On Quotient Rings in Alternative Rings: Fountain-Gould left quotients rings

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Abstract

We give a a necessary and sufficient condition on an alternative ring R in order to have a Fountain-Gould left order.

Keywords: Alternative rings, maximal ring of quotient, Fountain-Gould left order.

1 Introduction

The theory of rings of quotients has its origins between 1930 and 1940, in the works of O. Ore and K. Osano on the construction of the total ring of fractions. In that decade, Ore proved that a necessary and sufficient condition for a ring R to have a (left) classical ring of quotients is that for any regular element a in R, and any $b \in R$ there exist a regular $c \in R$ and $d \in R$ such that $cb = da$ (left Ore condition). At the end of the 50's, Goldie, Lesieur and Croisot characterized the (associative) rings that are classical left orders in semiprime and left Artinian rings [9, Chapter IV], (result known as Goldie's Theorem).

Later on in 1956, Y. Utumi introduced the notion of general left quotient rings [11] and proved that the rings without right zero divisors are precisely those which have a maximal left quotient ring.

Following Goldie's idea of characterizing certain types of rings via a suitable envelope, R. E. Johnson characterized those rings R whose maximal left quotient rings are von Neumann regular, see [9, (13.36)], and P. Gabriel specialized it further by giving characterizations for those rings whose maximal left quotient rings are semisimple, i.e., isomorphic to a finite direct product of rings of the form $End_{\Delta_i}(V_i)$ for suitable finite dimensional left vector spaces V_i over division rings Δ_i, see [9, (13.40)].

In 1990, J. Fountain and V. Gould, basing on ideas from semigroup theory, see [7], introduced a notion of order in a ring which need not to have an identity and gave, see [8], a Goldie-like characterization of two-sided orders in semiprime rings with descending chain conditions on principal one-sided ideals (equivalently, coinciding with their socles). Later, P. Ánh and L. Márki, see [1], extended this result to one-sided orders and, more recently, the same authors developed a general theory of Fountain-Gould left quotient rings, see [2] (we point out that the maximal left quotient ring plays a fundamental role in this work).

It is natural to ask whether similar notions (and results) can be obtained for alternative rings.

K.L. Beidar and A.V. Mikhalev, interested in the structure of nondegenerate and purely alternative algebras, introduced what they referred to as the almost classical localization of an algebra and described, using the theory of orthogonally complete algebraic systems, the structure of this type of algebras, see [5, (2.12)].

The question of Goldie's Theorems for alternative algebras was posed by H. Essannouni and A. Kaidi, see [6], for Noetherian alternative rings. Later, in 1994, the same authors established a Goldie-like theorem for alternative rings without elements of order three in its associator ideal. In [10], M. Gómez Lozano and M. Siles Molina introduced Fountain-Gould left orders in alternative rings and gave a Goldie-like characterization of alternative rings which are Fountain-Gould left orders in nondegenerate alternative rings which coincide with their socle (this result generalizes the classical Goldie's Theorems for alternative rings without additional conditions). In this work the authors introduced, as a tool, the notion of general left quotient rings (for nondegenerate alternative rings) and related properties of a ring to any of its general rings of quotients. In [3] we gave a general theory of maximal left quotient ring: We constructed a maximal left quotient ring of any alternative ring which is a left quotient ring of itself and proved that this is an alternative ring when $D(R)$ is semiprime or 2-torsion free. Furthermore, in [4], we introduced a notion of left nonsingularity for alternative rings and proved that an alternative ring is left nonsingular if and only if every essential left ideal is dense, if and only if its maximal left quotient ring is von Neumann regular (a Johnson-like Theorem) and we obtained a Gabriel-like Theorem for alternative rings.

In this paper, following the associative theory, see [2], we give a necessary and sufficient condition on an alternative ring R in order to have a Fountain-Gould left order (an Ore-like condition for Fountain-Gould left orders in alternative rings).❧

2 Main

2.1 The following three basic central subsets can be considered in a ring R: The associative center $N(R)$, the commutative center $K(R)$, and the center $Z(R)$, defined by:

$$\begin{aligned} N(R) &= \{x \in R \mid (x, R, R) = (R, x, R) = (R, R, x) = 0\}, \\ K(R) &= \{x \in R \mid [x, R] = 0\}, \\ Z(R) &= N(R) \cap K(R), \end{aligned}$$

where $[x, y] = xy - yx$ denotes the commutator of two elements $x, y \in R$ and $(x, y, z) = (xy)z - x(yz)$ is the associator of three elements x, y, z of R.

2.2 The defining axioms for an alternative ring R are the left and the right alternative laws:

$$(x, x, y) = 0 = (y, x, x)$$

for every $x, y \in R$. As a consequence, we have that the associator is an alternating function of its arguments. The standard reference for alternative rings is [12].

2.3 For a ring R, R^1 will denote its unitization, i.e., R if the ring is unital, or $\mathbb{Z} \times R$ with product $(m, x)(n, y) := (mn, nx + my + xy)$ if R has no unity.

2.4 We recall that for every nonempty subset X of an alternative ring R, the left annihilator of X is defined as the set

$$lan(X) := \{a \in R \mid ax = 0 \text{ for all } x \in X\},$$

written $lan_R(X)$ when it is necessary to emphasize the dependence on R. Similarly, the right annihilator of X, $ran(X) = ran_R(X)$, is defined by

$$ran(X) := \{a \in R \mid xa = 0 \text{ for all } x \in X\}.$$

We also write $ann(X) = ann_R(X) := lan(X) \cap ran(X)$ to denote the annihilator of X. In general, the left (right) annihilator of a subset X of an alternative ring R does not have to be a left (right) ideal. However, it is true if X is a right (left) ideal of R or if $X \subset N(R)$.

2.5 The notion of left quotient ring of an alternative ring was introduced in [10], where the relationship among classical, Fountain-Gould and this type of rings of quotients was established.

Let R be a subring of an alternative ring Q. We recall that Q is a left quotient ring of R, if:

(1) $N(R) \subset N(Q)$ and

(2) for every $p, q \in Q$, with $p \neq 0$, there exists $r \in N(R)$ such that $rp \neq 0$ and $rq \in R$.

In [3] we proved that an alternative ring R has a unique, up isomorphism, maximal left quotient ring if and only if R is a left quotient ring of itself. We denote the maximal left quotient ring of an alternative ring R as $Q^l_{max}(R)$. The maximal left quotient ring of an alternative ring R can be constructed as follows:

We will say that a left ideal I of R is dense if for every $p, q \in R$, with $p \neq 0$, there exists $a \in N(R)$ such that $ap \neq 0$ and $aq \in I$.

We denote by $\mathcal{F}^*$ the set of all left ideals A of $N(R)$ such that for every $0 \neq x \in R$ and $\mu \in N(R)$ there exists $\lambda \in N(R)$ such that $\lambda x \neq 0$ and $\lambda\mu \in A$.

Let $\mathcal{F} := \{R^1 A \mid A \in \mathcal{F}^*\}$, by [3, Proposition 2.6] every element of $\mathcal{F}$ is a dense left ideal of R and if I is a dense ideal of R, $N(I) \in \mathcal{F}^*$.

Let us consider

$$S := \{(I, f) \mid I \in \mathcal{F} \text{ and } f \in Hom^*_{N(R)}(I, R)\}$$

where $Hom^*_{N(R)}(I, R)$ denotes the set of all homomorphism of left $N(R)$–modules from I to R such that for every $x \in R$ and $\lambda, \mu \in N(I)$, $(x\lambda)f = x((\lambda)f)$ and $([\lambda, \mu])f \in N(R)$. Then $Q^l_{max}(R) = S/\approx$, where $\approx$ is the equivalence relation, $(I, f) \approx (I', f')$ if and only if there exists $I'' \in \mathcal{F}$, with $I'' \subset I \cap I'$, such that $f|_{I''} = f'|_{I''}$, and operations: For any $q = [A_q, f_q], q' = [A_{q'}, f_{q'}]$,

- Sum: $q + q' = [R^1(A_q \cap A_{q'}), f_q + f_{q'}]$.
- Product: $qq' = [R^1 A_{qq'}, f_{qq'}]$ where $A_{qq'} := \{\lambda \in A_q \text{ such that } (\lambda)f_q \in I_{q'}\}$ and $(\sum x_i a_i)f_{qq'} := \sum x_i\, ((a_i)f_q)f_{q'}$.

2.6 Let a be an element of an alternative ring R. We recall that:

(i). An element $b \in R$ is a group inverse of a if the following conditions hold:

$$aba = a, bab = b, ab = ba$$

It is easy to prove that the group inverse is unique. So we denote by $a^\sharp$ the group inverse of an element $a \in R$.

(ii). We say that a is left semiregular or left square cancellable (respectively right semiregular or right square cancellable) if $a^2x = a^2y$ implies $ax = ay$ ($xa^2 = ya^2$ implies $xa = ya$) for any $x, y \in R \cup \{1\}$. We will say that a is semiregular if it is left and right semiregular. We denote by

$$\mathcal{S} := \{a \in N(R) \text{ such that } a \text{ is semiregular in } R\}.$$

2.7 Let R be a subring of an alternative ring Q. We recall that R is a Fountain-Gould left order in Q if:

(i). Every element of $\mathcal{S}$ has a group inverse in Q, and

(ii). Every $q \in Q$ can be written in the form $q = a^\sharp x$, where $a \in \mathcal{S}$ and $x \in R$.

When only condition (ii) is satisfied we speak about weak Fountain-Gould left order.

The next lemma is a review of properties of Fountain-Gould left orders that can be found in [10, 6.3, 6.4, 6.5, 6.6] and [3, 3.1].

Lemma 2.8 If R is a Fountain-Gould left order in Q, then:

(i). If $q \in Q$, there exist $a \in \mathcal{S}$ and $b \in R$ with $q = a^\sharp b$ and $aa^\sharp b = b$.

(ii). $N(R) \subset N(Q)$, and $a^\sharp \in N(Q)$ for every $a \in \mathcal{S}$.

(iii). Common Denominator Property: For a finite number of elements of Q, $q_1, q_2, ..., q_n$, there exist $a \in \mathcal{S}$ and $b_1, b_2, ..., b_n \in R$ such that $q_i = a^\sharp b_i$ with $aa^\sharp b_i = b_i$.

(iv). Every semiregular element of R is semiregular in Q.

(v). Q is a left quotient ring of R, so $Q \subset Q^l_{max}(R)$.

Corollary 2.9 If R is a Fountain-Gould left order in Q and $a \in \mathcal{S}$, then

$$lan_Q(a) = lan_Q(a^2) = lan_Q(a^\sharp) = lan_Q((a^\sharp)^2).$$

Proposition 2.10 Let R be an alternative ring and let $a \in N(R)$ such that a has a group inverse in $Q = Q^l_{max}(R)$. Then:

(i). If $q = a^\sharp x \in Q$, then $lan_Q(a) \subset lan_Q(q)$. In particular, if $q \in R$, we have that $lan_R(a) \subset lan_R(q)$.

(ii). Let us suppose that $Ra + lan_R(a)$ is a dense left ideal of R and let $x \in R$, is such that $lan_R(a) \subset lan_R(x)$, then $a^\sharp ax = x$.

Proof: By definition $a \in N(Q)$. Moreover for every $p, q \in Q^l_{max}(R)$,

$$((a^\sharp)^2, p, q)a^4 = ((a^\sharp)^2 a^4, p, q) = (a^2, p, q) = 0.$$

So $0 = (((a^\sharp)^2, p, q)a^4)(a^\sharp)^3 = ((a^\sharp)^2, p, q)a^4)a = ((a^\sharp)^2 a, p, q)a^4) = (a^\sharp, p, q)$ and therefore, $a^\sharp \in N(Q^l_{max}(R))$.

(i). If $pa = 0$, $pq = p(a^\sharp b) = (pa^\sharp)b = (pa(a^\sharp)^2)b = 0$.

(ii). Since $(Ra + lan_R(a))(x - a^\sharp ax) = Ra(x - a^\sharp ax) = R(ax - ax) = 0$, we have that $x = a^\sharp ax$ since Q is a left quotient ring of $Ra + lan_R(a)$.❧

The next theorem is a generalization to alternative rings of [2, Theorem 6]. Moreover, the proof of [2, Theorem 6] has been adapted to this case. So we have a Ore-like condition for Fountain-Gould left orders in alternative rings.

Theorem 2.11 Let R be an alternative ring with $N = N(R)$. Then R has a Fountain-Gould left quotient ring if and only if it satisfies the following conditions:

(i). For every $x \in R$ there is $a \in \mathcal{S}$ such that $lan_R(a) \subset lan_R(x)$;

(ii). For every $a \in \mathcal{S}$ and $x \in R$, $(lan_N(a) + Na)x = 0$ implies $x = 0$;

(iii). For every $a, b \in \mathcal{S}$ there exist $c \in \mathcal{S}$ and $r, s \in N$ such that

$$lan_R(c) \subset lan_R(a) \cap lan_R(b),\ ca = ra^2,\ cb = sb^2;$$

(iv). For every $a, b \in \mathcal{S}$ and $x \in R$ there exist $u \in \mathcal{S}$, $z \in R$ and $r \in N$ such that

$$lan_R(u) \subset lan_R(a),\ ua = ra^2,\ rxb = zb^2$$

Remark:

(iii') In condition (iii), we can choose $r \in Ra$, $s \in Rb$: Applying the condition to $a^2, b^2 \in \mathcal{S}$ there exists $c \in \mathcal{S}$ and $r, s \in N$ such that $ca^2 = ra^4$, hence (since

$a \in \mathcal{S}$) $ca = ra^3 = (ra)a^2$, and $cb = sb^3 = (sb)b^2$. Moreover, $lan_R(a) = lan_R(a^2)$ and $lan_R(b) = lan_R(b^2)$ implies that $lan_R(c) \subset lan_R(a^2) \cap lan_R(b^2) = lan_R(a) \cap lan_R(b)$. So $c \in \mathcal{S}$ and $ra, sb \in N$ satisfies (iii).

(iv') In condition (iv), u, z, r can be chosen so that $lan_R(u) \subset lan_R(z) \cap lan_R(r)$: In the same way as above, we first choose u, y, r and then replace them by u^2, uz, ur. Furthermore, z can be chosen in Rb and r can be chosen in Ra as one can see in the same way as in the previous remark.

Proof: NECESSITY: (1) Let $y \in R$, $a \in \mathcal{S}$ such that $x = a^\sharp y$. Then, by 2.10 (i), $lan_R(a) \subset lan_R(x)$.

(2) If $ax \neq 0$, then there exist $c \in \mathcal{S}, y \in R$ such that $ax = c^\sharp y$ with $c(ax) = cc^\sharp y = y \neq 0$, so $Nax \neq 0$. if $ax = 0$, then $0 \neq x = (1 - a^\sharp a)x$. So, by the common denominator property, there exist $c_1 \in \mathcal{S}, y_1, y_2 \in R$ such that $x = c_1^\sharp y_1$ and $a^\sharp a = c_1^\sharp y_2$ with $0 \neq y_1 = c_1 x = c_1(1 - aa^\sharp)x$. Therefore, since $c_1(1 - aa^\sharp) \in lan_N(a)$, we have that $lan_N(a)x \neq 0$.

(3) Put $a^\sharp = c^\sharp r$ and $b^\sharp = c^\sharp s$ with $cc^\sharp r = r$, $cc^\sharp s = s$, then $ca^\sharp = r$, $cb^\sharp = s$, hence $r, s \in N$ by 2.8 (ii), and $ca = c(a^\sharp a^2) = (ca^\sharp)a^2 = ra^2$, and similarly $cb = sb^2$.

(4) Choose $u \in \mathcal{S}$, $r, z \in R$ such that $a^\sharp x b^\sharp = u^\sharp z$, $a^\sharp = u^\sharp r$ with $z = uu^\sharp z$ and $r = uu^\sharp r = ua^\sharp \in N(Q)$, so $r \in R \cap N(Q) = N(R)$ by 2.8 (ii). Moreover, $lan_R(u) \subset lan_R(a)$ by 2.10(i). Now, as above, we can see that $ua = ra^2$, so

$$(u^\sharp r)xb^\sharp = a^\sharp x b^\sharp = u^\sharp z$$

and hence

$$rxb = (uu^\sharp r)x(b^\sharp b^2) = u(u^\sharp r x b^\sharp)b^2 = (uu^\sharp z)b^2 = zb^2$$

SUFFICIENCY: The proof has 7 steps:

(1) For every $b \in \mathcal{S}$, $lan_R(b) + Rb$ is a dense left ideal of R: Let $x, y \in R$ with $x \neq 0$. Applying condition (i) to x there exists $a \in \mathcal{S}$ such that $lan_R(a) \subset lan_R(x)$. Now, if we apply condition (iv) to $a, b \in \mathcal{S}$ and $ay \in R$, there exist $u \in \mathcal{S}$, $z \in R$, $r \in N$ verifying $lan_R(u) \subset lan_R(a)$, $ua = ra^2$, $r(ay)b = rayb = zb^2$. Since $(u - ra)a = 0$ and $lan_R(a) \subset lan_R(x)$ we obtain $rax = ux$. Now, this element is nonzero (in other case since $lan_R(u) \subset lan_R(a) \subset lan_R(x)$, $(lan_N(u) + Nu)x = 0$ a contradiction by (ii)), so $rax \neq 0$. Now, since $(ray - zb)b = 0$, we have that $ray \in lan_R(b) + Rb$, so $n = ra \in N$ verifies $nx \neq 0$, $ny \in lan_R(b) + Rb$.

(2) Now by [3, 2.3] and (1), for every $b \in \mathcal{S}$, R is a left quotient ring of $lan_R(b) + Rb$ and therefore by [3, 1.12], R is a left quotient ring of itself. Then, by the construction given in 2.5 (see [3]) there exists, $Q^l_{max}(R)$, the maximal left quotient ring of R, where R is a dense $N(R)$-submodule of $Q^l_{max}(R)$ with $N(R) \subset N(Q^l_{max}(R))$ and where $Q^l_{max}(R)$ verifies the linearizations of the alternative laws.

(3) We prove that every element of $\mathcal{S}$ has a group inverse in $Q^l_{max}(R)$: Let $a \in \mathcal{S}$. We have seen, by (1) since $a^2 \in \mathcal{S}$, that $lan_R(a^2) + Ra^2 = lan_R(a) + Ra^2$ is a dense left ideal of R. So, by [3, 2.6(i)], $N(lan_R(a) + Ra^2) \in \mathcal{F}^*$.

Let us define the homomorphism of $N(R)$-modules: $a^\sharp : lan_R(a) + Ra^2 \longrightarrow R$, $(l + xa^2)a^\sharp = xa$. This map is well defined since a is semiregular. Now, we want to see that $[R^1 N(lan_R(a) + Ra^2),\ a^\sharp]$ is an element of $Q^l_{max}(R)$. By construction of $Q^l_{max}(R)$, see 2.5, we have to prove that for every $y \in R$ and $\lambda, \mu \in N(lan_R(a) \oplus Ra^2)$, $(y\lambda)a^\sharp = y((\lambda)a^\sharp)$: If $l' + x'a^2 \in N(R)$,

$$((l + xa^2)(l' + x'a^2))a^\sharp = (ll' + xa^2l' + lx'a^2 + xa^2x'a^2)a^\sharp = lx'a + xa^2x'a$$
$$= (l + xa^2)x'a = (l + xa^2)((l' + x'a^2)a^\sharp),$$

and $([\lambda, \mu])a^\sharp \in N(R)$: If $\lambda = l + xa^2, \mu = l' + x'a^2 \in N(lan_R(a) \oplus Ra^2) \subset N(R)$, then for every $y, z \in R$ we have $0 = (l+xa^2, y, z) = (l, y, z)+(xa^2, y, z)$, so multiply by a, $0 = (l, y, z)a+(xa^2, y, z)a = (la, y, z)+(xa^3, y, z) = (x, y, z)a^3$ which implies, since a is semiregular in R that $0 = (x, y, z)a = (xa, y, z)$ then $xa \in N(R)$ and $(l, y, z) = 0$, hence $l \in N(R)$. Similarly, $l', x'a \in N(R)$. Therefore,

$$[\lambda, \mu]a^\sharp = l(x'a) + (xa)a(x'a) - l'(xa) - (x'a)a(xa) \in N(R).$$

And it is straightforward that $aa^\sharp = a^\sharp a$, $a(a^\sharp)^2 = a^\sharp$ and $a^\sharp a^2 = a$.

(4) If $a \in \mathcal{S}$ then $a^\sharp \in N(Q^l_{max}(R))$: For every $p, q \in Q^l_{max}(R)$, we have that

$$a^3((a^\sharp)^2, p, q) = (a^3(a^\sharp)^2, p, q) = (a, p, q) = 0$$

which implies that

$$0 = (a^\sharp)^2(a^3((a^\sharp)^2, p, q)) = a((a^\sharp)^2, p, q) = (a^\sharp, p, q).$$

The other conditions are verified since $Q^l_{max}(R)$ satisfies the linearizations of the alternative laws.

(5) We prove that the set $Q := \{a^\sharp x \text{ with } a \in S,\ x \in R\}$ is a subring of $Q^l_{max}(R)$: By condition (iii'), given $a, b \in \mathcal{S}$, there exist $c \in \mathcal{S}$, $r, s \in N$ such that $ca = ra^2$, $cb = sb^2$ and $lan_R(c) \subset lan_R(a) \cap lan_R(b)$ with $r \in Ra$, $s \in Rb$. Then by 2.10 (ii) and 2.9, $a^\sharp = c^\sharp ca^\sharp = c^\sharp ca(a^\sharp)^2 = c^\sharp ra^2(a^\sharp)^2 = c^\sharp r$, and similarly $b^\sharp = c^\sharp s$, so $a^\sharp x_1 + b^\sharp x_2 = c^\sharp(rx_1 + sx_2) \in Q$. Next, given $a, b \in \mathcal{S}$ and $x \in R$, by condition (iv') there exist $u \in \mathcal{S}$, $z \in R$ and $r \in N$ such that $lan_R(u) \subset lan_R(a)$, $ua = ra^2$ and $rxb = zb^2$. Then, by 2.10 (ii) and 2.9,

$$a^\sharp = u^\sharp ua^\sharp = u^\sharp ua(a^\sharp)^2 = u^\sharp ra^2(a^\sharp)^2 = u^\sharp r,$$

and therefore

$$a^\sharp xb^\sharp = u^\sharp rxb(b^\sharp)^2 = u^\sharp zb^2(b^\sharp)^2 = u^\sharp z,$$

so $(a^\sharp x)(b^\sharp y) = (a^\sharp xb^\sharp)y = u^\sharp zy \in Q$.

(6) R is contained in Q. Let $r \in R$, by condition (i) there exists $c \in \mathcal{S}$ such that $lan_R(c) \subset lan_R(r)$. So by 2.10 (ii), $r = c^\sharp cr \in Q$.

(7) $N(R) \subset N(Q)$. It is straightforward since $N(R) \subset N(Q^l_{max}(R))$.

(8) Now, given $p, q \in Q$ there exist $a, b \in \mathcal{S}$ and $x, y \in R$ such that $p = a^\sharp x$ and $q = b^\sharp y$. So

$$(p, p, q) = (a^\sharp x, a^\sharp x, b^\sharp y) = (a^\sharp)^2 b^\sharp(x, x, y) = 0$$
$$(q, p, p) = (b^\sharp y, a^\sharp x, a^\sharp x) = b^\sharp(a^\sharp)^2(y, x, x) = 0$$

So R is a Fountain-Gould left order in the alternative ring Q. ❧

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