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Quotients in Graded Lie Algebras. Martindale-like Quotients for Kantor Pairs and Lie Triple Systems

Esther García · Miguel Gómez Lozano

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Abstract In this paper we prove that the maximal algebra of quotients of a nondegenerate Lie algebra with a short $\mathbb{Z}$-grading is $\mathbb{Z}$-graded with the same support. As a consequence, we introduce a notion of Martindale-like quotients for Kantor pairs and Lie triple systems and construct their maximal systems of quotients.

Keywords Martindale-like quotients · Lie triple systems · Kantor pairs · Lie algebras · Maximal algebra of quotients

Mathematics Subject Classifications (2010) Primary 17B70; Secondary 17A40 · 17B60 · 17B65

1 Introduction

Martindale rings of quotients were introduced by W. S. Martindale in 1969 for prime rings [8]. This concept was designed for applications to rings satisfying a generalized polynomial identity (GPI for short). In his work, Martindale showed that every prime

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E. García (✉)
Departamento de Matemática Aplicada, Universidad Rey Juan Carlos,
28933 Móstoles (Madrid), Spain
e-mail: esther.garcia@urjc.es

M. Gómez Lozano
Departamento de Álgebra, Geometría y Topología, Universidad de Málaga,
29071 Málaga, Spain
e-mail: magomez@agt.cie.uma.es

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ring satisfying a GPI is a subring of a primitive ring Q with nonzero socle and, moreover, the division ring associated to the socle of Q is finite dimensional over its center (this result generalizes both Amistur's and Posner's theorems).

In 1972, S. A. Amitsur generalized the construction of Martindale rings of quotients to the setting of semiprime rings, see [2]. This notion has proven to be useful not only for the theory of rings with identities, but also for Galois theory on non-commutative rings and for the study of prime ideals under ring extensions in general.

In 1989, when studying Jordan algebras and triple systems of symmetric elements, K. McCrimmon generalized Martindale's quotients to non-necessarily semiprime rings by introducing the notion of Martindale rings of quotients with respect to a filter of "denominators", see [10].

In 2004, the authors introduced the notion of Martindale-like systems of quotients of a Jordan system with respect to a filter of ideals, see [6]. Moreover, inspired by Martínez' idea of moving from a Jordan setting to a Lie one through the Tits-Kantor-Koecher construction, see [9], and using the construction of maximal Lie algebras of quotients of Siles Molina [13], they gave explicit constructions of maximal Jordan systems of Martindale-like quotients for nondegenerate Jordan pairs, triple systems and algebras with respect to power filters of ideals. A fundamental point in their construction is the fact that the maximal Lie algebra of quotients of a nondegenerate $\mathbb{Z}$-graded Lie algebra $L = L_{-1} \oplus L_0 \oplus L_1$ is $\mathbb{Z}$-graded with the same support, see [6, Corollary 1.9].

In this paper we prove that if G is an abelian group and L is a G-graded Lie algebra with finite support, then its maximal algebra of quotients with respect to a power filter of sturdy graded ideals is G-graded. In particular, if L is a G-graded Lie algebra with finite support (for example, if L is finite dimensional or if G is finite), the notion of maximal graded algebra of quotients introduced in [12] coincides in fact with the notion of maximal algebra of quotients with respect to the filter of all essential graded ideals given in [6, 2.2]. Moreover, if we specialize to a $\mathbb{Z}$-graded Lie algebra with finite support, the maximal graded algebra of quotients [12] is nothing but the maximal algebra of quotients in the sense of [13]. Furthermore, in this case, both the Lie algebra and its maximal algebra of quotients have the same support.

These results make possible to define Martindale-like quotients for both Kantor pair and Lie triple systems with respect to power filters of sturdy ideals, and to construct their maximal systems in the nondegenerate cases.

2 Quotients in Graded Lie Algebras

Given a (non necessarily associative) algebra A, we say that a linear map $d : A \to A$ is a *derivation* if d satisfies $d(ab) = d(a)b + ad(b)$ for every $a, b \in A$. If I is an ideal of A and $d : I \to A$ satisfies the above condition, d is called *partial derivation* on A. If G is an abelian group, $A = \sum_{g \in G} A_g$ is a G-graded algebra and I is a G-graded ideal, a partial derivation $d : I \to A$ is *homogeneous of degree* k if $d(I_g) \subset A_{g+k}$ for every $g \in G$.

Lemma 2.1 *Let A be a (non-necessarily associative) algebra over Φ. Let G be an abelian group and suppose that $A = \sum_{g \in G} A_g$ is G-graded with finite support. Then* *any partial derivation $d : I \to A$ defined on a graded ideal I of A is a sum of homogeneous partial derivations. In particular,* Der A *is a G-graded algebra.*

Proof Let I be a G-graded ideal of A, and let $d : I \to A$ be a partial derivation. Then $d = \mathrm{id}\, d\, \mathrm{id}$, where id denotes the identity map. But $\mathrm{id} = \sum \pi_i$ where $\pi_i : A \to A_i$ is the projection onto A_i, so $d = \sum_k \sum_{i-j=k} \pi_i d \pi_j$. Since the support of A is finite, the sums appearing in the last formula are finite. Moreover, for each k, $d_k = \sum_{i-j=k} \pi_i d \pi_j$ is a partial derivation of A: For any $a, b \in I$, $a \cdot b = \sum_{g \in G} \sum_m a_m \cdot b_{g-m}$ so

$$\begin{aligned}
d_k(a \cdot b) &= \sum_{i-j=k} \pi_i d \pi_j (a \cdot b) = \sum_{i-j=k} \pi_i d (\sum_m a_m \cdot b_{j-m}) \\
&= \sum_{i-j=k} \sum_m \pi_i (d(a_m) \cdot b_{j-m}) + \sum_{i-j=k} \sum_m \pi_i (a_m \cdot d(b_{j-m})) \\
&= \sum_{i-j=k} \sum_m \pi_{i-j+m} d \pi_m (a_m) \cdot b_{j-m} + \sum_{i-j=k} \sum_m a_m \cdot \pi_{i-m} d \pi_{j-m} (b_{j-m}) \\
&= d_k(a) \cdot b + a \cdot d_k(b).
\end{aligned}$$

□

Definition 2.2 The annihilator of an ideal I of a Lie algebra L is defined as $\mathrm{Ann}_L(I) = \{x \in L \mid [x, I] = 0\}$. It is an ideal of L. We say that an ideal of L is sturdy if it has zero annihilator.

A *filter* $\mathcal{F}$ on a Lie algebra is a nonempty family of nonzero ideals such that for any $I_1, I_2 \in \mathcal{F}$ there exists $I \in \mathcal{F}$ such that $I \subseteq I_1 \cap I_2$. Moreover, $\mathcal{F}$ is a *power filter* if for any $I \in \mathcal{F}$ there exists $K \in \mathcal{F}$ such that $K \subseteq [I, I]$.

A *Lie algebra of quotients* Q of a Lie algebra L with respect to a power filter of sturdy ideals $\mathcal{F}$ is $L \subset Q$ such that for every nonzero element $q \in Q$ there exists an ideal $I_q \in \mathcal{F}$ such that $0 \neq [q, I_q] \subseteq L$.

The maximal Lie algebra of quotients $Q_{\mathcal{F}}(L)$ of L with respect to $\mathcal{F}$ is built as the direct limit of derivations on ideals of $\mathcal{F}$,

$$Q_{\mathcal{F}}(L) = \lim_{\rightarrow} \mathrm{Der}(I, L), \quad I \in \mathcal{F},$$

i.e., $Q_{\mathcal{F}}(L)$ consists on equivalence classes of partial derivations $[d, I]$ for $I \in \mathcal{F}$, where $[d_1, I_1] = [d_2, I_2]$ if d_1 coincides with d_2 on I of $\mathcal{F}$ such that $I \subset I_1 \cap I_2$. This Lie algebra $Q_{\mathcal{F}}$ is maximal among all Lie algebras of quotients of L with respect to $\mathcal{F}$: for any nonzero element s in a Lie algebra of quotients S of L with respect to $\mathcal{F}$ there exists $I_s \in \mathcal{F}$ such that $0 \neq [s, I_s] \subseteq L$, so the map $s \mapsto [\mathrm{ad}\, s, I_s]$ is a monomorphism of S into $Q_{\mathcal{F}}(L)$, see [6, 13].

From Lemma 2.1 we obtain

Corollary 2.3 *Let G be an abelian group and let $L = \sum_{g \in G} L_g$ be a graded Lie algebra with finite support. If we consider a power filter $\mathcal{F}$ of sturdy graded ideals of L, then the maximal algebra of quotients of L with respect to $\mathcal{F}$ is G-graded.*

Remark 2.4 If G is an abelian group and L is a semiprime G-graded Lie algebra with finite support, the maximal graded algebra of quotients given in [12] is in fact the maximal algebra of quotients of L with respect to the power filter of all sturdy graded ideals.

An element x in a Lie algebra L is an absolute zero divisor of L if $\mathrm{ad}_x^2 = 0$, and L is nondegenerate if it has no nonzero absolute zero divisors.

Proposition 2.5 *Let L be a nondegenerate Lie algebra. If we consider a power filter $\mathcal{F}$ of sturdy ideals of L and the maximal algebra of quotients Q of L with respect to $\mathcal{F}$, then*

$$[q, [q, I]] = 0 \implies q = 0$$

for every $q \in Q$ and every ideal $I \in \mathcal{F}$.

Proof Suppose that $q \neq 0$, let $J \in \mathcal{F}$ be an absorbing ideal for q, and consider an ideal K of $\mathcal{F}$ contained in $I \cap J$ (notice that also K is an absorbing ideal for q, i.e., $0 \neq [q, K] \subset L$). From now on, we are going to denote adjoint maps by capital letters, i.e., $Q = \mathrm{ad}\, q$, $C = \mathrm{ad}\, c$, etc. Since $Q^2 = 0$ on K, for any $c \in K$ we have

$$QCQ = 0 \text{ on } K$$

Therefore, $0 = Q[C, [C, [C, [C, Q]]]]Q = 4QC^2QC^2Q$, i.e.,

$$QC^2QC^2Q = 0 \text{ on } K.$$

Moreover $\mathrm{ad}^2_{[q,c]} = -QC^2Q$ so $[q, c] \in L$ is ad-nilpotent of index at most 3 on K. The fundamental identity for Lie algebras gives

$$\mathrm{ad}^2_{\mathrm{ad}^2_{[q,c]} c} = \mathrm{ad}^2_{[q,c]} C^2 \mathrm{ad}^2_{[q,c]} = QC^2QC^2QC^2Q = 0 \text{ on } K,$$

i.e., $[[q, c], [[q, c], c]] \in K$ is an absolute zero divisor of K for every $c \in K$, i.e, $[[q, c], [[q, c], c]] = 0$ for every $c \in K$ by nondegeneracy of K.

Then $[[q, a + \lambda b], [[q, a + \lambda b], a + \lambda b]] = 0$ for every $a, b \in K$ and $\lambda \in \Phi$. But

$$\begin{aligned} 0 = [[q, a + \lambda b], [[q, a + \lambda b], a + \lambda b]] &= [[q, a], [[q, a], a]] + 3\lambda[[q, a], [[q, a], b]] \\ &\quad + 3\lambda^2[[q, b], [[q, b], a]] + \lambda^3[[q, b], [[q, b], b]] \end{aligned}$$

since $[[q, a], [[q, b], c]]$ is symmetric in its three arguments a, b, c, see [5, 2.1]. There are enough elements in Φ, so $[[q, a], [[q, a], b]] = 0$ for any $b \in K$, which implies that for every $a \in K$, $[q, a] \in \mathrm{Ann}_L(K) = 0$ (since K is sturdy). But this leads to a contradiction since K must absorb q nontrivially. □

The following result extends [6, 2.3] to Lie algebras with finite $\mathbb{Z}$-gradings.

Lemma 2.6 *Let L be a nondegenerate Lie algebra with finite $\mathbb{Z}$-grading $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$. Then every power filter $\mathcal{F}$ of sturdy ideals of L gives rise to a power filter $\mathcal{F}^{gr}$ of sturdy graded ideals.*

Proof Any nonzero ideal I of L contains a nonzero graded ideal: consider the ideal $\pi(I) = \pi_{-n}(I) \oplus \cdots \oplus \pi_0(I) \oplus \cdots \oplus \pi_n(I)$. By nondegeneracy of L, hence of

I, if k is the greatest integer for which $\pi_k(I) \neq 0$ then $-k$ is the least integer for which $\pi_{-k}(I) \neq 0$, so $0 \neq [L_k, [L_k, I]] = [L_k, [L_k, \pi_{-k}(I)]] \subset I$ and I contain nonzero homogeneous elements, which generate a nonzero graded ideal contained in I.

Let now I be a sturdy ideal of L, and let I^{gr} be the biggest graded ideal contained in I. We claim that I^{gr} is sturdy: for any ideal J of L, $I \cap J$ is nonzero by sturdiness of I, so the ideal $I \cap J$ contains a nonzero graded ideal of L, which is contained in $I^{gr} \cap J$. □

In [6, 2.4] it is shown that the maximal algebra of quotients of a 3-graded Lie algebra is 3-graded. This result is extended in the next corollary to Lie algebras with finite $\mathbb{Z}$-gradings.

Corollary 2.7 *Let L be a nondegenerate Lie algebra with finite $\mathbb{Z}$-grading $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$. If we consider a power filter $\mathcal{F}$ of sturdy ideals of L, then the maximal algebra of quotients Q of L with respect to $\mathcal{F}$ has a finite $\mathbb{Z}$-grading $Q = Q_{-n} \oplus \cdots \oplus Q_0 \oplus \cdots \oplus Q_n$.*

Proof By Corollary 2.3, Q is $\mathbb{Z}$-graded. If $q \in Q$ were an homogeneous element of degree k for $|k| > n$ and $I \in \mathcal{F}^{gr}$ were a graded absorbing ideal for q, then $[q, [q, [I, I]]] \subset L$ would be zero by grading, so $q = 0$ by Proposition 2.5. □

Remark 2.8 If L is a nondegenerate $\mathbb{Z}$-graded Lie algebra with finite support, then the maximal graded algebra of quotients Q of L is in fact the maximal algebra of quotients of L with respect to the power filter of all essential ideals of L. Moreover, L and Q have the same support.

Remark 2.9 Let F be a field and V an F-vector space with base $B = \{v_i,\ i \in \mathbb{N}\}$. Let us consider $W \leq V^*$ the subspace generated by the linearly independent maps $w_i : V \to F$ defined as $(v_j)w_i := \delta_{ij}$ (Kronecker delta). Then $(V, W, <, >, F)$ is a dual pair of vector spaces ($<, >: V \times W \to F$ given by $< v, w >:= (v)w$). Let $R := W \otimes_F V$ be the associative algebra with product

$$(w_1 \otimes v_1)(w_2 \otimes v_2) =< v_1, w_2 > w_1 \otimes v_2.$$

Note that $R \subset End_F(V)$ ($w \otimes v : V \to V$ with $x(w \otimes v) :=< x, w > v$) is a simple associative algebra with zero center. Moreover, R is a $\mathbb{Z}$-graded algebra $R = \oplus_{n\in\mathbb{Z}} R_n$ with R_n the vector subspace generated by the elements $w_i \otimes v_j$ with $j - i = n$. In these conditions, $L = [R^-, R^-]$ is a simple nondegenerate $\mathbb{Z}$-graded Lie algebra in which its maximal algebra of quotients $Q = \mathrm{Der}(L)$ is not $\mathbb{Z}$-graded. Indeed, if we consider $d : V \to V$ given by $d(v_i) = v_{2i}$, we claim that $\mathrm{ad}_d : L \to L$ is an element of $\mathrm{Der}(L)$ which can not be written as a finite sum of homogeneous derivations:

(1) $\mathrm{ad}_d : R^- \to R^-$ is a derivation, and therefore it is a derivation on L: $\mathrm{ad}_d(w_i \otimes v_j) = d(w_i \otimes v_j) - (w_i \otimes v_j)d = (dw_i) \otimes v_j$- $w_j \otimes (v_j d) = w_k \otimes v_j - w_j \otimes v_{2j}$ where $w_k = 0$ if i is odd and $w_k = w_{i/2}$ if i is even. Therefore, $\mathrm{ad}_d : R^- \to R^-$ is a well defined derivation on R^- (because $R^- \subset End_F(V)^-$), and hence it is a derivation on L.

(2) To show that ad_d is not a finite sum of homogeneous derivations, suppose on the contrary that d can be written as a finite sum of homogeneous derivations,

$d = \sum_{i=1}^{n} d_{\alpha_i}$, $d_{\alpha_i} \in R_{\alpha_i}$. Then $\mathrm{ad}_d(R_1) \subset \bigoplus \sum_{i=1}^{n} R_{\alpha_{i+1}}$, but for any $i \in \mathbb{N}$, $\mathrm{ad}_d(w_{2i} \otimes v_{2i+1}) = w_i \otimes v_{2i+1} - w_{2i} \otimes v_{4i+2} \in R_{i+1} \oplus R_{2i+2}$, a contradiction.

3 Quotients in Kantor Pairs

Definition 3.1 Recall that a Kantor pair $V = (V^+, V^-)$ consists of two Φ-modules with trilinear products $\{\ ,\ ,\ \} : V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma$, $\sigma = \pm$, that satisfy

$$[V_{x,y}, V_{z,w}] = V_{\{x,y,z\},w} - V_{z,\{y,x,w\}}, \tag{KP1}$$

$$K_{K_{z,w}x,y} = K_{z,w}V_{x,y} + V_{y,x}K_{z,w} \tag{KP2}$$

where $[A, B] := AB - BA$, $V_{x,y}z := \{x, y, z\}$, and $K_{a,b}z := \{a, z, b\} - \{b, z, a\}$, see [1, p.533].

If L is a $\mathbb{Z}$-graded Lie algebra of the form $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$, then the pair of Φ-modules (L_1, L_{-1}) with product $\{x, y, z\} := [[x, y], z]$, for every $x, z \in L_\sigma$, $y \in L_{-\sigma}$, is a Kantor pair, see [1, p.533]. Conversely, it follows from [1] or [3, Th. 4.3 and Cor. 4.6] that for any Kantor pair $V = (V^+, V^-)$ there exists a unique up to isomorphism 5-graded Lie algebra $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ with the following properties: (i) V is isomorphic to the Kantor pair (L_1, L_{-1}), (ii) $L_{2\sigma} = [L_\sigma, L_\sigma]$, $L_0 = [L_\sigma, L_{-\sigma}]$ for $\sigma = \pm$, and (iii) if $[x_{-2} + x_0 + x_2, L_1 \oplus L_{-1}] = 0$ then $x_{-2} + x_0 + x_2 = 0$. Thus, after identifying $V \equiv (L_1, L_{-1})$ the products in L are $[[x, z], y] = K_{x,z}y$, $[[x, y], z] = V_{x,y}z$ and $[[y, x], z] = -V_{x,y}z$ for $x, z \in L_\sigma$, $y \in L_{-\sigma}$. We will call this Lie algebra the standard embedding of V and denote it by $\mathcal{L}_V$. If both V^+ and V^- are contained in a Lie algebra L, then $\mathcal{L}_V$ can be regarded as

$$([V^+, V^+] \oplus V^+ \oplus [V^+, V^-] \oplus V^- \oplus [V^-, V^-])/C_V$$

where

$$C_V = \{x \in [V^+, V^+] \oplus [V^+, V^-] \oplus [V^-, V^-] \mid [x, V^+ \oplus V^-] = 0\}.$$

An element $x \in V^\sigma$ is an absolute zero divisor of $V = (V^+, V^-)$ if $\{x, V^{-\sigma}, x\} = 0$. A Kantor pair without nonzero absolute zero divisors is called nondegenerate. The Kantor pair V is nondegenerate if and only if $\mathcal{L}_V$ is a nondegenerate Lie algebra, [5, 2.6]. A pair $I = (I^+, I^-)$ of Φ-submodules of V is an ideal of V if $\{I^\sigma, V^{-\sigma}, V^\sigma\} + \{V^\sigma, I^{-\sigma}, V^\sigma\} + \{V^\sigma, V^{-\sigma}, I^\sigma\} \subset I^\sigma$ for $\sigma = \pm$. The annihilator of I in V is $\mathrm{Ann}_V(I) = (\mathrm{Ann}_V(I)^+, \mathrm{Ann}_V(I)^-)$ with

$$\begin{aligned}\mathrm{Ann}_V(I)^\sigma = \{x \in V^\sigma \mid \{V^\sigma, I^{-\sigma}, x\} = \{I^\sigma, V^{-\sigma}, x\} = \{x, I^{-\sigma}, V^\sigma\}\\ = \{V^{-\sigma}, x, I^{-\sigma}\} = \{I^{-\sigma}, x, V^{-\sigma}\} = \{x, V^{-\sigma}, I^\sigma\} = 0\}\end{aligned}$$

We have that $\mathrm{Ann}_V(I)$ is an ideal of V. We say that an ideal of V is sturdy if it has zero annihilator. If $\mathcal{I}$ is the ideal of $\mathcal{L}_V$ generated by $I^+ \cup I^-$ [5, 2.5],

$$\mathcal{I} = [I^+, V^+] \oplus I^+ \oplus ([I^+, V^-] + [V^+, I^-]) \oplus I^- \oplus [I^-, V^-],$$

then $\mathrm{Ann}_V(I) = V \cap \mathrm{Ann}_{\mathcal{L}_V}(\mathcal{I})$.

Definition 3.2 A *filter* $\mathcal{F}$ on a Kantor pair is a nonempty family of nonzero ideals such that for any $I_1, I_2 \in \mathcal{F}$ there exists $I \in \mathcal{F}$ such that $I \subseteq I_1 \cap I_2$. Moreover, $\mathcal{F}$ is a *power filter* if for any $I \in \mathcal{F}$ there exists $K \in \mathcal{F}$ such that $K \subseteq I_\sharp$, where $I_\sharp = (I_\sharp^+, I_\sharp^-)$ with $I_\sharp^\sigma = \{I^\sigma, V^{-\sigma}, I^\sigma\}, \sigma = \pm$.

Let V be a Kantor pair and consider a filter $\mathcal{F}$ on V. We say that a Kantor pair W is a *pair of quotients of V with respect to $\mathcal{F}$* if W is $\mathcal{F}$-absorbed into V, i.e., for each $0 \neq q \in W^\sigma$ there exists an ideal $I_q \in \mathcal{F}$ such that

$$\{q, I_q^{-\sigma}, V^\sigma\} + \{q, V^{-\sigma}, I_q^\sigma\} + \{V^\sigma, I_q^{-\sigma}, q\} + \{I_q^\sigma, V^{-\sigma}, q\} \subseteq V^\sigma$$

$$\{I_q^{-\sigma}, q, V^{-\sigma}\} + \{V_q^{-\sigma}, q, I^{-\sigma}\} \subseteq V^{-\sigma},$$

with $\{q, I_q^{-\sigma}, V^\sigma\} + \{q, V^{-\sigma}, I_q^\sigma\} + \{V^\sigma, I_q^{-\sigma}, q\} + \{I_q^\sigma, V^{-\sigma}, q\} \neq 0$ or $\{I_q^{-\sigma}, q, V^{-\sigma}\} + \{V_q^{-\sigma}, q, I^{-\sigma}\} \neq 0$.

Proposition 3.3 *Let V be a Kantor pair.*

(i) *If $\mathcal{F}$ is a filter on V and we consider*

$$\mathcal{F}_\mathcal{L} = \{\mathcal{K} = \mathrm{id}_{\mathcal{L}_V}((K^+ \cup K^-)) \mid (K^+, K^-) \in \mathcal{F}\},$$

then $\mathcal{F}_\mathcal{L}$ is a filter on $\mathcal{L}_V$, and it is a power filter if $\mathcal{F}$ is so.

(ii) *The ideal $\mathcal{K}$ generated in $\mathcal{L}_V$ by an ideal (K^+, K^-) of V is sturdy in $\mathcal{L}_V$ if and only if (K^+, K^-) is sturdy in V.*

(iii) *For any Kantor pair of quotients W of V with respect to a filter of sturdy ideals $\mathcal{F}$, the standard envelopes $\mathcal{L}_V \subseteq \mathcal{L}_W$.*

(iv) *W is a pair of quotients of V with respect to a power filter of sturdy ideals $\mathcal{F}$ if and only if $\mathcal{L}_W$ is a Lie algebra of quotients for $\mathcal{L}_V$ with respect to the power filter of sturdy ideals $\mathcal{F}_\mathcal{L}$.*

Proof

(i) Follows as in [6, 2.9(i)], adapting its proof for Kantor pairs and taking into account that for any ideal I of $\mathcal{F}$ and any ideal K contained in $I_\sharp$, $\mathcal{K} \subset [\mathcal{I}, \mathcal{I}]$.

(ii) Follows since $L_2 \oplus L_0 \oplus L_{-2}$ does not contain nonzero ideals of $\mathcal{L}_V$, see [5, 2.6], and $\mathrm{Ann}_V(I) = V \cap \mathrm{Ann}_{\mathcal{L}_V}(\mathcal{I})$.

(iii) The containment $\mathcal{L}_V \subseteq \mathcal{L}_W$ follows as in [6, 2.10] since the elements of C_W can be characterized as elements $x \in [W^+, W^+] \oplus [W^+, W^-] \oplus [W^-, W^-]$ such that $[x, V^+ \oplus V^-] = 0$.

(iv) It is enough to show that if $\mathcal{L}_W$ is a Lie algebra of quotients for $\mathcal{L}_V$ with respect to the power filter of sturdy ideals $\mathcal{F}_\mathcal{L}$ then W is a pair of quotients of V with respect to $\mathcal{F}$: For any $q \in W^\sigma$, $\sigma = \pm$, if (K^+, K^-) is an ideal of $\mathcal{F}_W$ contained into $I_\sharp$ for any absorbing ideal I of q, then $\mathcal{K}$ absorbs q into V. For any $[q_1, q_2] \in [W^\sigma, W^\sigma]$, $\sigma = \pm$, if I is an absorbing ideal for both q_1 and q_2 and K is an ideal of $\mathcal{F}_W$ contained into $(I_\sharp)_\sharp$, then $\mathcal{K}$ absorbs $[[q_1, q_2]]$ into V. Finally, for any $[q_1, q_2] \in [W^+, W^-]$, if I is an absorbing ideal for both q_1 and q_2 and K is an ideal of $\mathcal{F}_W$ contained into $(I_\sharp)_\sharp$, then $\mathcal{K}$ absorbs $[q_1, q_2]$ into V. □

Corollary 3.4 *Let V be a nondegenerate (strongly prime) Kantor pair, consider a power filter $\mathcal{F}$ on V and let W be a pair of quotients of V with respect to $\mathcal{F}$. Then W is nondegenerate (strongly prime).*

Proof A Kantor pair V is nondegenerate (strongly prime) if and only if its standard embedding $\mathcal{L}_V$ is nondegenerate (strongly prime), see [5, 2.5, 2.7(2)]. Moreover, since $\mathcal{L}_W$ is a Lie algebra of quotients for $\mathcal{L}_V$, $\mathcal{L}_W$ is nondegenerate (strongly prime) by [13, 2.7]. Then, W is nondegenerate (strongly prime) again by [5, 2.5, 2.7(2)]. □

Construction of the maximal Kantor pair of quotients Let V be a nondegenerate Kantor pair and $\mathcal{F}$ a power filter of sturdy ideals on V. Let $\mathcal{F}_{\mathcal{L}_V}$ be the power filter of sturdy ideals of $\mathcal{L}_V$ induced by $\mathcal{F}$. Let $Q_{\mathcal{F}_{\mathcal{L}_V}}(\mathcal{L}_V)$ be the maximal algebra of quotients of $\mathcal{L}_V$ with respect to $\mathcal{F}_{\mathcal{L}_V}$, which is a $\mathbb{Z}$-graded Lie algebra of the form $Q_{-2} \oplus Q_{-1} \oplus Q_0 \oplus Q_1 \oplus Q_2$ by Corollary 2.7. We claim that $Q = (Q_1, Q_{-1})$ is a Kantor pair of quotients of V with respect to $\mathcal{F}$: for any $q \in Q_{\sigma 1}$, $\sigma = \pm$, there exists an ideal $\mathcal{I} \in \mathcal{F}_{\mathcal{L}_V}$ such that $[q, \mathcal{I}] \subset \mathcal{L}_V$, so the ideal $(I^+, I^-) \in \mathcal{F}$ absorbs q into V.

Moreover, Q is maximal in the sense that if $S = (S^+, S^-)$ is another Kantor pair of quotients of V with respect to $\mathcal{F}$, then $(S^+, S^-) \subset (Q_1, Q_{-1})$. Indeed, if S is a Kantor pair of quotients of V with respect to $\mathcal{F}$, then the Lie algebra $\mathcal{L}_S$ is a Lie algebra of quotients of $\mathcal{L}_V$ with respect to $\mathcal{F}_{\mathcal{L}_V}$ by Proposition 3.3(iv). Therefore, by maximality of the Lie algebra of quotients, $\mathcal{L}_S \subset Q_{\mathcal{F}_{\mathcal{L}_V}}(\mathcal{L}_V)$ so $S \subset Q$.

The Kantor pair $Q = (Q_1, Q_{-1})$ will be called the maximal Kantor pair of quotients of V with respect to the filter $\mathcal{F}$, and will be denoted by $Q_{\mathcal{F}}(V)$.

4 Quotients in Lie Triple Systems

Definition 4.1 A Lie triple system T is a Φ-module with a trilinear product $T \times T \times T \to T$ satisfying

$$0 = [x, x, y], \tag{LTS1}$$

$$0 = [x, y, z] + [y, z, x] + [z, x, y], \tag{LTS2}$$

$$[x, y, [z, w, u]] = [[x, y, z], w, u] + [z, [x, y, w], u] + [z, w, [x, y, u]]. \tag{LTS3}$$

If L is a $\mathbb{Z}_2$-graded Lie algebra, $L = L_0 \oplus L_1$, then L_1 with product $[x, y, z] := [[x, y], z]$ is a Lie triple system. Conversely, T is a Lie triple system if and only if there exists a $\mathbb{Z}_2$-graded Lie algebra $L = L_0 \oplus L_1$ with $T = L_1$. In particular, there exists a unique up to isomorphism $\mathbb{Z}_2$-graded Lie algebra $\mathcal{L}_T = L_0 \oplus L_1$ such that T is isomorphic to L_1, $L_0 = [L_1, L_1]$ and $[x_0, L_1] \neq 0$ for every nonzero $x_0 \in L_0$. This Lie algebra is called the standard embedding of T, see for example [7, p. 309], [11, IV] or [4, 2]. Each Kantor pair V gives rise to a Lie triple system $T(V) = V^+ \oplus V^-$, called the associated Lie triple system, see [1, Th. 7] .

An element in a Lie triple system T is an absolute zero divisor if $[x, T, x] = 0$. A Lie triple system without nonzero absolute zero divisors is called nondegenerate. A Lie triple system T is nondegenerate if and only if its standard envelope $\mathcal{L}_T$ is a nondegenerate Lie algebra, see [5, 2.4].

An ideal I of a Lie triple system T is a Φ-submodule of T satisfying $[I, T, T] \subset T$. We define the annihilator of I in T as $\mathrm{Ann}_T(I) = \{x \in T \mid [x, I, T] = [T, I, x] = 0\} = T \cap \mathrm{Ann}_{\mathcal{L}_T}([T, I] \oplus I)$ where $\mathcal{I} := [T, I] \oplus I$ is the ideal of $\mathcal{L}_T$ generated by I. The annihilator of an ideal of T is again an ideal of T. We say that an ideal of T is sturdy of it has zero annihilator.

Definition 4.2 A *filter* $\mathcal{F}$ on a Lie triple system is a nonempty family of nonzero ideals such that for any $I_1, I_2 \in \mathcal{F}$ there exists $I \in \mathcal{F}$ such that $I \subseteq I_1 \cap I_2$. Moreover, $\mathcal{F}$ is a *power filter* if for any $I \in \mathcal{F}$ there exists $K \in \mathcal{F}$ such that $K \subseteq \{I, T, I\}$.

Let T be a Lie triple system and let $\mathcal{F}$ be a filter on T. A Lie triple system Q is a *Lie triple system of quotients of T with respect to $\mathcal{F}$* if Q is $\mathcal{F}$-absorbed into T, i.e., for each $0 \neq q \in Q$ there exists an ideal $I_q \in \mathcal{F}$ such that

$$0 \neq [q, I_q, T] + [q, T, I_q] \subseteq T.$$

Proposition 4.3 *Let T be a Lie triple system.*

(i) *For any ideal $I \in \mathcal{F}$, let $\mathcal{I} = I + [T, I]$. Then $\mathcal{F}_{\mathcal{L}} = \{\mathcal{I} \mid I \in \mathcal{F}\}$ is a filter on $\mathcal{L}_T$, and it is a power filter if $\mathcal{F}$ is so.*
(ii) *The ideal $\mathcal{K}$ generated in $\mathcal{L}_T$ by an ideal K of T is sturdy in $\mathcal{L}_T$ if and only if K is sturdy in T.*
(iii) *For any Lie triple system of quotients Q of T with respect to a filter of sturdy ideals $\mathcal{F}$, the standard envelopes $\mathcal{L}_T \subseteq \mathcal{L}_Q$.*
(iv) *Q is a Lie triple system of quotients of T with respect to a power filter of sturdy ideals $\mathcal{F}$ if and only if $\mathcal{L}_Q$ is a Lie algebra of quotients for $\mathcal{L}_T$ with respect to the power filter of sturdy ideals $\mathcal{F}_{\mathcal{L}}$.*

Proof

(i) As in [6, 2.9(i)] using that for every ideal I of $\mathcal{F}$ and every ideal K contained in $I_\sharp$, $\mathcal{K} \subset [\mathcal{I}, \mathcal{I}]$.
(ii) Follows since $\mathrm{Ann}_T(I) = T \cap \mathrm{Ann}_{\mathcal{L}_T}([T, I] \oplus I)$ and every nonzero graded ideal of $\mathcal{L}_T$ has nonzero intersection with T.
(iii) The containment $\mathcal{L}_T \subseteq \mathcal{L}_Q$ follows as in [6, 2.10] since the elements of $(\mathcal{L}_Q)_0$ can be characterized as elements $z \in [Q, Q]$ such that $[z, T] = 0$.
(iv) It is enough to show that Q is a Lie triple system of quotients of T with respect to $\mathcal{F}$ when $\mathcal{L}_Q$ is a Lie algebra of quotients for $\mathcal{L}_T$ with respect to the power filter of sturdy ideals $\mathcal{F}_{\mathcal{L}}$: For any $q \in (\mathcal{L}_Q)_1 = Q$, if K is an ideal of $\mathcal{F}_T$ contained into $[I, T, I]$ for any absorbing ideal I of q, then $[q, K] \subset [q, [[I, T], I]] \subset [T, T]$, and $[q, [K, T]] \subset T$. For any $[q_1, q_2] \in (\mathcal{L}_Q)_0$, if I is an absorbing ideal for both q_1 and q_2 and K is an ideal of $\mathcal{F}_T$ contained into $[[I, T, I], T, [I, T, I]]$, then $[[q_1, q_2], K] \subset [[q_1, q_2], [[I, T, I], T, [I, T, I]]] \subset T$, and $[[q_1, q_2], [K, T]] \subset [T, T]$. □

Corollary 4.4 *Let T be a nondegenerate Lie triple system, consider a power filter $\mathcal{F}$ of T and let Q be a Lie triple system of quotients of T with respect to $\mathcal{F}$. Then Q is nondegenerate.*

Proof A Lie triple system is nondegenerate if and only if its standard embedding $\mathcal{L}_T$ is nondegenerate, [5, 2.4]. Moreover, since $\mathcal{L}_Q$ is a Lie algebra of quotients for $\mathcal{L}_T$, $\mathcal{L}_Q$ is nondegenerate by [13, 2.7(iii)]. From this, Q is nondegenerate again by [5, 2.4]. □

Construction of the maximal Lie triple system of quotients Let T be a nondegenerate Lie triple system and $\mathcal{F}$ a power filter of sturdy ideals on T. Let us consider $\mathcal{F}_{\mathcal{L}_T}$ the power filter of $\mathcal{L}_T$ obtained from $\mathcal{F}$, and let $Q_{\mathcal{F}_{\mathcal{L}_T}}(\mathcal{L}_T)$ be the maximal algebra of quotients of $\mathcal{L}_T$ with respect to $\mathcal{F}_{\mathcal{L}_T}$, which is a $\mathbb{Z}_2$-graded Lie algebra of the form $Q_0 \oplus Q_1$. We claim that Q_1 is a Lie triple system of quotients of T with respect to $\mathcal{F}$: for any $q \in Q_1$ there exists an ideal $\mathcal{I} = I + [T, I] \in \mathcal{F}_{\mathcal{L}_T}$ such that $[q, \mathcal{I}] \subset \mathcal{L}_T$, i.e., $[q, I] \subset [T, T]$ and $[q, [T, I]] \subset T$. From these two containments it is easy to see that $[q, I, T] + [q, T, I] \subset T$.

Moreover, Q_1 is maximal in the sense that if S is another Lie triple system of quotients of T with respect to $\mathcal{F}$, then $S \subset Q_1$: If S is a Lie triple system of quotients of T with respect to $\mathcal{F}$, then the Lie algebra $\mathcal{L}_S$ is a Lie algebra of quotients of $\mathcal{L}_T$ with respect to $\mathcal{F}_{\mathcal{L}_T}$ by Proposition 4.3(iv). Therefore, by maximality of the Lie algebra of quotients, $\mathcal{L}_S \subset Q_{\mathcal{F}_{\mathcal{L}_T}}(\mathcal{L}_T)$, so $S \subset Q_1$.

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