \mathrm{Ann}_J(\mathrm{Ker}\,2\,\mathrm{Id}_J)$: for any $y\in \mathrm{Ker}\,2\,\mathrm{Id}_J$, $x\in J$,

$$\begin{aligned}
U_yU_{a-b}x &= U_y\big[U_ax + U_bx - \{a,x,b\}\big] = U_y\big[U_ax + U_ax - \{a,x,b\}\big] \quad \text{(as above)}\\
&= U_y\big[2U_ax - \{a,x,b\}\big] = 0
\end{aligned}$$

by (1.2) and (II).

Finally, for any $y\in \mathrm{Ker}\,2\,\mathrm{Id}_J$, $U_{U_{a-b}y} = U_{a-b}U_yU_{a-b}$ (by (0.2)(iii)) $=0$, by (III), hence $U_{a-b}y=0$ by nondegeneracy, and $a-b\in \mathrm{Ann}_J(\mathrm{Ker}\,2\,\mathrm{Id}_J)$ (0.4). □

Under the assumption of characteristic 2, (1.3) turns into the following result of independent interest, though it is not explicitly needed in the sequel.

1.4. Corollary. *Let J be a nondegenerate Jordan algebra of characteristic two ($2J=0$), $a,b\in J$. If $U_a=U_b$, then $a=b$.*

Proof. Use (1.3) and the fact that $\mathrm{Ann}_J(\mathrm{Ker}\,2\,\mathrm{Id}_J)=\mathrm{Ann}_J(J)=0$ by nondegeneracy. □

2. Martindale-like covers

2.1. When J and Q are Jordan algebras such that J is a subalgebra of Q, we will say that Q is a *cover* of J. We will consider the following *ideal absorption properties* for a cover Q of J:

the *outer* ideal absorption properties:
(IA1) for any $0\neq q\in Q$ there exists an essential ideal I of J such that $0\neq U_Iq\subseteq J$,
(IA2) for any $q\in Q$ there exists an essential ideal I of J such that $I\circ q\subseteq J$,
and the *inner* ideal absorption property:
(IA3) for any $q\in Q$ there exists an essential ideal I of J such that $U_qI\subseteq J$.

A cover Q of J will be said a *Martindale-like cover* if it satisfies (IA1)–(IA3).

2.2. Remark. Assuming (IA1), condition (IA2) can be replaced by

(IA2′) For any $q\in Q$ there exists an essential ideal I of J such that $\{q,I,I\}\subseteq J$.

Indeed, (0.2)(i) implies that $\{q, I, I\} \subseteq (q \circ I) \circ I + \{I, q, I\} \subseteq J$ when I is the intersection of the ideals in (IA1) and (IA2) for the element q. Conversely, if I and L are essential ideals satisfying $U_I q + \{q, L, L\} \subseteq J$, then $K := U_{I\cap L}(I \cap L)$ is an essential ideal of J by [12, 1.2(a)], and (0.2)(ii) yields

$$q \circ K = q \circ U_{I\cap L}(I \cap L) \subseteq \{q, I \cap L, I \cap L\} \circ (I \cap L) + (I \cap L) \circ U_{I\cap L} q$$
$$\subseteq \{q, L, L\} \circ J + J \circ U_I q \subseteq J.$$

2.3. Remark. Notice that any cover Q of J satisfying (IA1) is tight over J, i.e., any nonzero ideal of Q hits J. As a consequence, if J is nondegenerate then Q is also nondegenerate (cf. [9, 2.9(iii)]). Similarly, J is free of 2-torsion if and only if Q is free of 2-torsion, using tightness, (0.6)(ii), and the obvious fact that $\operatorname{Ker} 2\operatorname{Id}_J = J \cap \operatorname{Ker} 2\operatorname{Id}_Q$.

In the next result we go further in the tightness of Martindale-like covers, in fact of covers just satisfying (IA1).

2.4. Proposition. *Let J be a nondegenerate Jordan algebra and Q be a cover of J satisfying* (IA1). *Then, for any $0 \neq q \in Q$, and any essential ideal L of J, $U_L q \neq 0$ and $U_q L \neq 0$. If J has not 2-torsion, then also $L \circ q \neq 0$.*

Proof. Given $0 \neq q \in Q$, let I be an essential ideal of J such that $0 \neq U_I q \subseteq J$, so that we can take $x \in I$ such that $0 \neq U_x q$. For any essential ideal L of J, $0 \neq U_{U_x q} L$ since $\operatorname{Ann}_J(L) = 0$. But $U_{U_x q} L = U_x U_q U_x L$ (by (0.2)(iii)) $\subseteq U_x U_q L$, which implies $U_q L \neq 0$.

If $U_L q = 0$, then $U_{L[t]} q = 0$ in the algebra $Q[t]$ of polynomials over Q. Notice that Q is nondegenerate by (2.3), which readily implies that $Q[t]$ is also nondegenerate. For any $h \in L[t]$, let $a := U_h U_q h \in Q[t]$. By (0.2)(iii),

$$U_a Q[t] = U_h U_q U_h U_q U_h Q[t] = U_{U_h q} U_q U_h Q[t] = 0$$

since $U_h q = 0$, hence $a = 0$ by nondegeneracy. For $x, y \in L$, the coefficient of t in $U_{x+ty} U_q (x + ty)$ is $U_x U_q y + U_{x,y} U_q x$, which is then zero. But, on the other hand, $U_{x,y} U_q x = \{U_x q, q, y\}$ (by (0.2)(iv)) $= 0$, hence we obtain $U_L U_q L = 0$. Fixing any $x \in L$ such that $U_q x \neq 0$, we then have $0 \neq U_{U_q x} L = U_q U_x U_q L \subseteq U_q U_L U_q L$, which contradicts $U_L U_q L = 0$. This shows $U_L q \neq 0$.

Finally, in case J has not 2-torsion, $0 \neq 2U_L q \subseteq L \circ (L \circ q) + L^2 \circ q$ (by (0.2)(v)) $\subseteq L \circ (L \circ q) + L \circ q$ implies $L \circ q \neq 0$. □

As a consequence, we can choose a single ideal to nontrivially absorb any given finite set of elements in the cover.

2.5. Corollary. *Let J be a nondegenerate Jordan algebra and Q be a cover of J satisfying* (IA1). *Given a finite set $q_1, \dots, q_n$ of nonzero elements in Q, there exists an essential ideal I of J such that $0 \neq U_I q_i \subseteq J$, for all $i = 1, \dots, n$.*

If Q also satisfies (IA2) *and/or* (IA3)*, then the ideal I above can also be assumed to satisfy $I \circ q_i + \{q_i, I, I\} \subseteq J$ (with $0 \neq I \circ q_i$ in case J has not 2-torsion), and/or $0 \neq U_{q_i} I \subseteq J$, respectively, for all $i = 1, \dots, n$.*

Proof. Apply (2.4) and (2.2) together with the fact that the finite intersection of essential ideals is also essential. □

2.6. If J is a nondegenerate Jordan algebra without 2-torsion, a cover Q of J is a Martindale-like cover of J if and only if for any $0 \neq q \in Q$ there exists an essential ideal I of J such that $0 \neq I \circ q \subseteq J$ (when $1/2 \in \Phi$, this just amounts to saying that Q is a Jordan algebra of Martindale-like quotients of J with respect to the filter of all essential ideals of J in the sense of [4, 5.1]).

Indeed, a Martindale-like cover of J satisfies (IA2) and, moreover, $I \circ q \neq 0$ for any $0 \neq q \in Q$ by (2.4) in the absence of 2-torsion. Conversely, assume that, for any $0 \neq q \in Q$, there exists an essential ideal I of J such that $0 \neq I \circ q \subseteq J$. Clearly, $M := 2I$ is an essential ideal of J and

$$\begin{aligned} U_M q = 2(2U_I q) &\subseteq 2\big(I \circ (I \circ q) + I^2 \circ q\big) \quad \big(\text{by } (0.2)(\text{v})\big) \\ &\subseteq I \circ J + I \circ q \subseteq J. \end{aligned}$$

Moreover, for $x \in I$ such that $x \circ q \neq 0$, we have, by sturdiness of I (cf. (0.4)),

$$\begin{aligned} 0 \neq U_I(x \circ q) &\subseteq \{I \circ x, q, I\} + x \circ U_I q \quad \big(\text{by } (0.2)(\text{vii})\big) \\ &\subseteq U_I q + x \circ U_I q, \end{aligned}$$

which implies $U_I q \neq 0$, hence $0 \neq 4U_I q = U_M q$, and we have established (IA1).

Furthermore, $M \circ q \subseteq J$, and $\{q, M, M\} \subseteq J$ as in the proof of (2.2). We now just need to show (IA3). Let L be an essential ideal of J such that $q^2 \circ L \subseteq J$, and let $K := U_M M \cap L$, which is an essential ideal of J by [12, 1.2(a)], and we will show $U_q 2K \subseteq J$. First, $q \circ U_M M \subseteq M$: for any $x, y \in M$,

$$\begin{aligned} q \circ U_x y &= \{q, x, y\} \circ x - y \circ U_x q \quad \big(\text{by } (0.2)(\text{ii})\big) \\ &\subseteq \{q, M, M\} \circ M + M \circ U_M q \subseteq J \circ M \subseteq M. \end{aligned}$$

Thus, by (0.2)(v), $U_q 2K = 2U_q K \subseteq q \circ (q \circ K) + q^2 \circ K \subseteq q \circ (q \circ U_M M) + q^2 \circ L \subseteq q \circ M + q^2 \circ L \subseteq J$.

3. Center inheritance in Martindale-like covers

The proof of the next result is just the quadratic version of the proof of [1, 4.1]. In the generalization a factor 2 comes out.

3.1. Lemma. *Let J be a nondegenerate Jordan algebra, Q be a cover of J satisfying* (IA1) *and* (IA2)*, and $z \in C(J)$. Then, $2z \circ (p \circ q) = 2(z \circ p) \circ q$, for any $p, q \in Q$, i.e., $2V_z V_q = 2V_q V_z$, for any $q \in Q$.*

Proof. (I) For any $q \in Q$ and any $x \in J$ such that $x \circ q \in J$, $z \circ (x \circ q) = (z \circ x) \circ q$:

Use (2.5) to find an essential ideal I of J such that $U_I q + \{q, I, I\} \subseteq J$. For any $y_1, y_2 \in I$, and $t \in \hat{J}$,

$$
\begin{aligned}
\{z\circ t, q, U_{y_1}y_2\} &= \bigl\{z\circ t, \{q, y_1, y_2\}, y_1\bigr\} - \{z\circ t, y_2, U_{y_1}q\} \quad \bigl(\text{by (0.2)(vi)}\bigr)\\
&= z\circ\bigl\{t, \{q, y_1, y_2\}, y_1\bigr\} - z\circ\{t, y_2, U_{y_1}q\}\\
&= z\circ\{t, q, U_{y_1}y_2\} \quad \bigl(\text{by (0.2)(vi)}\bigr)
\end{aligned}
\tag{1}
$$

since $\{q, y_1, y_2\}, U_{y_1}q \in J$, $z \in C(J)$, and $C(J) \subseteq C(\hat{J})$ [3, Corollary 1]. Now, if $K := U_I I$ and $y \in K$,

$$
\begin{aligned}
U_y\bigl((z\circ x)\circ q\bigr) &= \bigl\{y\circ(z\circ x), q, y\bigr\} - (z\circ x)\circ U_y q \quad \bigl(\text{by (0.2)(vii)}\bigr)\\
&= \bigl\{z\circ(y\circ x), q, y\bigr\} - z\circ(x\circ U_y q) \quad \bigl(\text{since } x, y, U_y q \in J,\ z\in C(J)\bigr)\\
&= z\circ\{y\circ x, q, y\} - z\circ(x\circ U_y q) \quad \bigl(\text{by (1)}\bigr)\\
&= z\circ\bigl(U_y(x\circ q)\bigr) \quad \bigl(\text{by (0.2)(vii)}\bigr)\\
&= U_y\bigl(z\circ(x\circ q)\bigr)
\end{aligned}
$$

since $y, x\circ q \in J$ and $z \in C(J)$. We have shown that $U_K((z\circ x)\circ q - z\circ(x\circ q)) = 0$, which implies $(z\circ x)\circ q - z\circ(x\circ q) = 0$ by (2.4) since K is an essential ideal of J by [12, 1.2(a)].

(II) Let $q \in Q$, and I be an essential ideal of J satisfying $I\circ q + U_I q + \{q, I, I\} \subseteq J$, that can be found by (2.5). Then $(z\circ q)\circ x = z\circ(q\circ x)$ for any $x \in U_I I$:

$$
\begin{aligned}
(z\circ q)\circ x &= 2\{z, q, x\} - z\circ(q\circ x) + (z\circ x)\circ q \quad \bigl(\text{by linearized (0.2)(v)}\bigr)\\
&= 2\{z, q, x\} \quad \bigl(\text{by (I)}\bigr)\\
&= \{z\circ 1, q, x\}\\
&= z\circ\{1, q, x\} \quad \bigl(\text{by (1)}\bigr)\\
&= z\circ(q\circ x).
\end{aligned}
$$

(III) For any $p, q \in Q$, $2(z\circ p)\circ q = 2z\circ(p\circ q)$:

By (2.5), we can find an essential ideal I of J such that $I\circ p + U_I p + \{p, I, I\} + I\circ q + U_I q + \{q, I, I\} + I\circ(p\circ q) + U_I(p\circ q) + \{p\circ q, I, I\} \subseteq J$. Let $K := U_I I$ and $L := U_K K$. Notice that

$$
U_L q \subseteq K. \tag{2}
$$

Indeed, $U_L q$ is spanned by elements of the form $U_{U_a b}q$ and $\{U_{a'}b', q, U_a b\}$, where $a, b, a', b' \in K$, and

$$
\begin{aligned}
U_{U_a b}q &= U_a U_b U_a q \quad \bigl(\text{by (0.2)(iii)}\bigr)\\
&\subseteq U_K U_K U_I q \subseteq U_K U_K J \subseteq K,
\end{aligned}
$$

whereas

$$
\begin{aligned}
\{U_{a'}b', q, U_a b\} &\subseteq \{K, q, U_a b\} \subseteq \bigl\{K, \{q, a, b\}, a\bigr\} + \{K, b, U_a q\} \quad \bigl(\text{by (0.2)(vi)}\bigr)\\
&\subseteq \bigl\{K, \{q, I, I\}, K\bigr\} + \{K, K, U_I q\} \subseteq \{K, J, K\} + \{K, K, J\} \subseteq K.
\end{aligned}
$$

Now, for any $y \in L$,

$$\begin{aligned}
U_y\big(2(z\circ p)\circ q\big) &= 2\big[\big\{y\circ(z\circ p), q, y\big\} - (z\circ p)\circ U_y q\big] \quad \big(\text{by (0.2)(vii)}\big)\\
&= 2\big[\big\{z\circ(y\circ p), q, y\big\} - z\circ(p\circ U_y q)\big] \quad \big(\text{by (II) since } y, U_y q\in K \text{ by (2)}\big)\\
&= 2\big[z\circ\{y\circ p, q, y\} - z\circ(p\circ U_y q)\big] \quad \big(\text{by (1) since } y\circ p\in J \text{ and } y\in K\big)\\
&= 2z\circ U_y(p\circ q) \quad \big(\text{by (0.2)(vii)}\big)\\
&= z\circ\big[\big(y\circ(p\circ q)\big)\circ y - y^2\circ(p\circ q)\big] \quad \big(\text{by (0.2)(v)}\big)\\
&= \big[\big(y\circ\big(z\circ(p\circ q)\big)\big)\circ y - y^2\circ\big(z\circ(p\circ q)\big)\big] \quad \big(\text{by (II)}\big)\\
&= 2U_y\big(z\circ(p\circ q)\big) \quad \big(\text{by (0.2)(v)}\big)\\
&= U_y\big(2z\circ(p\circ q)\big).
\end{aligned}$$

We have shown $U_L(2(z\circ p)\circ q - 2z\circ(p\circ q)) = 0$, which implies $2(z\circ p)\circ q - 2z\circ(p\circ q) = 0$ by (2.4), since L is an essential ideal of J by [12, 1.2(a)]. □

3.2. Theorem. *Let J be a nondegenerate Jordan algebra, Q be a cover of J satisfying* (IA1) *and* (IA2)*, and $z\in C(J)$. Then,*

$$2c_i(z, Q, Q) = 0, \quad \textit{for } i = 1,2,3,5,6,8 \quad \textit{and} \quad 2c_i(z, Q) = 0, \quad \textit{for } i = 4,7,9.$$

Proof. By (1.1), it is enough to prove $2^n c_i(z, Q, \ldots) = 0$ for some positive integer n. On the other hand, we claim that, for any c_i, $i = 1, \ldots, 9$, there exists a positive integer n such that $2^n c_i$ can be expressed in terms of 2 times "∘-products." As an example, for $p, q \in Q$, using (0.2)(v) yields

$$\begin{aligned}
8c_3(z,p,q) &= 8\big[U_{z\circ p}q - z\circ(z\circ U_p q)\big]\\
&= 4\big[(z\circ p)\circ\big((z\circ p)\circ q\big) - (z\circ p)^2\circ q - z\circ\big(z\circ\big[p\circ(p\circ q) - p^2\circ q\big]\big)\big]\\
&= 2\big[2(z\circ p)\circ\big((z\circ p)\circ q\big) - \big[(z\circ p)\circ(z\circ p)\big]\circ q\\
&\quad - z\circ\big(z\circ\big[2p\circ(p\circ q) - (p\circ p)\circ q\big]\big)\big].
\end{aligned}$$

Now, our result follows from (3.1). □

The above result is enough to obtain a generalization of [1, 4.1] for 2-torsion free Jordan algebras.

3.3. Corollary. *Let J be a nondegenerate Jordan algebra without* 2*-torsion, Q be a cover of J satisfying* (IA1) *and* (IA2)*. Then, $C(J) \subseteq C(Q)$.*

Proof. Use (0.7), (3.2), and the fact that Q has not 2-torsion by (2.3). □

3.4. Corollary. *Let J be a nondegenerate Jordan algebra, Q be a cover of J satisfying* (IA1) *and* (IA2)*. Then, $2C(J) \subseteq C(Q)$.*

Proof. For any $z \in C(J)$, and any $i = 1, \ldots, 7$, $c_i(2z, Q, \ldots) = 2^k c_i(z, Q, \ldots)$ (for some positive integer k) $= 0$ by (3.2), hence $2z \in C(Q)$ by (0.7). □

In order to extend (3.3) to the general quadratic case we will proceed in two steps. In the first one we will study the centrality in Q of the operator V_z for a central element of J, and show that only conditions (IA1) and (IA2) are needed. Our first result is the natural generalization of [3, Corollary 2].

3.5. Lemma. *In a nondegenerate Jordan algebra J, $C(J) \circ \operatorname{Ker} 2\operatorname{Id}_J = 0$.*

Proof. Let $z \in C(J)$, $x \in \operatorname{Ker} 2\operatorname{Id}_J$, and $y \in J$. By (0.2)(viii),

$$\begin{aligned} U_{z\circ x} y &= U_z U_x y + U_x U_z y + z \circ U_x(y \circ z) - \{U_z x, y, x\} \\ &= 2U_z U_x y + (z \circ z) \circ U_x y - 2U_z U_x y = 2z^2 \circ U_x y \in 2J. \end{aligned}$$

But $4U_{z\circ x} y = U_{z\circ 2x} y = 0$, hence $U_{z\circ x} y = 0$ by (1.1). We have shown $U_{z\circ x} J = 0$, hence $z \circ x = 0$ by nondegeneracy. □

The next two results are meant to "lift" (3.5) to covers satisfying (IA1).

3.6. Lemma. *If J is a nondegenerate Jordan algebra and Q is a cover of J satisfying* (IA1)*, then $C(J) \circ \operatorname{Ker} 2\operatorname{Id}_Q = 0$.*

Proof. Let $z \in C(J)$, $q \in \operatorname{Ker} 2\operatorname{Id}_Q$, and let I be an essential ideal of J such that $U_I q \subseteq J$. Notice that $L := I \cap (2J + \operatorname{Ker} 2\operatorname{Id}_J)$ is an essential ideal of J by (0.6)(iii). For any $y \in L$, using (0.2)(vii),

$$U_y(z \circ q) = \{y \circ z, q, y\} - z \circ U_y q.$$

But writing $y = 2a + b$ for $a \in J$, $b \in \operatorname{Ker} 2\operatorname{Id}_J$, $\{y \circ z, q, y\} = \{2a \circ z, q, y\} + \{b \circ z, q, y\} = \{a \circ z, 2q, y\}$ (since $b \circ z = 0$ by (3.5)) $= 0$ since $q \in \operatorname{Ker} 2\operatorname{Id}_Q$. On the other hand, $2U_y q = U_y 2q = 0$, hence $U_y q \in J \cap \operatorname{Ker} 2\operatorname{Id}_Q = \operatorname{Ker} 2\operatorname{Id}_J$, so that $z \circ U_y q = 0$ by (3.5). We have shown $U_L(z \circ q) = 0$, which implies $z \circ q = 0$ by (2.4). □

3.7. Lemma. *If J is a nondegenerate Jordan algebra and Q is a cover of J satisfying* (IA1)*, then $C(J) \circ Q \subseteq \operatorname{Ann}_Q(\operatorname{Ker} 2\operatorname{Id}_Q)$.*

Proof. Let $z \in C(J)$, $p \in \operatorname{Ker} 2\operatorname{Id}_Q$, and $q \in Q$. By (0.2)(vii),

$$U_p(z \circ q) = \{p \circ z, q, p\} - z \circ U_p q = 0$$

by (3.6) since $p, U_p q \in \operatorname{Ker} 2\operatorname{Id}_Q$. This shows $z \circ q \in \operatorname{Ann}_Q(\operatorname{Ker} 2\operatorname{Id}_Q)$ (cf. (0.4) since Q is nondegenerate by (2.3)). □

3.8. Theorem. *If J is a nondegenerate Jordan algebra and Q is a cover of J satisfying* (IA1) *and* (IA2)*, then $V_z \in \Gamma(Q)$ for any $z \in C(J)$, equivalently,*

$$c_1(z, Q, Q) = c_2(z, Q, Q) = c_3(z, Q, Q) = c_4(z, Q) = 0.$$

Proof. Notice that, $c_1(z,Q,Q)$, $c_2(z,Q,Q)$, $c_3(z,Q,Q)$, $c_4(z,Q)$ are contained in $\mathrm{Ann}_Q(\mathrm{Ker}\,2\,\mathrm{Id}_Q)$ by (3.7), since they lie in the ideal of Q generated by $z\circ Q$. Now, the result follows by using (3.2) and (1.1). □

3.9. Theorem. *Let J be a nondegenerate Jordan algebra, Q be a cover of J satisfying* (IA1) *and* (IA2)*, and $z\in C(J)$. Then*

(i) $\{z,p,q\}=\{z,q,p\}=\{p,z,q\}$, *for any $p,q\in Q$,*
(ii) $c_6(z,Q,Q)=c_7(z,Q)=c_9(z,Q)=0$.

Proof. (i) By (0.2)(i) and (3.8),

$$\{z,p,q\}=-\{p,z,q\}+(p\circ z)\circ q=-\{p,z,q\}+p\circ(z\circ q)=\{z,q,p\},\quad \text{and}$$
$$\{z,p,q\}=-\{z,q,p\}+(p\circ q)\circ z=-\{z,q,p\}+p\circ(q\circ z)=\{p,z,q\}.$$

(ii) If $c_9(z,Q)=0$ then, for any $p\in Q$,

$$c_7(z,p)=(U_zp)^2-U_z^2p^2=(U_zp)^2-U_zU_pz^2=0$$

by (0.2)(ix). Thus we will show $c_6(z,Q,Q)=c_9(z,Q)=0$, and we just need to prove that $c_6(z,Q,Q), c_9(z,Q)\subseteq \mathrm{Ann}_Q(\mathrm{Ker}\,2\,\mathrm{Id}_Q)$ by (3.2) and (1.1). For any $p,q\in Q$, $y\in \mathrm{Ker}\,2\,\mathrm{Id}_Q$,

$$\begin{aligned}U_yc_6(z,p,q)&=U_y\big(U_z(p\circ q)-p\circ U_zq\big)\\&=U_y\big(U_z(p\circ q)+p\circ U_zq\big)\quad\big(\text{by (1.2) since } Q \text{ is nondegenerate by (2.3)}\big)\\&=U_y\big(\{z\circ p,q,z\}\big)\quad\big(\text{by (0.2)(vii)}\big)\\&=0\end{aligned}$$

since $z\circ p\in \mathrm{Ann}_Q(\mathrm{Ker}\,2\,\mathrm{Id}_Q)$ (by (3.7)) implies $\{z\circ p,q,z\}\in \mathrm{Ann}_Q(\mathrm{Ker}\,2\,\mathrm{Id}_Q)$. Also

$$\begin{aligned}U_yc_9(z,p)&=U_y\big(U_zp^2-U_pz^2\big)\\&=U_y\big(U_zp^2+U_pz^2\big)\quad\big(\text{by (1.2) since } Q \text{ is nondegenerate by (2.3)}\big)\\&=U_y\big((z\circ p)^2-z\circ U_pz\big)\quad\big(\text{by (0.2)(x)}\big)\\&=0\end{aligned}$$

since $z\circ Q\subseteq \mathrm{Ann}_Q(\mathrm{Ker}\,2\,\mathrm{Id}_Q)$ by (3.7). □

3.10. Lemma. *Let J be a nondegenerate Jordan algebra, Q be a cover of J satisfying* (IA1) *and* (IA2)*, and $z\in C(J)$. Then*

$$U_zU_pq=-U_pU_zq+\{U_zp,q,p\},\quad \textit{for any } p,q\in Q.$$

Proof. Notice that $c_{10}(z,p,q) := U_zU_pq + U_pU_zq - \{U_zp,q,p\} = -c_5(z,p,q) - c_8(z,p,q)$, hence $2c_{10}(z,p,q) = 0$ by (3.2). Using (1.1), we just need to prove $c_{10}(z,Q,Q) \subseteq \mathrm{Ann}_Q(\mathrm{Ker}\,2\,\mathrm{Id}_Q)$. But using (0.2)(viii) yields $c_{10}(z,p,q) = U_{z\circ p}q - z \circ U_p(q \circ z) \in \mathrm{Ann}_Q(\mathrm{Ker}\,2\,\mathrm{Id}_Q)$ by (3.7). □

Notice that, up to now, only the outer ideal absorption properties have been needed. The next results, aimed at studying the centrality of U_z, will make explicit use of inner ideal absorption.

3.11. Lemma. *Let J be a nondegenerate Jordan algebra, Q be a Martindale-like cover of J, and $z \in C(J)$. Then $c_5(z,J,Q) = 0$.*

Proof. Let $x \in J$, $q \in Q$, and I be an essential ideal of J such that $U_qI \subseteq J$, $U_I(U_zU_xq) \subseteq J$, and $U_I(U_xU_zq) \subseteq J$, which exists by (2.5). For any $y \in I$, $a \in J$,

$$\begin{aligned} U_{U_yU_zU_xq}a &= U_yU_zU_xU_qU_xU_zU_ya \quad \big(\text{by (0.2)(iii)}\big)\\ &= U_yU_xU_zU_qU_xU_zU_ya \quad \big(\text{since } U_qU_xU_zU_ya \subseteq U_qI \subseteq J, \text{ and } U_z \in \Gamma(J)\big)\\ &= U_yU_xU_zU_qU_zU_xU_ya \quad \big(\text{since } U_ya \in J, \text{ and } U_z \in \Gamma(J)\big)\\ &= U_{U_yU_xU_zq}a \quad \big(\text{by (0.2)(iii)}\big). \end{aligned}$$

By (1.3), we have $U_yc_5(z,x,q) = U_yU_zU_xq - U_yU_xU_zq \in \mathrm{Ann}_J(\mathrm{Ker}\,2\,\mathrm{Id}_J)$. But $2U_yc_5(z,x,q) = U_y2c_5(z,x,q) = 0$ by (3.2), hence $U_yc_5(z,x,q) = 0$ by (1.1).

We have shown that $U_Ic_5(z,x,q) = 0$, which implies $c_5(z,x,q) = 0$ by (2.4). □

3.12. Lemma. *Let J be a nondegenerate Jordan algebra, Q be a Martindale-like cover of J, and $z \in C(J)$. Then, for any $x, y \in J$, $q \in Q$,*

(i) $\{U_zx,q,x\} = 2U_zU_xq \in 2Q$, *so that* $\{U_zx,q,x\} \in \mathrm{Ann}_Q(\mathrm{Ker}\,2\,\mathrm{Id}_Q)$,
(ii) $\{U_zx,q,y\} + \{x,q,U_zy\} \in \mathrm{Ann}_Q(\mathrm{Ker}\,2\,\mathrm{Id}_Q)$.

Proof. By (3.10), $U_zU_xq = -U_xU_zq + \{U_zx,q,x\}$, which implies (i) using (3.11), (2.3), and (1.2), whereas (ii) follows by linearizing (i). □

3.13. Lemma. *Let J be a nondegenerate Jordan algebra, Q be a Martindale-like cover of J, $z \in C(J)$, $q \in Q$, and I be an essential ideal of J such that $U_qI + U_Iq \subseteq J$. Then,*

$$\{U_zq,y,q\} = 2U_zU_qy \in 2Q, \quad \textit{so that } \{U_zq,y,q\} \in \mathrm{Ann}_Q(\mathrm{Ker}\,2\,\mathrm{Id}_Q),$$

for any $y \in I$.

Proof. For any $x \in I$, $u \in \mathrm{Ker}\,2\,\mathrm{Id}_Q$,

$$\begin{aligned} U_uU_x\{U_zq,y,q\} &= U_u\big[\big\{x,U_zq,\{y,q,x\}\big\} - \{U_xU_zq,q,y\}\big] \quad \big(\text{by (0.2)(xiv)}\big)\\ &= U_u\big[U_z\big\{x,q,\{y,q,x\}\big\} - \{U_zU_xq,q,y\}\big]\\ &\qquad \big(\text{by applying (3.11) to both terms since } \{y,q,x\} \in U_Iq \subseteq J\big) \end{aligned}$$

$$\begin{aligned}
&= U_u\big[U_z\big\{x, q, \{y, q, x\}\big\} - \{U_x q, q, U_z y\}\big] \\
&\quad \big(\text{by (3.12)(ii) since } U_x q \in U_I q \subseteq J, \text{ and (1.2)}\big) \\
&= U_u\big[U_z\big\{x, q, \{y, q, x\}\big\} - \{x, U_q x, U_z y\}\big] \quad \big(\text{by (0.2)(iv)}\big) \\
&= U_u\big[U_z\big\{x, q, \{y, q, x\}\big\} - U_z\{x, U_q x, y\}\big] \\
&\quad \big(\text{since } U_q x \in U_q I \subseteq J \text{ and } U_z \in \Gamma(J)\big) \\
&= U_u\big[U_z\big\{x, q, \{y, q, x\}\big\} - U_z\{U_x q, q, y\}\big] \quad \big(\text{by (0.2)(iv)}\big) \\
&= U_u U_z U_x \{q, y, q\} \quad \big(\text{by (0.2)(xiv)}\big) \\
&= U_u 2 U_z U_x U_q y = 0
\end{aligned}$$

by (1.2). Also $U_u U_x 2U_z U_q y = 0$ by (1.2), hence we have shown

$$U_I c_8(z, q, y) = U_I\big[\{U_z q, y, q\} - 2U_z U_q y\big] \subseteq \mathrm{Ann}_Q(\mathrm{Ker}\, 2\, \mathrm{Id}_Q).$$

But $2U_I c_8(z, q, y) = U_I 2c_8(z, q, y) = 0$ by (3.2), so that $U_I c_8(z, q, y) = 0$ by (1.1). Therefore $c_8(z, q, y) = 0$ by (2.4), i.e., $\{U_z q, y, q\} = 2U_z U_q y$. □

3.14. Lemma. *Let J be a nondegenerate Jordan algebra, Q be a Martindale-like cover of J, $z \in C(J)$, $q \in Q$, and I be an essential ideal of J such that $U_q I + U_I q \subseteq J$. Then, for any $y \in I$, $c_5(z, q, y) = 0$.*

Proof. By (3.10), $U_q U_z y = -U_z U_q y + \{U_z q, y, q\} = U_z U_q y$ using (3.13). □

3.15. Proposition. *Let J be a nondegenerate Jordan algebra, Q be a Martindale-like cover of J, and $z \in C(J)$. Then $c_5(z, Q, Q) = 0$.*

Proof. Let $p, q \in Q$ and I be an essential ideal of J such that $U_I q + \{q, I, I\} + U_I p + U_p I \subseteq J$, which exists by (2.5). If we take $K := U_I I$, we also have that K is an essential ideal of J [12, 1.2(a)], and $U_K q \subseteq I$, as in (III)(2) of the proof of (3.1).

For any $x \in K$,

$$\begin{aligned}
U_x U_z U_p q &= U_{\{p,z,x\}} q - U_p U_z U_x q - \big\{p, z, U_x\{z, p, q\}\big\} + \{U_p U_z x, q, x\} \quad \big((0.2)(\text{xi})\big) \\
&= U_{\{z,p,x\}} q - U_z U_p U_x q - \big\{z, p, U_x\{p, z, q\}\big\} + \{U_z U_p x, q, x\} \\
&\quad \big(\text{by (3.9)(i) and (3.14) since } U_x q \in I\big) \\
&= U_x U_p U_z q
\end{aligned}$$

using again (0.2)(xi). We have shown that $U_K c_5(z, p, q) = 0$, which implies that $c_5(z, p, q) = 0$ by (2.4). □

3.16. Theorem. *Let J be a nondegenerate Jordan algebra, Q be a Martindale-like cover of J. Then $C(J) \subseteq C(Q)$.*

Proof. Put together (3.8), (3.9)(ii), and (3.15), and use (0.7). □

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