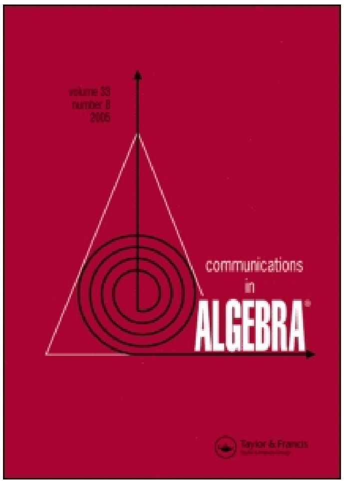

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A NOTE ON A RESULT OF KOSTRIKIN

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Let L be a Lie algebra over a ring of scalars in which $2, 3, \dots, r$ ***are invertible, and let x be an ad-nilpotent element of index n with*** $n + \left[\frac{n}{2}\right] - 1 \le r$. ***We prove that*** $\mathrm{ad}_x^{n-1}(L)$ ***is an abelian inner ideal of L. In particular, for every*** $a \in L$, $\mathrm{ad}_x^{n-1}(a)$ ***is ad-nilpotent of index at most 3, which extends a result of Kostrikin [2, Lemma 1.1, p. 31].***

Key Words: Ad-nilpotent element; Inner ideal; Lie algebra.

2000 Mathematics Subject Classification: 17B30.

1. INTRODUCTION

Kostrikin's Lemma is a fundamental result on Lie algebras.

Kostrikin's Lemma ([2, Lemma 1.1, p. 31]). *Let L be a Lie algebra over a field of characteristic p and* $x \in L$ *be an ad-nilpotent element of L of index n, with* $4 \le n < p$. *Then, for every* $a \in L$, $\mathrm{ad}_x^{n-1}(a)$ *is an ad-nilpotent element of L of index at most* $n - 1$.

This result has been used to obtain ad-nilpotent elements of index at most 3 from ad-nilpotent elements of greater indexes. Ad-nilpotent elements of index at most 3 play a fundamental role in Benkart's proof of Kostrikin's conjecture that any finite-dimensional simple nondegenerate Lie algebra (over a field of characteristic greater than 5) is classical, see [1, 3].

We say that a submodule B of a Lie algebra L over a ring of scalars Φ is an inner ideal of L if $[B, [L, B]] \subset B$, and it is an abelian inner ideal if it is an inner ideal with $[B, B] = 0$, [1, §1]. Abelian inner ideals are closely related with ad-nilpotent elements of index 3: an element $x \in L$ is ad-nilpotent of index 3 if and only if it is contained in the abelian inner ideal $\Phi x + [x, [x, L]]$, see [1].

In this article, we prove that every ad-nilpotent element of index n of a Lie algebra L over a ring of scalars in which $2, 3, \dots, r$ are invertible, $r \ge n + \left[\frac{n}{2}\right] - 1$, gives rise to the abelian inner ideal $\mathrm{ad}_x^{n-1}(L)$. This result extends [1, 1.10] where this

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fact was proved for ad-nilpotent elements of index 3. Moreover, Kostrikin's Lemma is also generalized in the sense that we show that $\mathrm{ad}_x^{n-1}a$ is already ad-nilpotent of index at most 3 for any $a \in L$.

2. MAIN

The following lemma is part of the proof of [2, Theorem 3.1, p. 40].

Lemma 2.1. *Let $n, r, s \in \mathbb{N}$ with $2r + s \leq n$ and $A(r, s)$ the matrix $A(r, s) = (\alpha_{ij})_{i,j=0,\ldots,r}$ where $\alpha_{ij} := (-1)^{s+i+j}\binom{n}{s+i+j}$. Then the determinant of A is equal to*

$$|A(r, s)| = (-1)^{s(r+1)+\binom{r+1}{2}} \frac{\prod_{i=0}^{r+s-1}\binom{n+r-i}{r+1}}{\prod_{i=0}^{r+s-1}\binom{2r+s-i}{r+1}}.$$

From now on, to simplify the notation, let us denote by capital letters the adjoint maps, i.e., $X = \mathrm{ad}_x$, $A = \mathrm{ad}_a$ for any $x, a \in L$.

Lemma 2.2. *Let L be a Lie algebra over a ring of scalars in which $2, 3, \ldots, r$ are invertible, and let $x \in L$ be an ad-nilpotent element of L of index n. Let $0 < m < n$, $a \in L$, and let us denote*

$$y_1 = X^m A X^{n-1},\ y_2 = X^{m+1} A X^{n-2}, \ldots, y_{n-m} = X^{n-1} A X^m.$$

(i) *If $2m \geq n$ and $r \geq 2n - m - 1$, then all $y_1, \ldots, y_{n-m}$ are zero.*
(ii) *If $2m < n$ and $r \geq n + m - 1$, then for any $1 \leq i \leq n - 2m + 1$, we can express the m elements $y_i, \ldots, y_{i+m-1} \in \{y_1, \ldots, y_{n-m}\}$ as a linear combination of the rest $n - 2m$ elements of $\{y_1, \ldots, y_{n-m}\}$.*

Proof. Since $X^n = 0$, we have that $\mathrm{ad}_{X^n(a)} = 0$. So, for $\alpha_i = (-1)^i\binom{n}{i}$,

$$\alpha_1 X A X^{n-1} + \alpha_2 X^2 A X^{n-2} + \cdots + \alpha_{n-1} X^{n-1} A X = 0. \tag{1}$$

Let us multiply (1) by X^{m-1-i} on the left, and by X^i on the right, $i = 0, 1, 2, \ldots, m - 1$:

$$\begin{aligned} \alpha_1 X^m A X^{n-1} + \cdots + \alpha_{n-m}\, X^{n-1} A X^m &= 0 \\ \alpha_2 X^m A X^{n-1} + \cdots + \alpha_{n-m+1}\, X^{n-1} A X^m &= 0 \\ \vdots \qquad\qquad \vdots \\ \alpha_m X^m A X^{n-1} + \cdots + \alpha_{n-1}\, X^{n-1} A X^m &= 0. \end{aligned}$$

If we adopt the notation $y_1 = X^m A X^{n-1}$, $y_2 = X^{m+1} A X^{n-2}, \ldots, y_{n-m} = X^{n-1} A X^m$, these equations can be written as a linear system with m equations and $n - m$

unknowns $y_1, \ldots, y_{n-m}$. The matrix of the system is

$$A = \begin{pmatrix} \alpha_1 & \cdots & \cdots & \alpha_{n-m} \\ \alpha_2 & \cdots & \cdots & \alpha_{n-m+1} \\ \vdots & & & \vdots \\ \alpha_m & \cdots & \cdots & \alpha_{n-1} \end{pmatrix} \in \mathrm{Mat}_{m\times(n-m)}.$$

(i) If $2m \geq n$, the first $n-m$ equations give an homogeneous linear system of equations with matrix of the form $A(n-m-1, 1)$ in the notation of 2.1. Looking at its determinant, we conclude that this matrix is invertible in the ring of scalars if $r \geq 2n-m-1$. In this case, all unknowns of the homogeneous system are zero.

(ii) If $2m < n$, for any $1 \leq i \leq n-2m+1$, the columns $(i, \ldots, i+m-1)$ of A form the matrix $A(m-1, i)$ in the notation of 2.1. This matrix is invertible in the ring of scalars when $r \geq n+m-1$, so we can express the unknowns $y_i, \ldots, y_{i+m-1} \in \{y_1, \ldots, y_{n-m}\}$ as a linear combination of the rest $n-2m$ unknowns. □

Theorem 2.3. *Let L be a Lie algebra over a ring of scalars in which $2, 3, \ldots, r$ are invertible, and let $x \in L$ be an ad-nilpotent element of index n, with $n + \left[\frac{n}{2}\right] - 1 \leq r$. Then $\mathrm{ad}_x^{n-1}(L)$ is an abelian inner ideal of L.*

Proof. Let $a \in L$. We are going to show that $(\mathrm{ad}_{X^{n-1}(a)})^2 = X^{n-1}\xi$ for some ξ in the subalgebra of $\mathrm{End}\, L$ generated by X and A, so $\mathrm{ad}_x^{n-1}(L)$ is an inner ideal of L:

$$(\mathrm{ad}_{X^{n-1}(a)})^2 = \left(\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}X^{n-1-i}AX^i\right)\left(\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}X^jAX^{n-1-j}\right)$$
$$= \sum_{i,j=0}^{n-1}(-1)^{i+j}\binom{n-1}{i}\binom{n-1}{j}X^{n-1-i}AX^{i+j}AX^{n-1-j} \qquad (1)$$

Notice that $X^{n-1-i}AX^{i+j} = 0$ if $2j \geq n$ by 2.2(i) and, similarly, $X^{i+j}AX^{n-1-j} = 0$ if $2i \geq n$, so we can assume in (1) that both $i, j = 0, \ldots, \left[\frac{n}{2}\right]$.

(I) We consider a generic term $X^{n-1-i}AX^{i+j}AX^{n-1-j}$, and we claim that it can be expressed as a linear combination of elements of the form

$$X^{n-1}AX^{\alpha}AX^{n-1-\alpha} \qquad \text{for some } \alpha$$

and elements of the form

$$X^rAX^{2n-2-r-s}AX^s,$$

where the last exponent s is lower than $n-1-j$.

- We are going to show that $X^{n-1-i}AX^{i+j}$ can be expressed as a linear combination of the terms

$$\left\{X^{n-1}AX^j, X^{n-j-2}AX^{2j+1}, X^{n-j-3}AX^{2j+2}, \ldots, X^{j+1}AX^{n-2}, X^jAX^{n-1}\right\}.$$

(i) If $i \geq j+1$ or $i = 0$, the element $X^{n-1-i}AX^{i+j}$ is already one of

$$\{X^{n-1}AX^{j}, X^{n-j-2}AX^{2j+1}, X^{n-j-3}AX^{2j+2}, \ldots, X^{j+1}AX^{n-2}, X^{j}AX^{n-1}\}.$$

(ii) If $i < j+1$, by 2.2(ii) the set of j elements

$$\{X^{n-2}AX^{j+1}, \ldots, X^{n-1-j}AX^{2j}\}$$

can be expressed as a linear combination of

$$\{X^{n-1}AX^{j}, X^{n-j-2}AX^{2j+1}, X^{n-j-3}AX^{2j+2}, \ldots, X^{j+1}AX^{n-2}, X^{j}AX^{n-1}\}.$$

In particular, $X^{n-1-i}AX^{i+j} \in \{X^{n-2}AX^{j+1}, \ldots, X^{n-1-j}AX^{2j}\}$ can be expressed as a linear combination of

$$\{X^{n-1}AX^{j}, X^{n-j-2}AX^{2j+1}, X^{n-j-3}AX^{2j+2}, \ldots, X^{j+1}AX^{n-2}, X^{j}AX^{n-1}\}.$$

- This means that the monomial $X^{n-1-i}AX^{i+j}AX^{n-1-j}$ is expressed as a linear combination of $X^{n-1}AX^{j}AX^{n-1-j}$ and the terms

$$\{X^{n-j-2}AX^{2j+1}AX^{n-1-j}, X^{n-j-3}AX^{2j+2}AX^{n-1-j}, \ldots, X^{j}AX^{n-1}AX^{n-1-j}\}.$$

- Take any monomial appearing in the last set

$$X^{n-j-1-k}AX^{2j+k}AX^{n-1-j}, \qquad k = 1, \ldots, n-1-2j.$$

By 2.2(i) and (ii), the $j+k$ elements in

$$\{X^{2j+2k-1}AX^{n-j-k}, \ldots, X^{j+k}AX^{n-1}\}$$

are either zero (when $2j+2k \geq n$) or can be expressed as a linear combination of the terms

$$\{X^{2j+2k}AX^{n-1-j-k}, \ldots, X^{n-1}AX^{j+k}\},$$

so in particular $X^{2j+k}AX^{n-1-j} \in \{X^{2j+2k-1}AX^{n-j-k}, \ldots, X^{j+k}AX^{n-1}\}$ can be expressed as a linear combination of

$$\{X^{2j+2k}AX^{n-1-j-k}, \ldots, X^{n-1}AX^{j+k}\}.$$

- This means that any monomial $X^{n-1-j-k}AX^{2j+k}AX^{n-1-j}$ can be expressed as a linear combination of

$$\{X^{n-1-j-k}AX^{2j+2k}AX^{n-1-j-k}, \ldots, X^{n-1-j-k}AX^{n-1}AX^{j+k}\}.$$

- Since $k \geq 1$, all the monomials appearing in the set are elements of the form $X^{r}AX^{2n-2-r-s}AX^{s}$, where the last exponent s is lower than $n-1-j$.

(II) We repeat this process with all monomials of (1) until they are a linear combination of terms of the form $X^{n-1}AX^{\alpha}AX^{n-1-\alpha}$ for some α, and terms whose last exponent is lower than $\frac{n}{2}-1$; in this last case, the sum of the first two exponents is greater than $n-1+\frac{n}{2}$ and the corresponding monomial is zero by 2.2(i).

To see that $\mathrm{ad}_x^{n-1}(L)$ is abelian, consider two elements $X^{n-1}(a), X^{n-1}(b) \in \mathrm{ad}_x^{n-1}(L)$. Then

$$\left[X^{n-1}(a), X^{n-1}(b)\right] = \mathrm{ad}_{X^{n-1}a}X^{n-1}(b) = \sum_i (-1)^i \binom{n-1}{i} X^{n-i}AX^{i+n-1}(b),$$

but the sum of the two exponents appearing in the monomials of this last formula is $2n-1$ and by 2.2(i) they are all zero. □

If L is a Lie algebra over a ring of scalars Φ without k-torsion, $k \leq n + [n/2] - 1$, we can regard L as a Lie algebra over the scalar extension $\widetilde{\Phi} = S^{-1}\Phi$, for $S = \left\{1, \ldots, n + \left[\frac{n}{2}\right] - 1\right\}$. Let us denote this Lie algebra by $L_{\tilde{\Phi}}$. Given an ad-nilpotent element $x \in L$ of index n, by Theorem 2.3, $\mathrm{ad}_x^{n-1}L$ is an abelian inner ideal of $L_{\tilde{\Phi}}$. All elements of an abelian inner ideal are ad-nilpotent of index at most 3, so we get the following corollary.

Corollary 2.4 (Generalization of Kostrikin's Lemma). *Let L be a Lie algebra over a ring of scalars without k-torsion, $k \leq n + \left[\frac{n}{2}\right] - 1$, and let $x \in L$ be an ad-nilpotent element of L of index n. Then, for every $a \in L$, $\mathrm{ad}_x^{n-1}(a)$ is ad-nilpotent of index at most* 3.

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