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Journal of Algebra

www.elsevier.com/locate/jalgebra



## A characterization of the Kostrikin radical of a Lie algebra

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### ARTICLE INFO

#### Article history:

Received 7 March 2011

Available online 16 September 2011

Communicated by Efim Zelmanov

#### MSC:

primary 17B05

secondary 17B60

#### Keywords:

Lie algebra

Absolute zero divisor

Kostrikin radical

Strongly prime ideal

m-Sequence

### ABSTRACT

In this paper we study if the Kostrikin radical of a Lie algebra is the intersection of all its strongly prime ideals, and prove that this result is true for Lie algebras over fields of characteristic zero, for Lie algebras arising from associative algebras over rings of scalars with no 2-torsion, for Artinian Lie algebras over arbitrary rings of scalars, and for some others. In all these cases, this implies that nondegenerate Lie algebras are subdirect products of strongly prime Lie algebras, providing a structure theory for Lie algebras without any restriction on their dimension.

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The theory of radicals constitutes an important tool in the study of rings. This notion appears firstly in the context of non-associative rings: in a work of E. Cartan about finite dimensional Lie algebras  $A$  over  $\mathbb{C}$ , he defined the maximal solvable ideal of  $A$  as the sum of all solvable ideals of  $A$  and proved that  $A$  is semisimple (direct sum of simple ideals) if and only if its radical is zero.

For an associative ring  $R$ , the Baer radical  $r(R)$  is defined as the intersection of all prime ideals of  $R$ , so  $R/r(R)$  is a subdirect product of prime rings, and  $r(R)$  coincides with the smallest ideal of  $R$  such that  $R/r(R)$  is semiprime, see [17]. Similarly, for Jordan systems  $J$  one finds the notion of McCrimmon radical  $Mc(J)$ , which is the least ideal of  $J$  such that  $J/Mc(J)$  is nondegenerate. It coincides with the intersection of all strongly prime ideals of  $J$ ,  $J/Mc(J)$  is a subdirect product of

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<sup>1</sup> The author was partially supported by the MEC and Fondos FEDER MTM2010-16153, by FMQ 264, and by MICINN I3-2010/00075/001.

<sup>2</sup> The author was partially supported by the MEC and Fondos FEDER MTM2010-19482, by FMQ 264 and FQM 3737, and by MICINN I3-2010/00075/001.

strongly prime Jordan systems, and  $Mc(J)$  can be characterized as the set of elements such that any  $m$ -sequence starting with any of them has finite length, see [22] and [19].

For a Lie algebra  $L$ , the smallest ideal inducing a nondegenerate quotient is the Kostrikin radical  $K(L)$ . This radical was first studied by Filippov in [10]. We highlight the works of E. Zelmanov [24,23] where the properties of  $K(L)$  were established and used intensively. Among other properties, it is shown that the Kostrikin radical is inherited by subalgebras ( $K(A) = A$  for any subalgebra  $A \subset K(L)$ ) and by ideals ( $K(I) = I \cap K(L)$  for any ideal  $I$  of  $L$ ) for Lie algebras over fields of characteristic zero.

The goal of this paper is to answer the question: *Is the Kostrikin radical of a Lie algebra  $L$  the intersection of all strongly prime ideals of  $L$ ?* A positive answer to this question would imply that any nondegenerate Lie algebra is a subdirect product of strongly prime Lie algebras, providing a structure theory for Lie algebras without any restriction on their dimension.

In this paper we show that this question has a positive answer for the following types of Lie algebras:

- (1) Nondegenerate Lie algebras  $L$  satisfying that every submodule invariant under inner automorphisms is an ideal of  $L$  and such that every nonzero ideal of  $L$  contains nonzero Jordan elements, Theorem 2.9. In particular, nondegenerate Lie algebras of the form  $L = L_n \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_{-n}$ ,  $L_0 = \sum_{i=1}^n [L_i, L_{-i}]$ , over a ring of scalars  $\Phi$  with  $\frac{1}{k} \in \Phi$  for every  $0 \leq k \leq 4n$ , Corollary 2.10. Furthermore, we relate the Kostrikin radical and the McCrimmon radical when the Lie and the Jordan structures are connected, Corollaries 2.5 and 2.6, and Proposition 2.7.
- (2) Lie algebras over fields of characteristic zero, Theorem 3.10.
- (3) Lie algebras arising from associative algebras over rings of scalars with no 2-torsion, Theorems 4.3, 4.7 and Remark 4.9. Moreover, in these cases we relate the Kostrikin radical of the Lie algebras with the Baer radical of the associative algebras.
- (4) Nondegenerate Lie algebras with chain condition on annihilator ideals over arbitrary rings of scalars, Proposition 5.3; in particular, Artinian Lie algebras, Corollary 5.4.

The key point to prove that the Kostrikin radical is the intersection of all strongly prime ideals is to define  $m$ -sequences for Lie algebras (a notion similar to that of Jordan systems), and to characterize the elements of the Kostrikin radical as those for which every  $m$ -sequence starting with them has finite length. This characterization is true for Lie algebras of type (1) and (3). For Lie algebras as in (2) the notion of  $m$ -sequence needs to be generalized. Generalized  $m$ -sequences for Lie algebras are defined in 3.5, and it is proved that for Lie algebras over fields of characteristic zero the Kostrikin radical coincides with the set of elements such that every generalized  $m$ -sequence starting with them has finite length, Corollary 3.9.

The paper is organized as follows. Section 1 consists on a preliminary section where we recall several notions and results that will be used in the paper. In order to relate the Kostrikin radical of a Lie algebra  $L$  and the McCrimmon radical of some Jordan structures associated to  $L$ , in Section 2 we deal with Lie algebras where every submodule invariant under inner automorphisms is an ideal, which is satisfied by large families of Lie algebras such as Lie algebras generated as algebras by ad-nilpotent elements of index at most  $n$  over a ring of scalars  $\Phi$  with  $\frac{1}{k} \in \Phi$  for  $k = 1, 2, \dots, 2n - 2$ , in particular Lie algebras with a finite  $\mathbb{Z}$ -grading  $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$  with  $L_0 = \sum_{i=1}^n [L_i, L_{-i}]$  over a ring of scalars  $\Phi$  with  $\frac{1}{k} \in \Phi$  for  $k = 1, 2, \dots, 4n$ . There are different constructions to relate Lie and Jordan structures: associated to any ad-nilpotent element  $x$  of index  $\leq 3$  of a Lie algebra  $L$  one can build a Jordan algebra  $L_x$ , and the Kostrikin radical of  $L$  and the McCrimmon radical of  $L_x$  can be compared:  $Mc(L_x) = \{\bar{a} \in L_x \mid [x, [x, a]] \in K(L)\}$ . Similarly, one has the notion of subquotient of a Lie algebra, which is a Jordan pair: if  $V = (M, L/\text{Ker } M)$  is the subquotient, then  $Mc(V)^+ = M \cap K(L)$ , and  $Mc(V)^- = \{a + \text{Ker } M \mid [M, [M, a]] \subset K(L)\}$ . This result generalizes the one given by E. Zelmanov in [23, Lemma 3] where he proved that the McCrimmon radical of the Jordan pair  $(V^+, V^-)$  consisting of two abelian inner ideals  $V^+$  and  $V^-$  of a Lie algebra  $L$  satisfies  $[[Mc(V)^\sigma, V^{-\sigma}], V^\sigma] \subset K(L)$  and where he related the Kostrikin radical of a Lie algebra  $L$  with a short  $\mathbb{Z}$ -grading  $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$  and the McCrimmon radical of the Jordan pair  $V = (L_{-n}, L_n)$ .

Under the technical property that every submodule invariant under inner automorphisms is an ideal, the ad-nilpotent elements of index 3 contained in the Kostrikin radical of  $L$  satisfy that any  $m$ -sequence starting with them has finite length. This makes possible to prove that Lie algebras with enough ad-nilpotent elements are nondegenerate if and only if they are subdirect product of strongly prime ones. In particular, this result applies to any Lie algebra with a finite  $\mathbb{Z}$ -grading,  $L = \bigoplus_{i=-n}^n L_i$ ,  $L_0 = \sum_{i=1}^n [L_i, L_{-i}]$ , over a ring of scalars of characteristic bigger than  $4n$ .

Section 3 of the paper follows a private communication with E. Zelmanov where he dropped the hypothesis of having enough ad-nilpotent elements when dealing with Lie algebras over a field of characteristic zero. Basically Section 3 is [20] with some minors changes made by us. We are grateful to E. Zelmanov for allowing us to include them in the final version of this paper. We highlight the notion of generalized  $m$ -sequence, which is the key point for the results contained in this section.

In Section 4 we relate the Baer radical of an associative algebra  $R$  and the Kostrikin radical of Lie algebras of the form  $R^-$  or  $\text{Skew}(R, *)$  when  $R$  is an associative algebra with involution over a ring of scalars with no 2-torsion. Roughly speaking, the Kostrikin radical of these algebras coincides with the center of  $R^-$  or  $\text{Skew}(R, *)$  modulo the Baer radical  $r(R)$  of  $R$ .

Finally, in Section 5 we study Lie algebras satisfying chain conditions on annihilator ideals and defined over arbitrary rings of scalars; in particular, Artinian Lie algebras and Lie algebras with essential socle.

We remark that each Sections 2, 3, 4 and 5 can be read independently.

## 1. Nondegenerate radicals

**1.1.** We will be dealing with Lie algebras  $L$ , (linear) Jordan algebras  $J$  and (linear) Jordan pairs. As usual, given a Lie algebra  $L$ ,  $[x, y]$  will denote the Lie bracket, with  $\text{ad}_x$  (sometimes denoted by  $X$ ) the adjoint map determined by  $x$ , Jordan algebras  $J$  have bilinear product  $a \bullet b$ , with quadratic operator  $U_a b = 2(a \bullet b) \bullet a - a^2 \bullet b$ , and Jordan pairs  $V = (V^+, V^-)$  have triple products  $\{x, y, z\} \in V^\sigma$ , for  $x, z \in V^\sigma, y \in V^{-\sigma}, \sigma = \pm$ .

**1.2.** We recall that a (non-necessarily associative) algebra  $A$  is a subdirect product of algebras  $\{A_\alpha\}_{\alpha \in \Lambda}$  if there exists a monomorphism  $f : A \rightarrow \prod_{\alpha \in \Lambda} A_\alpha$  such that for every  $\beta \in \Lambda, \pi_\beta \circ f : A \rightarrow A_\beta$  is onto, where  $\pi_\beta : \prod_{\alpha \in \Lambda} A_\alpha \rightarrow A_\beta$  denotes the canonical projection. Notice that this is equivalent to the existence of a family of ideals  $\{I_\alpha\}_{\alpha \in \Lambda}$  of  $A$  such that  $\bigcap_{\alpha \in \Lambda} I_\alpha = 0$  and  $A_\alpha \cong A/I_\alpha$  for all  $\alpha \in \Lambda$ . A subdirect product of  $\{A_\alpha\}_{\alpha \in \Lambda}$  will be called an essential subdirect product if  $A$  contains an essential ideal of the direct product  $\prod_{\alpha \in \Lambda} A_\alpha$ . Recall that an ideal  $I$  of an algebra  $A$  is essential if it intersects nontrivially any nonzero ideal  $K$  of  $A$ , i.e.,  $I \cap K \neq 0$  for every nonzero ideal  $K$  of  $A$ .

**1.3.** A (non-necessarily associative) algebra  $A$  is semiprime if for every nonzero ideal  $I$  of  $A, I^2 := \{xy \mid x, y \in I\} \neq 0$ , and it is prime if  $IJ := \{yx \mid y \in I, x \in J\} \neq 0$  for every nonzero ideals  $I, J$  of  $A$ . Moreover, an ideal  $I$  of  $A$  is semiprime (prime) if the quotient algebra  $A/I$  is semiprime (prime). It is well known that every semiprime ideal  $I$  of an algebra  $A$  is the intersection of all prime ideals of  $A$  which contain  $I$ , see [3,17]. This result implies that the Baer or semiprime radical  $r(A)$  of an algebra  $A$  is the intersection of all prime ideals of  $A$  and therefore that semiprime algebras are exactly subdirect products of prime ones.

**1.4.** An important characterization of primeness and semiprimeness in the associative setting appears in [17]: A ring  $R$  is prime if and only if  $aRb \neq 0$  for arbitrary nonzero elements  $a, b \in R$  and it is semiprime if and only if  $aRa \neq 0$  for every nonzero element  $a \in R$ . Unfortunately (or fortunately) in a general non-associative setting, due to the difficulty of building ideals, these characterizations do not hold. Nevertheless, the above characterizations give rise to new concepts in the Lie and Jordan settings, nondegeneracy and strong primeness (these notions have not been defined in a general non-associative context): An absolute zero divisor in a Jordan algebra  $J$  is an element  $x \in J$  such that the quadratic operator  $U_x = 0$ . A Jordan algebra  $J$  is called nondegenerate if it has no nonzero absolute

zero divisors and it is strongly prime if  $J$  is nondegenerate and prime. An element  $x$  in a Lie algebra  $L$  is ad-nilpotent of index  $k \in \mathbb{N}$  if  $\text{ad}_x^k L = 0$  but  $\text{ad}_x^{k-1} L \neq 0$ . An absolute zero divisor of  $L$  is an ad-nilpotent element of index  $\leq 2$ . A Lie algebra  $L$  is nondegenerate if it has no nonzero absolute zero divisors and it is strongly prime if  $L$  is nondegenerate and prime. Note that if a Lie or Jordan algebra is nondegenerate, then it is semiprime.

**1.5.** Let  $L$  be a Lie algebra. By a nondegenerate (strongly prime) ideal of  $L$  we mean an ideal  $I$  of  $L$  such that the quotient algebra  $L/I$  is nondegenerate (strongly prime). The Kostrikin radical  $K(L)$  of  $L$  is the smallest ideal of  $L$  whose associated quotient algebra  $L/K(L)$  is nondegenerate. It is radical in the sense of Amitsur–Kurosh, see [10], and can be constructed in the following way:  $K_0(L) = 0$  and let  $K_1(L)$  be the ideal of  $L$  generated by all absolute zero divisors of  $L$ ; using transfinite induction we define a chain of ideals  $K_\alpha(L)$  by  $K_\alpha(L) = \bigcup_{\beta < \alpha} K_\beta(L)$  for a limit ordinal  $\alpha$ , and  $K_\alpha(L)/K_{\alpha-1}(L) = K_1(L/K_{\alpha-1}(L))$  otherwise. The Kostrikin radical of  $L$  is defined as  $K(L) = \bigcup_\alpha K_\alpha(L)$ . By construction,  $K(L)$  is the smallest nondegenerate ideal of  $L$ , see [23].

**1.6.** Let  $J$  be a Jordan algebra. We will say that an ideal  $I$  of  $J$  is a nondegenerate (strongly prime) ideal of  $J$  if the quotient algebra  $J/I$  is nondegenerate (strongly prime). The McCrimmon radical or small radical  $Mc(J)$  of a Jordan algebra  $J$  is the smallest ideal of  $J$  whose associated quotient algebra  $J/Mc(J)$  is nondegenerate. It is radical in the sense of Amitsur–Kurosh, see [19, Theorem 4], and can be constructed in the following way:  $Mc_0(J) = 0$  and let  $Mc_1(J)$  be the subalgebra of  $J$  generated by all absolute zero divisors of  $J$  ( $Mc_1(J)$  is an ideal of  $J$ , see [18, Theorem 9]); using transfinite induction we define a chain of ideals  $Mc_\alpha(J)$  by  $Mc_\alpha(J) = \bigcup_{\beta < \alpha} Mc_\beta(J)$  for a limit ordinal  $\alpha$ , and  $Mc_\alpha(J)/Mc_{\alpha-1}(J) = Mc_1(J/Mc_{\alpha-1}(J))$  otherwise. Then the McCrimmon radical of  $J$  is defined as  $Mc(J) = \bigcup_\alpha Mc_\alpha(J)$ . Note that  $Mc(J)$ , by construction, is a nondegenerate ideal and is contained in any nondegenerate ideal of  $J$ , see [18,15].

**1.7.** For any Jordan system  $J$  one has the notion of m-sequence: It is a sequence  $\{a_n\}_{n \in \mathbb{N}}$  such that  $a_{n+1} = U_{a_n} b$  for some  $b \in J$ . We will say that an m-sequence of  $J$  has length  $k$  if  $a_k \neq 0$  and  $a_{k+1} = 0$ . There is a beautiful characterization of the elements of the McCrimmon radical in terms of m-sequences: An element  $x \in J$  is contained in  $Mc(J)$  if and only if any m-sequence  $\{a_n\}_{n \in \mathbb{N}}$  with  $a_1 = x$  has finite length, i.e., there exists  $k \in \mathbb{N}$  such that  $a_k = 0$ . From this property it is shown that the McCrimmon radical of a Jordan algebra  $J$  coincides with the intersection of all strongly prime ideals of  $J$  or, equivalently, that every nondegenerate ideal  $I$  of a Jordan algebra  $J$  is the intersection of all strongly prime ideals of  $J$  containing  $I$ , see [22] for the linear case and [19] for its quadratic extension.

**1.8.** Following the notion of m-sequence introduced in the previous paragraph for Jordan algebras, we define an analogous concept in the context of Lie algebras: Let  $L$  be a Lie algebra. An m-sequence is a set  $\{a_n\}_{n \in \mathbb{N}}$  such that  $a_{n+1} = [a_n, [a_n, b_n]]$  for some  $b_n \in L$ . We will say that an m-sequence of  $L$  has length  $k$  if  $a_k \neq 0$  and  $a_{k+1} = 0$ . Note that, if  $x \in L$  satisfies that  $[x, [x, L]] \subset K(L)$ , then  $x \in K(L)$  (because  $\bar{x} = x + K(L)$  is an absolute zero divisor in the nondegenerate Lie algebra  $L/K(L)$ ). So if any m-sequence of  $L$  starting with  $x$  has finite length, then  $x \in K(L)$ .

## 2. Lie algebras with enough ad-nilpotent elements

**2.1.** Let  $L$  be a Lie algebra over a ring of scalars  $\Phi$  such that  $\frac{1}{2}, \frac{1}{3} \in \Phi$ . We say that an element  $x$  in  $L$  is a Jordan element if  $x$  is ad-nilpotent of index  $\leq 3$ , i.e., if  $\text{ad}_x^3 = 0$ . Every Jordan element gives rise to a Jordan algebra, called the Jordan algebra of  $L$  at  $x$ , see [7]: Let  $L$  be a Lie algebra and let  $x \in L$  be a Jordan element. Then  $L$  with the new product given by  $a \bullet b := \frac{1}{2}[[a, x], b]$  is an algebra such that

$$\ker(x) := \{z \in L \mid [x, [x, z]] = 0\}$$

is an ideal of  $(L, \bullet)$ . Moreover,  $L_x := (L/\ker(x), \bullet)$  is a Jordan algebra. In this Jordan algebra the U-operator has this very nice expression:

$$U_{\bar{a}}\bar{b} = \frac{1}{8} \overline{\text{ad}_a^2 \text{ad}_x^2 b}, \quad \text{for all } a, b \in L,$$

and

$$\{\bar{a}, \bar{b}, \bar{c}\} = -\frac{1}{4} \overline{[a, [\text{ad}_x^2 b, c]]} \quad \text{for all } a, b, c \in L.$$

A Lie algebra is nondegenerate if and only if  $L_x$  is nonzero for every Jordan element  $x \in L$ . Moreover,  $L_x$  inherits nondegeneracy from  $L$  [7, 2.15(i)].

An inner ideal of  $L$  is a subspace  $M$  of  $L$  such that  $[M, [M, L]] \subset M$ . It is an abelian inner ideal if it is also an abelian subalgebra, i.e.,  $[M, M] = 0$ . The kernel of  $M$  is the set  $\text{Ker}_L M = \{x \in L : [M, [M, x]] = 0\}$ . If  $M$  is abelian, then  $\text{Ker}_L M = \{x \in L : [m, [m, x]] = 0 \text{ for every } m \in M\}$ . For any abelian inner ideal  $M$  of  $L$ , the pair  $V = (M, L/\text{Ker}_L M)$  with the triple products given by

$$\begin{aligned} \{m, \bar{a}, n\} &:= [[m, a], n] \quad \text{for every } m, n \in M \text{ and } a \in L, \\ \{\bar{a}, m, \bar{b}\} &:= \overline{[a, m], b} \quad \text{for every } m \in M \text{ and } a, b \in L, \end{aligned}$$

where  $\bar{x}$  denotes the coset of  $x$  relative to the submodule  $\text{Ker}_L M$ , is a Jordan pair called the subquotient of  $L$  with respect to  $M$ . When  $L$  is nondegenerate, the notion of subquotient generalizes that of Jordan algebra of a Lie algebra: if  $x$  is a Jordan element,  $M$  is the abelian inner ideal generated by  $x$ , and we consider the subquotient  $V = (M, L/\text{Ker} M)$  defined by  $M$ , then the Jordan homotope algebra  $V^{(x)}$  coincides with the Jordan algebra  $L_x$  of  $L$  at  $x$ , cf. [8, §3].

**Proposition 2.2.** *Let  $L$  be a Lie algebra over a ring of scalars  $\Phi$  such that  $\frac{1}{2}, \frac{1}{3} \in \Phi$  and let  $x \in L$  be a Jordan element. Then for every  $a \in L$  every  $m$ -sequence of  $L$  of length  $k$  starting with  $[x, [x, a]]$  gives rise to an  $m$ -sequence of  $L_x$  starting with  $\bar{a}$  with the same length, and vice versa.*

**Proof.** Let  $\{\bar{c}_n\}$  be an  $m$ -sequence in  $L_x$ . Let us prove that  $\{a_n\}$ , with  $a_n := [x, [x, c_n]]$  is an  $m$ -sequence of  $L$  with the same number of nonzero terms: we know that for every  $n \in \mathbb{N}$  there exists  $b_n \in L_x$  such that  $\bar{c}_{n+1} = U_{\bar{c}_n}\bar{b}_n = \overline{\text{ad}_{c_n}^2 \text{ad}_x^2 b_n}$ . So

$$\text{ad}_{a_n} b_n = \text{ad}_x^2 \text{ad}_{c_n}^2 \text{ad}_x^2 b_n = \text{ad}_x^2 c_{n+1} = a_{n+1}.$$

Moreover, by construction,  $\bar{a}_n \neq \bar{0}$  if and only if  $[x, [x, a_n]] \neq 0$ .

Conversely, let  $\{a_n\}_{n \in \mathbb{N}}$  be an  $m$ -sequence of  $L$  with  $a_1 = [x, [x, a]]$  and let us consider  $b_n \in L$  such that  $a_{n+1} = [a_n, [a_n, b_n]]$  for every  $n \in \mathbb{N}$ . Let us prove that for every  $n$  there exists  $c_n \in L$  such that  $a_n = [x, [x, c_n]]$ : The case  $n = 1$  holds by hypothesis. So let us suppose that there exists  $c_n \in L$  such that  $a_n = [x, [x, c_n]]$ . Then

$$a_{n+1} = [a_n, [a_n, b_n]] = \text{ad}_{\text{ad}_x^2 c_n}^2 b_n = \text{ad}_x^2 \text{ad}_{c_n}^2 \text{ad}_x^2 b_n. \tag{1}$$

Now, formula (1) implies that  $\{\bar{c}_n\}_{n \geq 2}$  is an  $m$ -sequence of  $L_x$  because

$$U_{\bar{c}_n}\bar{b}_n = \overline{\text{ad}_{c_n}^2 \text{ad}_x^2 b_n} = \bar{c}_{n+1}$$

with  $\bar{c}_n \neq \bar{0}$  if  $a_n \neq 0$ .  $\square$

From now on we will suppose that  $L$  is a Lie algebra where every submodule invariant under inner automorphisms is an ideal of  $L$ . For example:

- (i) Every Lie algebra  $L$  over a ring of scalars  $\Phi$  with no torsion which is generated as an algebra by ad-nilpotent elements; also if  $L$  is generated by ad-nilpotent elements of index at most  $m$  and  $\frac{1}{k} \in \Phi$  for  $1 \leq k \leq 2m - 2$  (by using a Vandermonde argument).
- (ii) Every  $\mathbb{Z}$ -graded Lie algebra  $L = \bigoplus_{i=-n}^n L_i$  with  $L_0 = \sum_{i=1}^n [L_i, L_{-i}]$ , defined over a ring of scalars  $\Phi$  with  $\frac{1}{k} \in \Phi$  for  $1 \leq k \leq 4n$ .

Next proposition was suggested to the authors by Prof. Artem Golubkov.

**Proposition 2.3.** *Let  $L$  be a Lie algebra over a ring of scalars  $\Phi$  with  $1/7! \in \Phi$ , let  $C_1$  be the submodule generated by all absolute zero divisors of  $L$ , and let  $x_1, x_2, \dots, x_n$  be absolute zero divisors of  $L$  such that  $x = x_1 + \dots + x_n \in C_1$  is a Jordan element. Then  $Mc(L_x) = L_x$  and every  $m$ -sequence of  $L_x$  finishes in a finite number of steps.*

**Proof.** Let us denote by capital letters the adjoint maps associated to elements of  $L$ , i.e.,  $A = \text{ad}_a$ ,  $a \in L$ , etc. First let us prove that for every absolute zero divisor  $z \in L$  and any  $a \in L$ ,  $[a, z]$  is ad-nilpotent of index  $\leq 3$  and  $[a, [a, z]]$  is ad-nilpotent of index  $\leq 5$ : using that  $Z^2 = ZAZ = 0$

$$\text{ad}_{[a,z]}^2 = -ZA^2Z, \quad \text{ad}_{[a,z]}^3 = -ZA^2Z(AZ - ZA) = 0.$$

Moreover, since  $Z \text{ad}_A^k(Z)Z = 0$  for  $k \in \mathbb{N}$  we get

$$6ZA^2ZA^2Z = 0 \quad \text{for } k = 4,$$

$$10ZA^2ZA^3Z - 10ZA^3ZA^2Z = 0 \quad \text{for } k = 5$$

and

$$20ZA^3ZA^3Z - 15ZA^2ZA^4Z - 15ZA^4ZA^2Z = 0 \quad \text{for } k = 6.$$

Therefore, since  $\frac{1}{2}, \frac{1}{3}, \frac{1}{5} \in \Phi$

$$\text{ad}_{[a,[a,z]]}^3 = 2AZA^2ZA^3Z - ZA^4ZA^2Z - ZA^2ZA^4Z + 2ZA^2ZA^3ZA,$$

$$\text{ad}_{[a,[a,z]]}^4 = ZA^2ZA^4ZA^2Z \quad \text{and} \quad \text{ad}_{[a,[a,z]]}^5 = 0.$$

Now, since  $C_1$  is invariant under inner automorphisms and  $\frac{1}{k} \in \Phi$  for  $k = 2, 3, 5, 7$ ,  $[t, z] \in C_1$  for  $z \in C_1$  and  $t$  an ad-nilpotent element of  $L$  of index  $\leq 5$ .

Every absolute zero divisor  $z$  of  $L$  gives rise to an absolute zero divisor  $\bar{z}$  of  $L_x$ :  $U_{\bar{z}}\bar{a} = \overline{\text{ad}_z^2 \text{ad}_x^2 a} = \bar{0}$ , so

$$\bar{C}_1 \subset Mc(L_x). \tag{1}$$

Moreover,  $C_1$  is an inner ideal of  $L$ : for any  $x = x_1 + \dots + x_k \in C_1$ , with  $x_i$  absolute zero divisors of  $L$  and any  $b \in L$ ,  $([x_j, b])$  is ad-nilpotent of index  $\leq 3$ ,

$$\text{ad}_x^2 b = \sum_{i,j=1}^n [x_i, [x_j, b]] \in C_1. \tag{2}$$

Furthermore,

$$U_{\overline{\text{ad}_a^2 z}} \bar{b} = \overline{[\text{ad}_a^2 z, [\text{ad}_a^2 z, \text{ad}_x^2 b]]} \in \overline{[\text{ad}_a^2 z, [\text{ad}_a^2 z, C_1]]} \subset \overline{C_1} \subset \text{Mc}(L_x), \tag{3}$$

and therefore  $\overline{\text{ad}_a^2 z} \in \text{Mc}(L_x)$ , which implies that  $\overline{\text{ad}_a^2 C_1} \subset \text{Mc}(L_x)$ . Finally, by (2) and (3),  $U_{\bar{a}} \bar{b} = \overline{\text{ad}_a^2 \text{ad}_x^2 b} \in \text{Mc}(L_x)$  for every  $\bar{b} \in L_x$ , so  $\bar{a} \in \text{Mc}(L_x)$  for every  $\bar{a} \in L_x$ , i.e.,  $\text{Mc}(L_x) = L_x$ .  $\square$

**Proposition 2.4.** *Let  $L$  be a Lie algebra such that every submodule of  $L$  invariant under inner automorphisms is an ideal and let  $x \in K(L)$  be a Jordan element. Then every  $m$ -sequence of  $L_x$  has finite length and  $\text{Mc}(L_x) = L_x$ .*

**Proof.** By hypothesis  $C_1$ , the submodule generated by all absolute zero divisors of  $L$ , is an ideal of  $L$  and coincides with  $K_1(L)$ . Similarly, by induction, if  $\alpha$  is not a limit ordinal  $K_\alpha(L) = C_\alpha$ , the submodule generated by all  $x \in L$  such that  $[x, [x, L]] \subset K_{\alpha-1}(L)$ , which is an ideal since it is invariant under inner automorphisms.

Therefore, if  $x \in K(L) = \bigcup_\beta K_\beta(L)$  is a Jordan element,  $x \in K_\alpha(L) = C_\alpha$  for some  $\alpha$  which is not a limit ordinal. Now, for every  $\bar{a} \in L_x$  and every  $m$ -sequence  $\{\bar{a}_n\}_{n \in \mathbb{N}}$  of elements of  $L_x$  with  $\bar{a}_1 = \bar{a}$ , by 2.3 there exists  $n \in \mathbb{N}$  such that  $\bar{a}_n \in K_{\alpha-1}$ . Then by induction there exists  $m \in \mathbb{N}$  with  $\bar{a}_m = \bar{0}$ , i.e., the  $m$ -sequence vanishes in a finite number of steps, which implies that  $L_x = \text{Mc}(L_x)$ .  $\square$

**Corollary 2.5.** *Let  $L$  be a Lie algebra such that every submodule of  $L$  invariant under inner automorphisms is an ideal and let  $x \in L$  be a Jordan element. Then  $\text{Mc}(L_x) = \{\bar{a} \in L_x \mid [x, [x, a]] \in K(L)\}$ .*

**Proof.** If  $\bar{a} \in \text{Mc}(L_x)$ , then every  $m$ -sequence of  $L_x$  starting with  $\bar{a}$  has finite length. Therefore, by Proposition 2.2, every  $m$ -sequence of  $L$  starting with  $[x, [x, a]]$  has finite length and therefore, by 1.8,  $[x, [x, a]] \in K(L)$ . Conversely, since  $x$  is a Jordan element, for every  $a \in L$ ,  $\text{ad}_x^2 a$  is a Jordan element. So, if  $[x, [x, a]] \in K(L)$ , by Proposition 2.4 every  $m$ -sequence starting with  $[x, [x, a]]$  has finite length in  $L$ , so the  $m$ -sequences of  $L_x$  starting with  $\bar{a}$  have finite length by Proposition 2.2, which implies that  $\bar{a} \in \text{Mc}(L_x)$ .  $\square$

**Corollary 2.6.** *Let  $L$  be a Lie algebra such that every submodule of  $L$  invariant under inner automorphisms is an ideal, let  $M$  be an abelian inner ideal of  $L$  and consider the subquotient  $V = (M, L/\text{Ker } M)$ . Then*

$$\text{Mc}(V)^+ = M \cap K(L),$$

and

$$\text{Mc}(V)^- = \{a + \text{Ker } M \mid [M, [M, a]] \subset K(L)\}.$$

**Proof.** Notice that  $\text{Mc}(V)^+$  consists on the elements of  $M$  for which every  $m$ -sequence has finite length, so  $\text{Mc}(V)^+ \subset K(L)$ . Conversely, since every element of  $M$  is a Jordan element, if  $x \in K(L) \cap M$  then it satisfies the  $m$ -sequence condition by Proposition 2.4, so it belongs to  $\text{Mc}(V)$ .

For the second equality, if  $a + \text{Ker } M$  belongs to  $\text{Mc}(V)^-$ , then  $[m_1, [m_2, a]] = -\{m_1, a, m_2\} \in \text{Mc}(V)^+ \subset K(L)$  for every  $m_1, m_2 \in M$ . Conversely, if  $a \in L$  has  $[M, [M, a]] \in K(L)$  then  $\{M, (a + \text{Ker } L), M\} \subset K(L) \cap M = \text{Mc}(V)^+$ , but this implies  $a + \text{Ker } M \in \text{Mc}(V)^-$  by [1, 3.4].  $\square$

The next result can be found in [24]. We give here an alternative proof.

**Proposition 2.7.** *Let  $V$  be a Jordan pair over a ring of scalars  $\Phi$  with  $\frac{1}{2}, \frac{1}{3} \in \Phi$  and consider the Lie algebra  $L = \text{TKK}(V)$ . Then, the Kostrikin radical  $K(L)$  of  $L$  is a 3-graded ideal with  $\pi_{\sigma_1}(K(L)) = \text{Mc}(V)^\sigma$ ,  $\sigma = \pm$ , where  $\pi_{\sigma_1}$  denotes the canonical projection of  $L$  onto  $L_{\sigma_1}$ , and is isomorphic to the center of  $L/\text{id}_L(\text{Mc}(V)^+ \oplus \text{Mc}(V)^-)$ .*



**Proof.** Under these conditions,  $L$  satisfies that every submodule invariant under inner automorphisms is an ideal of  $L$ . We will show that

$$K(L) = Mc(V)^+ \oplus (K(L) \cap [V^+, V^-]) \oplus Mc(V)^-. \tag{1}$$

Clearly,  $Mc(V)^+ \oplus Mc(V)^- \subset K(L)$  since all  $m$ -sequences starting with these elements have finite length. Conversely, let  $y = y_1 + y_0 + y_{-1} \in K(L)$ . If  $[V^+, [y, V^+]] = [V^+, [y_{-1}, V^+]] \not\subset Mc(V)^+$ , then we would have Jordan elements in  $K(L) \cap V^+$  which do not belong to the McCrimmon radical of  $V$ , a contradiction with Theorem 2.4. Therefore,  $[V^+, [y, V^+]] = \{V^+, y_{-1}, V^+\} \subset Mc(V)^+$ , so  $y_{-1} \in Mc(V)^-$  by [1, 3.4]. Similarly,  $y_1 \in Mc(V)^+$ .

Suppose that now that  $y_0 \neq 0$ . Then at least  $[y_0, V^+] \neq 0$  or  $[y_0, V^-] \neq 0$ . Suppose  $[y_0, V^+] \neq 0$ . Since  $[V^-, [V^-, [y, V^+]]] = [V^-, [V^-, [y_0, V^+]]] \subset I \cap V^- \subset Mc(V)^-$ , then the Jordan triple product

$$\{V^-, [y_0, V^+], V^-\} \subset Mc(V)^-,$$

so  $[y_0, V^+] \subset Mc(V)^+$  by [1, 3.4]. Therefore,  $[y_0, [y_0, V^+]] \subset Mc(V)^+ \subset K(L)$ , and similarly  $[y_0, [y_0, V^-]] \subset K(L)$ . Therefore  $[y_0, [y_0, L]] \subset K(L)$ , so also  $y_0 \in K(L)$ .

From (1), every  $x \in K(L)$  satisfies  $[x, L] \in \text{id}_L(Mc(V)^+ \oplus Mc(V)^-)$ , so  $x + \text{id}_L(Mc(V)) \in Z(L/\text{id}_L(Mc(V)))$ . Conversely, if  $x \in L$  satisfies

$$[x, L] \in \text{id}_L(Mc(V)^+ \oplus Mc(V)^-),$$

then  $[x, [x, L]] \in \text{id}_L(Mc(V)) \subset K(L)$ , so  $x \in K(L)$  by 1.8.  $\square$

**Proposition 2.8.** *Let  $L$  be a Lie algebra such that every submodule invariant under inner automorphisms is an ideal of  $L$ , and let  $M$  be an  $m$ -system of  $L$  of nonzero Jordan elements. Then every maximal ideal  $P$  of  $L$  with respect to the property  $P \cap M = \emptyset$  is nondegenerate. Moreover, if  $M$  is an  $m$ -sequence of  $L$ , then  $P$  is strongly prime.*

**Proof.** Let  $P$  be a maximal ideal with respect to the property  $P \cap M = \emptyset$ .

Let us prove that  $P$  is nondegenerate: consider the canonical projection  $\pi : L \rightarrow L/P$ . Let us suppose that  $L/P$  is degenerate and let  $K := \pi^{-1}(K(L/P))$  where  $K(L/P)$  is the Kostrikin radical of  $L/P$ . By construction, since  $P$  is maximal, there exists  $x \in M \cap K$  and therefore an  $m$ -sequence  $\{x_i\}$  which starts with  $x$  contained in  $M$ . But  $\{\bar{x}_i\}$  is an infinite  $m$ -sequence in  $L/P$ . So, by Theorem 2.4,  $\bar{x} \notin K(L/P) \supseteq L/P$ , a contradiction.

Now, let us suppose that  $M = \{a_n\}_{n \in \mathbb{N}}$  is an  $m$ -sequence and let  $I, J$  be two ideals of  $L$  with  $P \subsetneq I$  and  $P \subsetneq J$ . Then, since  $P$  is maximal with respect to  $P \cap M = \emptyset$ , there exist  $i, j \in \mathbb{N}$  such that  $a_i \in I$  and  $a_j \in J$ . Moreover, if  $k \geq \max\{i, j\}$ ,  $a_k \in I \cap J$  and  $0 \neq a_{k+1} = [a_k, [a_k, b_k]] \in [I, J]$  with  $a_{k+1} \notin P$  which proves that  $P$  is a strongly prime ideal of  $L$ .  $\square$

In the following results we will require that every nonzero ideal of  $L$  contains nonzero Jordan elements. If the ring of scalars  $\Phi$  has  $\frac{1}{k} \in \Phi$  for every  $0 \leq k \leq r$ , this hypothesis can be achieved as soon as every ideal of  $L$  contains nonzero ad-nilpotent elements of index at most  $n$  for  $n + \lfloor \frac{n}{2} \rfloor - 1 \leq r$ , see [16, Lemma 1.1, p. 31].

**Theorem 2.9.** *Let  $L$  be a nondegenerate Lie algebra such that every submodule invariant under inner automorphisms is an ideal of  $L$ , and such that every nonzero ideal of  $L$  contains nonzero Jordan elements. Then, the intersection of all strongly prime ideals of  $L$  is zero. Consequently,  $L$  is nondegenerate if and only if it is a subdirect product of strongly prime Lie algebras.*

**Proof.** We will show that for any nonzero element  $x$  of  $L$  we can always find a strongly prime ideal of  $L$  that does not contain  $x$ . Let  $I := \text{id}_L(x)$  be the ideal of  $L$  generated by  $x$ . By hypothesis there is

a nonzero Jordan element  $y$  of  $L$  contained in  $I$ . Now, we can construct the following  $m$ -sequence of  $L$  of infinite length  $N = \{a_i\}_{i \in \mathbb{N}}$ :  $a_1 = y$ , and given any  $a_i \neq 0$  define  $a_{i+1} = [a_i, [a_i, x_i]]$  for any  $x_i \in L$  such that  $0 \neq [a_i, [a_i, x_i]]$  (there exists such  $x_i$  because  $L$  is nondegenerate). By Zorn Lemma there exists a maximal ideal in  $\{J \triangleleft L \mid J \cap N = \emptyset\}$ , which is strongly prime ideal of  $L$  by Proposition 2.8 and, by construction, it does not contain  $y$  and therefore it does not contain  $x$ .  $\square$

In particular, all nonzero ideals of a nondegenerate Lie algebra with a finite  $\mathbb{Z}$ -grading of the form  $L = L_n \oplus \dots \oplus L_0 \oplus \dots \oplus L_{-n}$ ,  $L_0 = \sum_{i=1}^n [L_i, L_{-i}]$ , and  $\frac{1}{k} \in \Phi$  for every  $0 \leq k \leq 4n$ , always contain nonzero Jordan elements, and therefore, for such Lie algebras Theorem 2.9 reads as follows:

**Corollary 2.10.** *Let  $L = L_n \oplus \dots \oplus L_0 \oplus \dots \oplus L_{-n}$ ,  $L_0 = \sum_{i=1}^n [L_i, L_{-i}]$ , be a nondegenerate Lie algebra with a finite  $\mathbb{Z}$ -grading over a ring of scalars  $\Phi$  with  $\frac{1}{k} \in \Phi$  for every  $0 \leq k \leq 4n$ . Then, the intersection of all strongly prime ideals of  $L$  is zero. Consequently,  $L$  is nondegenerate if and only if it is a subdirect product of strongly prime Lie algebras.*

**Proof.** It is enough to prove that nonzero ideals of  $L$  have nonzero Jordan elements: let  $I$  be a nonzero ideal of  $L$  and consider the biggest natural  $k \in \mathbb{N}$  such that  $\pi_s(I) = 0$  for all  $|s| > k$ . Then, by nondegeneracy of the ideal  $\pi(I) = \pi_k(I) \oplus \dots \oplus \pi_0(I) \oplus \dots \oplus \pi_{-k}(I)$  of  $L$ ,  $0 \neq [\pi_k(I), [\pi_k(I), \pi(I)]] = [\pi_k(I), [\pi_k(I), \pi_{-k}(I)]] = [\pi_k(I), [\pi_k(I), I]] \subset I \cap \pi_k(I)$  consists of Jordan elements, and thus the claim follows by Theorem 2.9.  $\square$

### 3. Lie algebras over fields of characteristic zero

The results contained in this section were outlined by E. Zelmanov in a private communication [20] to the authors. We are grateful to him for allowing us to include them in the final version of this paper.

**Lemma 3.1.** *Given a Lie algebra  $L$  over a field of characteristic zero and  $0 \neq a \in L$  an ad-nilpotent element of index  $s > 3$ , there exists  $a_1 \in L$  such that  $[a, a_1] \neq 0$  is ad-nilpotent of index at most 3.*

**Proof.** In characteristic zero every element of the form  $\text{ad}_a^{s-1} x \in [a, L]$  is ad-nilpotent of index at most 3 for any  $x \in L$  [12].  $\square$

Lemma 9 of [23] gives conditions that guarantee that an element of  $L$  belongs to the Kostrikin radical of  $L$ . In the next proposition we weaken these conditions.

**Proposition 3.2.** *Given a Lie algebra  $L$  over a field of characteristic zero, if  $a \in L$  is such that there exists  $q \in \mathbb{N}$  with*

$$\text{ad}_a^q x_0 = \text{ad}_{[a, x_1]}^q x_0 = \text{ad}_{[[a, x_1], x_2]}^q x_0 = 0, \quad \text{for all } x_0, x_1, x_2 \in L$$

then  $a \in K(L)$ .

**Proof.** We can work in  $L/K(L)$ , assume that  $L$  is nondegenerate, and show that  $a = 0$ . Suppose that  $a \neq 0$  and let  $s$  be the index of ad-nilpotency of  $a$ . If  $s > 3$ , take  $a_1 \in L$  given by Lemma 3.1, and let  $b = [a, a_1] \neq 0$ , which is ad-nilpotent of index 3; if  $s \leq 3$ , let  $b = a$ . Since by hypothesis  $[x, b]$  is ad-nilpotent of index at most  $q$  for all  $x \in L$ , every element  $\bar{x}$  of the Jordan algebra  $L_b$ , see 2.1, is nilpotent of index at most  $q + 1$ . Indeed, since  $\bar{x}^{(n,b)} = \bar{x} \bullet \bar{x}^{(n-1,b)}$ , one readily has that

$$\bar{x}^{(2,b)} = \frac{1}{2} \overline{[x, b], x}, \quad \bar{x}^{(3,b)} = \frac{1}{4} \overline{[x, b], [x, b], x}, \quad \dots, \quad \bar{x}^{(n,b)} = \frac{1}{2^{n-1}} \overline{\text{ad}_{[x,b]}^{n-1} x}.$$

Therefore  $L_b$  is radical in the sense of McCrimmon, see [21, Lemma 17, p. 849]. But the Jordan algebras of nondegenerate Lie algebras are nondegenerate, see 2.1, so  $L_b = Mc(L_b) = 0$ , which implies that  $\text{Ker } b = L$ , so  $[b, [b, L]] = 0$ , i.e.,  $b$  is an absolute zero divisor, hence  $b = 0$ , a contradiction.  $\square$

**3.3.** Given  $n \in \mathbb{N}$  and a Lie algebra  $L$ , let

$$B_n(L) = \left\{ \sum_{i=1}^n [[a_i, b_{i_1}], \dots, b_{i_{k_i}}] \mid 0 \leq k_i \leq n, b_{i_j} \in L, \text{ad}_{a_i}^2 = 0 \right\}$$

be the sums of  $n$  monomials in  $L$  whose distance to an absolute zero divisor of  $L$  is less than or equal to  $n$ . Notice that  $B_1 \subset B_2 \subset \dots \subset B_n$  and  $K_1(L) = \bigcup_n B_n$ .

**Lemma 3.4.** For each  $n, r \in \mathbb{N}$  there exists  $f(n, r) \in \mathbb{N}$  with  $f(n, r) \geq 3$  such that for every Lie algebra  $L$  over a field of characteristic zero and for every  $a \in B_n(L)$

$$\text{ad}_{[[a, b_1], \dots, b_k]}^{f(n,r)} = 0 \quad \text{for every } b_1, \dots, b_k \in L, 0 \leq k \leq r.$$

**Proof.** This proof is inspired by [23, Lemma 8]. Let

$$X := \{x_0\} \cup \{x_i \mid i \in \mathbb{N}\} \cup \{x_{ij} \mid i, j \in \mathbb{N}\} \cup \{y_i \mid i \in \mathbb{N}\}$$

and consider the free Lie algebra  $\mathcal{L}[X]$ . Let  $\bar{\mathcal{L}}[X] = \mathcal{L}[X]/\text{Id}_{\mathcal{L}[X]}(\text{ad}_{x_i}^2 \mathcal{L}[X] \mid i \in \mathbb{N})$ , in which every  $\bar{x}_i$  is an absolute zero divisor. For every  $n, r \in \mathbb{N}$ , define

$$A_{n,r} := \left\{ \sum_{i=1}^n [[[\bar{x}_i, \bar{x}_{i_1}], \dots, \bar{x}_{i_{k_i}}], \bar{y}_1], \dots, \bar{y}_k \mid 0 \leq k_i \leq n, 0 \leq k \leq r \right\}.$$

Notice that  $A_{n,r} \subset K_1(\bar{\mathcal{L}}[X])$ , and it is a finite set, hence also the set  $A_{n,r} \cup [A_{n,r}, x_0] \subset K_1(\bar{\mathcal{L}}[X])$  has of a finite number of elements. For fixed  $n, r \in \mathbb{N}$ , the set  $D_{n,r} = \text{Subalg}_{\bar{\mathcal{L}}[X]}(A_{n,r} \cup [A_{n,r}, x_0])$  is nilpotent by a result of Grishkov [13], so there exists  $f(n, r) \geq 3$  such that  $D_{n,r}^{f(n,r)} = 0$ .

Let now  $L$  be a Lie algebra, let  $a \in B_n(L)$ , fix  $r \in \mathbb{N}$  and let  $b_1, \dots, b_k$  be arbitrary elements of  $L$ ,  $1 \leq k \leq r$ , and  $c \in L$ . We want to show that  $\text{ad}_{[[a, b_1], \dots, b_k]}^{f(n,r)} c = 0$ . Since  $a \in B_n(L)$ ,  $a = \sum_{i=1}^n [[a_i, b_{i_1}], \dots, b_{i_{k_i}}]$  for certain absolute zero divisors  $a_i \in L$ , and certain  $b_{ij} \in L$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, k_i$ ,  $0 \leq k_i \leq n$ . There exists a unique homomorphism of Lie algebras  $\varphi : \mathcal{L}[X] \rightarrow L$  such that  $\varphi(x_0) = c$ ;  $\varphi(x_i) = a_i$  if  $1 \leq i \leq n$  and  $\varphi(x_i) = 0$  otherwise;  $\varphi(x_{ij}) = b_{ij}$  if  $1 \leq i \leq n$ ,  $1 \leq j \leq k_i$ , and  $\varphi(x_{ij}) = 0$  otherwise;  $\varphi(y_i) = b_i$  if  $1 \leq i \leq k$  and  $\varphi(y_i) = 0$  otherwise. Moreover, since

$$\varphi(\text{Id}_{\mathcal{L}[X]}(\text{ad}_{x_i}^2 \mathcal{L}[X] \mid i = 1, \dots, n)) \subset \text{Id}_L(\text{ad}_{a_i}^2 L \mid i = 1, \dots, n) = 0,$$

$\varphi$  gives rise to a unique homomorphism of Lie algebras  $\bar{\varphi} : \bar{\mathcal{L}}[X] \rightarrow L$  such that  $\bar{\varphi}(\bar{x}_0) = c$ ,  $\bar{\varphi}(\bar{x}_i) = a_i$ ,  $1 \leq i \leq n$ ,  $\bar{\varphi}(\bar{x}_{ij}) = b_{ij}$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq k_i$ , and  $\bar{\varphi}(\bar{y}_i) = b_i$ ,  $1 \leq i \leq k$ . Finally,

$$\begin{aligned} \text{ad}_{[[a, b_1], \dots, b_k]}^{f(n,r)} c &= \bar{\varphi}(\text{ad}_{[[\sum_{i=1}^n [[\bar{x}_i, \bar{x}_{i_1}], \dots, \bar{x}_{i_{k_i}}], \bar{y}_1], \dots, \bar{y}_k]}^{f(n,r)} \bar{x}_0) \\ &= \bar{\varphi}(\text{ad}_{\sum_{i=1}^n [[[\bar{x}_i, \bar{x}_{i_1}], \dots, \bar{x}_{i_{k_i}}], \bar{y}_1], \dots, \bar{y}_k]}^{f(n,r)} \bar{x}_0) \in \bar{\varphi}(\text{ad}_{A_{n,r}}^{f(n,r)-1} [A_{n,r}, x_0]) \\ &\subset \bar{\varphi}(D_{n,r}^{f(n,r)}) = 0. \quad \square \end{aligned}$$

**3.5.** Given a Lie algebra  $L$  over a field of characteristic zero, we say that the sequence  $\{c_i\}_{i \in \mathbb{N}}$  is a *generalized  $m$ -sequence* of  $L$  if  $c_1 \in L$  and each  $c_{i+1}$ ,  $i \geq 1$ , is an element of the form

$$\text{ad}_{c_i}^{q_i} x_0, \quad \text{ad}_{[c_i, x_1]}^{q_i} x_0, \quad \text{or} \quad \text{ad}_{[[c_i, x_1], x_2]}^{q_i} x_0$$

for some  $x_0, x_1, x_2 \in L$  and  $q_i = f(i, 3i + 2)$ .

Notice that for every  $i$ , since  $q_i \geq 3$ ,

$$\begin{aligned} \text{ad}_{c_i}^{q_i} x_0 &\in [c_i, [c_i, [c_i, L]]] \subset [[c_i, L], L], \\ \text{ad}_{[c_i, x_1]}^{q_i} x_0 &\in [[c_i, x_1], [c_i, x_1], L] \subset [[c_i, L], L], \\ \text{ad}_{[[c_i, x_1], x_2]}^{q_i} x_0 &\in [[c_i, x_1], x_2], L] \subset [[c_i, L], L] \end{aligned}$$

so in each step  $c_{i+1} \in [[c_i, L], L]$ .

**Proposition 3.6.** *If a generalized  $m$ -sequence  $\{c_i\}_{i \in \mathbb{N}}$  in a Lie algebra  $L$  over a field of characteristic zero contains an element  $c_i$  in  $K(L)$ , the sequence has finite length.*

**Proof.** Suppose first that  $c_i \in K_1(L) = \bigcup_m B_m$ , so  $c_i$  belongs to certain  $B_n$  (it can be assumed that  $n \geq i$ ). Let us show that  $c_{n+1} = 0$ : Since  $c_{i+1}$  is an element of the form  $\text{ad}_{c_i}^{q_i} x_0$ ,  $\text{ad}_{[c_i, x_1]}^{q_i} x_0$ , or  $\text{ad}_{[[c_i, x_1], x_2]}^{q_i} x_0$  for some  $x_0, x_1, x_2 \in L$ , it can be expressed as an element  $c_{i+1} \in \underbrace{[[c_i, L], L], L}_3$  by 3.5.

Similarly,

$$c_{i+2} \in \underbrace{[[c_{i+1}, L], L], L}_3 \subset \underbrace{[[c_i, L], \dots, L]}_{3 \cdot 2}.$$

Finally,  $c_n \in \underbrace{[[c_i, L], \dots, L]}_{3(n-i)}$ . Since  $q_n = f(n, 3n + 2)$

$$\text{ad}_{c_n}^{q_n} x_0 = 0, \quad \text{ad}_{[c_n, x_1]}^{q_n} x_0 = 0, \quad \text{and} \quad \text{ad}_{[[c_n, x_1], x_2]}^{q_n} x_0 = 0$$

for all  $x_0, x_1, x_2 \in L$ , so  $c_{n+1} = 0$ .

We will show by transfinite induction that if  $c_i \in K_\alpha(L)$ , then the generalized  $m$ -sequence  $\{c_i\}_{i \in \mathbb{N}}$  has finite length. We have already shown the case  $\alpha = 1$ . Now assume that our assertion is true for every  $\beta < \alpha$ .

If  $\alpha$  is a limit ordinal,  $c_i \in \bigcup_{\beta < \alpha} K_\beta(L)$  so there exists some  $\beta < \alpha$  such that  $c_i \in K_\beta(L)$  and the sequence has finite length by the induction hypothesis. Otherwise,  $\alpha = \beta + 1$  for some  $\beta$  and we can consider the corresponding generalized  $m$ -sequence in  $L/K_\beta(L)$ ,  $\{c_j + K_\beta(L)\}_{j \in \mathbb{N}}$  for which  $c_i + K_\beta(L) \in K_1(L/K_\beta(L))$ . By the case  $\alpha = 1$  this sequence has finite length and there exists  $c_k + K_\beta(L) = \bar{0}$ , so  $c_k \in K_\beta(L)$  and the result follows by induction.  $\square$

**Proposition 3.7.** *Let  $L$  be a Lie algebra over a field of characteristic zero, let  $\{c_i\}_{i \in \mathbb{N}}$  be a generalized  $m$ -sequence of  $L$ , and let  $P$  be an ideal of  $L$  which is maximal among those ideals of  $L$  not containing any element of  $\{c_i\}_{i \in \mathbb{N}}$ . Then  $P$  is a strongly prime ideal of  $L$ , i.e.,  $L/P$  is a strongly prime Lie algebra.*

**Proof.** To see that  $L/P$  is prime, if  $A/P$  and  $B/P$  are two nonzero ideals of  $L/P$ , there exist some  $c_j \in A$ , some  $c_k \in B$ , so  $c_l \in A \cap B$  for every  $l \geq j, k$ . Then,  $c_{\max(j,k)+1} \in [A, B]$  so  $[A/P, B/P] \neq \bar{0}$ .

To see that  $L/P$  is nondegenerate, suppose on the contrary that  $K(L/P) \neq 0$ . Consider  $\hat{K} = \pi^{-1}(K(L/P))$ , where  $\pi : L \rightarrow L/P$  denotes the canonical projection, which is an ideal of  $L$  properly containing  $P$ , so there exists some  $c_j \in \hat{K}$ , hence  $c_j + P \in K(L/P)$ . By Proposition 3.6 the sequence  $\{c_i + P\}_{i \in \mathbb{N}}$  has finite length, so there exists some  $c_k + P = \bar{0}$ , i.e.,  $c_k \in P$ , a contradiction.  $\square$

**Proposition 3.8.** *Given a Lie algebra  $L$  over a field of characteristic zero, if  $a \in L$  does not belong to  $K(L)$  then there exists an infinite generalized  $m$ -sequence starting with  $a$ .*

**Proof.** Consider  $\bar{0} \neq \bar{a} = a + K(L) \in L/K(L)$  and let  $\bar{c}_0 = \bar{a}$ . If  $\bar{c}_i \neq \bar{0}$  then there exists  $\bar{c}_{i+1} \neq \bar{0}$  since otherwise it would mean that  $\text{ad}_{\bar{c}_i}^{q_i} \bar{x}_0 = \text{ad}_{[\bar{c}_i, \bar{x}_1]}^{q_i} \bar{x}_0 = \text{ad}_{[[\bar{c}_i, \bar{x}_1], \bar{x}_2]}^{q_i} \bar{x}_0 = 0$ , for all  $\bar{x}_0, \bar{x}_1, \bar{x}_2 \in L/K(L)$ ,  $q_i = f(i, 3i + 2)$ , but by Proposition 3.2 this implies that  $\bar{c}_i \in K(L/K(L)) = \bar{0}$ , a contradiction. The infinite generalized  $m$ -sequence  $\{\bar{c}_i\}$  in  $L/K(L)$  induces an infinite generalized  $m$ -sequence in  $L$ .  $\square$

By Lemma 3.4 and Proposition 3.8 one readily has

**Corollary 3.9.** *Let  $L$  be a Lie algebra over a field of characteristic zero, and let  $K(L)$  denote its Kostrikin radical. Then*

$$K(L) = \{x \in L \mid \text{every generalized } m\text{-sequence starting with } x \text{ has finite length}\}.$$

**Theorem 3.10.** *The Kostrikin radical  $K(L)$  of a Lie algebra  $L$  over a field of characteristic zero is the intersection of all strongly prime ideals of  $L$ . Therefore,  $L/K(L)$  is isomorphic to a subdirect product of strongly prime Lie algebras.*

**Proof.** If  $\{P_i\}$  denotes the set of all strongly prime ideals of  $L$ , it is clear that  $K(L) \subset P_i$  for each  $i$  since  $L/P_i$  is nondegenerate, so  $K(L) \subset \bigcap P_i$ . Conversely, let  $a \in L$  be an element that does not belong to  $K(L)$ . By Proposition 3.8 there exists an infinite generalized  $m$ -sequence starting with  $a$ . Let  $P$  be an ideal of  $L$  maximal among those not containing any element of the  $m$ -sequence. By Proposition 3.7  $P$  is a strongly prime ideal of  $L$ , and  $a \notin P$ , so  $a \notin \bigcap P_i$ .  $\square$

#### 4. Lie algebras arising from associative algebras

There are two important ways of producing Lie algebras out of an associative algebra  $R$ :

- If  $R$  is an associative algebra,  $R^-$  with product  $[x, y] := xy - yx$  is a Lie algebra.
- If  $R$  is an associative algebra with involution  $*$ , the set of skew elements of  $R$ ,  $\text{Skew}(R, *) = \{x \in R \mid x^* = -x\}$ , becomes a Lie subalgebra of  $R^-$ .

We begin by studying some relations between the Baer radical  $r(R)$  of an associative algebra  $R$  and the Kostrikin radical of  $R^-$ .

**Lemma 4.1.** *Let  $R$  be an associative algebra and let  $x \in r(R)$ . Then any  $m$ -sequence  $\{a_n\}_{n \in \mathbb{N}}$  of  $R^-$  with  $a_1 = x$  has finite length, i.e., there exists  $k \in \mathbb{N}$  such that  $a_k = 0$ .*

**Proof.** It is well known that the Baer radical of  $R$  can be constructed as in 1.5 or 1.6. Moreover, since the (associative) ideal generated by all absolute zero divisors of  $R$  coincides with the submodule generated by all absolute zero divisors, we only need to show that the proposition holds when  $x$  is a sum of absolute zero divisors of  $R$ . Let  $x = a_1 + \dots + a_k$  where each  $a_i$  is an absolute zero divisor of  $R$ ,  $i = 1, 2, \dots, k$ . Then any product of elements of  $R$  in which  $x$  appears at least  $k + 1$  times is zero. Therefore, any  $m$ -sequence of  $R^-$  which starts with  $x$  has at most length  $n$ , for  $2^n \leq k$ .  $\square$

**Lemma 4.2.** *Let  $R$  be an associative algebra defined over a ring of scalars  $\Phi$  with no 2-torsion. If  $R$  is semiprime, the Lie algebra  $R^-/Z(R)$  is nondegenerate. Furthermore, if  $R$  is prime,  $R^-/Z(R)$  is strongly prime.*

**Proof.** We can suppose that  $R$  is not commutative, otherwise  $R = Z(R)$  and the result is trivial.

Let us first see that  $R^-/Z(R)$  is nondegenerate when  $R$  is semiprime: Suppose that  $x \in R$  satisfies  $[x, [x, R]] \in Z(R)$ . Given any  $a \in R$ ,

$$0 = [a, [x, [x, xa]]] = [a, [x, x[x, a]]] = [a, x[x, [x, a]]] = [a, x][x, [x, a]]$$

since  $[x, [x, a]] \in Z(R)$ , which implies  $0 = \text{ad}_x([a, x][x, [x, a]]) = -([x, [x, a]])^2$  and, therefore,  $[x, [x, a]] = 0$  because  $R$  is semiprime and  $[x, [x, a]]$  is a nilpotent element of  $Z(R)$ ; now, by [14, Sublemma, p. 5],  $[x, [x, R]] = 0$  implies  $x \in Z(R)$ .

Now suppose that  $R$  is prime. By [2, Theorem 3.4] if  $I/Z(R)$  is a nonzero ideal of  $R/Z(R)$  there exists a nonzero ideal  $I'$  of  $R$  such that  $[I', R] \subset I$ . Let us prove that for every nonzero ideal  $I'$  of  $R$ ,  $[I', R]$  is not contained in  $Z(R)$ . Otherwise,  $[I', [I', R]] = 0$  which implies  $0 \neq I' \subset Z(R)$  (because  $R^-/Z(R)$  is nondegenerate) and this is not possible because in a prime noncommutative associative algebra there are no nonzero ideals contained in the center. Finally, if  $I_1/Z(R)$  and  $I_2/Z(R)$  are two nonzero ideals of  $R^-/Z(R)$ , there exist two nonzero ideals  $I'_1, I'_2$  of  $R$  with  $[I'_i, R] \subset I_i$  for  $i = 1, 2$ . Now,  $0 \neq [(I'_1 \cap I'_2) + Z(R)]/Z(R), R/Z(R) \subset I_1/Z(R) \cap I_2/Z(R)$ , which implies that  $R^-/Z(R)$  is prime.  $\square$

**Theorem 4.3.** *Let  $R$  be an associative algebra defined over a ring of scalars  $\Phi$  with no 2-torsion, and denote by  $K(R^-)$  the Kostrikin radical of  $R^-$ . Then:*

- (1)  $K(R^-)$  coincides with the intersection of all strongly prime ideals of  $R^-$ .
- (2)  $K(R^-) = \pi^{-1}(Z(R/r(R)))$  where  $r(R)$  is the Baer radical of  $R$  and  $\pi : R \rightarrow R/r(R)$  denotes the (associative) canonical projection.
- (3)  $K(R^-) = \{x \in R \mid \text{every } m\text{-sequence starting with } x \text{ has finite length}\}$ .

**Proof.** The intersection of all prime ideals  $\{I_i\}_i$  of  $R$  coincides with the Baer radical  $r(R)$ . For every prime ideal  $I_i$  of  $R$ ,  $R/I_i$  is a prime algebra, and the maps

$$\Psi_i : R^- \rightarrow (R/I_i)/Z(R/I_i)$$

are epimorphisms of Lie algebras, which implies by Lemma 4.2 that  $\text{Ker}(\Psi_i)$  is a strongly prime ideal of  $R^-$ , and since the Kostrikin radical is contained in every strongly prime ideal of  $R^-$ ,  $K(R^-) \subset \text{Ker}(\Psi_i)$ . Now, if  $x \in \bigcap \text{Ker}(\Psi_i)$ ,  $x + I_i \in Z(R/I_i)$  for every prime ideal  $I_i$  of  $R$  and therefore  $[x, R] \subset \bigcap I_i = r(R)$ . Hence  $x \in \pi^{-1}(Z(R/r(R)))$ , and if  $\{\mathcal{I}_i\}$  denotes the family of all strongly prime ideals of  $R^-$ ,

$$K(R^-) \subset \bigcap \mathcal{I}_i \subset \bigcap \text{Ker}(\Psi_i) \subset \pi^{-1}(Z(R/r(R))).$$

Finally, if  $x \in \pi^{-1}(Z(R/r(R)))$ ,  $[x, [x, a]] \in r(R)$  for every  $a \in R$  and therefore, every  $m$ -sequence of  $R^-$  starting with  $[x, [x, a]]$  has finite length by Lemma 4.1, which implies that  $x \in K(R^-)$  by 1.8.  $\square$

**Corollary 4.4.** *Let  $R$  be a semiprime algebra over a ring of scalars  $\Phi$  with no 2-torsion, and let us consider the Lie algebra  $L = R^-/Z(R)$ . Then the intersection of all strongly prime ideals of  $L$  is zero.*

Now we turn associative algebras with involution and study the relation between the Kostrikin radical of  $\text{Skew}(R, *)$  and the Baer radical of  $R$ .

**Lemma 4.5.** *If  $Q$  is a simple Lie algebra with involution  $*$  over a ring of scalars  $\Phi$  with no 2-torsion,  $*$  is of the first kind and  $\dim_{Z(Q)} Q \leq 4$ , then the Lie algebra  $L = \text{Skew}(Q, *)$  is either strongly prime or central and, in the second case,  $L$  has dimension one over  $Z(Q)$ .*

**Proof.** Let  $0 \neq t \in L = \text{Skew}(Q, *)$  be an element such that  $[t, [t, \text{Skew}(Q, *)]] = 0$ . In [2, Theorem 2.10] it is shown that  $[t, \text{Skew}(Q, *)] = 0$ , and from this we get that  $t$  commutes with the subalgebra  $\text{Skew}(Q, *)$  generated by  $\text{Skew}(Q, *)$ . But Herstein in [14, Lemma 2.2] showed that either  $\text{Skew}(Q, *) = Q$ , leading to  $t \in Z(Q) \cap \text{Skew}(Q, *) = 0$ , or  $L$  is 1-dimensional over its center. Furthermore, if  $L$  is nondegenerate, it is prime since  $\dim_{Z(Q)} Q \leq 4$  and there cannot exist two nonzero ideals with zero intersection.  $\square$

**Proposition 4.6.** *Let  $R$  be a  $*$ -prime associative algebra with involution  $*$  over a ring of scalars  $\Phi$  with no 2-torsion and let  $L = \text{Skew}(R, *)$ .*

- *If the involution is of the second kind or the involution is of the first kind and  $R$  is not an order in a simple algebra  $Q$  of dimension at most 16 over its center, then  $L/Z(L)$  is strongly prime. In these cases,  $Z(R) \cap L = Z(L)$ .*
- *If the involution is of the first kind and  $R$  is an order in a simple algebra  $Q$  with  $\dim_{Z(Q)} Q = 9$  or 16, then  $Z(R) \cap L = Z(L) = K(L) = 0$  and the intersection of all strongly prime ideals of  $L/Z(L)$  is zero.*
- *If the involution is of the first kind and  $R$  is an order in a simple algebra  $Q$  with  $\dim_{Z(Q)} Q \leq 4$ , then either  $L$  is abelian or strongly prime.*

**Proof.** If  $R$  is a commutative algebra, all the results are trivial, so we can suppose that  $R$  is noncommutative.

First, let us suppose that the involution is of the second kind: Let us consider the Lie algebra  $L' := L/(Z(R) \cap L)$  and let  $\bar{t}$  be an absolute zero divisor of  $L'$ . If  $[t, [t, L]] = 0$ , then by [2, Theorem 2.13] (which also holds for  $*$ -prime algebras),  $t \in Z(R)$ , so  $\bar{t} = \bar{0}$  in  $L'$ . If  $0 \neq [t, [t, L]] \subset Z(R)$ , there exists  $x \in L$  such that  $0 \neq [t, [t, x]] = \alpha$ . Since  $\alpha \in Z(R)$ ,  $\alpha[t, [t, H(R, *)]] = [t, [t, \alpha H(R, *)]] \subset [t, [t, L]] \subset Z(R)$ , but then also  $[t, [t, H(R, *)]] \subset Z(R)$  since  $R$  is  $*$ -prime (notice that in any  $*$ -prime  $R$ ,  $0 \neq \alpha \in Z(R)$  and  $r \in R$  with  $\alpha r \in Z(R)$  implies  $r \in Z(R)$ ). Therefore,  $[t, [t, R]] \subset Z(R)$  and we get that  $t \in Z(R)$  by Lemma 4.2, i.e.,  $L'$  is nondegenerate. Therefore  $K(L') = 0$ , so  $K(L) = Z(R) \cap L$ , which implies, in particular, that  $Z(L) = Z(R) \cap L$ .

Now, let us suppose that the involution is of the first kind and  $R$  is not an order in a simple algebra of dimension less than 9 over its center. Then, by [2, Theorem 2.10] (notice that the proof of this result also works in the  $*$ -prime setting) the Lie algebra  $L$  is nondegenerate. So  $K(L) = 0$  which implies that  $Z(R) \cap L = Z(L) = 0$ .

Suppose that either the involution is of the second kind, or it is of the first kind but  $R$  is not an order in a simple algebra  $Q$  of dimension at most 16 over its center. To show that  $L/(Z(R) \cap L)$  is strongly prime, assume firstly that  $R$  is prime. Then, by [5, Theorem 1(a), p. 525] if  $*$  is of the second kind, or by [5, corollary, p. 533] if  $*$  is of the first kind and  $R$  is not an order in a simple algebra  $Q$  which is at most 16-dimensional over its center, given a nonzero ideal  $I'/(Z(R) \cap L)$  of  $L/(Z(R) \cap L)$ , there exists a nonzero  $*$ -ideal  $I$  of  $R$  such that  $[I \cap \text{Skew}(R, *), \text{Skew}(R, *)] \subset I'$ . Let us show that  $[I \cap \text{Skew}(R, *), \text{Skew}(R, *)] \neq 0$ . Otherwise,  $I \cap \text{Skew}(R, *)$  can be regarded as a nilpotent ideal of the nondegenerate Lie algebra  $L$ , so it is zero modulo  $Z(R)$ , in which case:

- (I) If  $I \cap \text{Skew}(R, *) = 0$ , then for every  $y \in I$ ,  $y - y^* \in \text{Skew}(R, *) \cap I = 0$ , so  $y = y^*$  for every  $y \in I$ , and given  $r, s \in R$ ,

$$yrs = (yrs)^* = s^*r^*y; \quad yrs = (yr)^*s = r^*ys = r^*(ys)^* = r^*s^*y,$$

hence  $(s^*r^* - r^*s^*)y = 0$ , and since  $R$  is prime,  $(rs)^* = (sr)^*$  for every  $r, s \in R$ , which implies  $R$  is commutative, a contradiction.

(II) If  $0 \neq I \cap \text{Skew}(R, *) \subset Z(R)$ , then there exists  $\alpha \in I \cap \text{Skew}(R, *) \cap Z(R)$ . Since  $I = I \cap \text{Skew}(R, *) \oplus I \cap H(R, *)$ , we have that  $I \subset Z(R)$  because also  $I \cap H(R, *) \subset Z(R)$  since  $\alpha(I \cap H(R, *)) \subset I \cap \text{Skew}(R, *) \subset Z(R)$ . But a noncommutative prime  $R$  cannot have nonzero  $*$ -ideals  $I$  contained in  $Z(R)$ , a contradiction.

Thus if  $I/(Z(R) \cap \text{Skew}(R, *))$  and  $J/(Z(R) \cap \text{Skew}(R, *))$  are ideals of  $L/(Z(R) \cap L)$ , there exist ideals  $I', J'$  of  $R$  such that

$$0 \neq [I' \cap J' \cap \text{Skew}(R, *), \text{Skew}(R, *)] \subset I \cap J,$$

so  $L/(Z(R) \cap L)$  is a prime nondegenerate algebra, i.e., it is strongly prime.

If  $R$  is  $*$ -prime but not prime, there exists a prime ideal  $I$  of  $R$  such that  $I \cap I^* = 0$ . The map  $f : R \rightarrow R/I \times R/I^*$  is a  $*$ -monomorphism of algebras with exchange involution

$$* : R/I \times R/I^* \rightarrow R/I \times R/I^*$$

given by  $(x, y)^* = (y^*, x^*)$ . Now,  $f(I \oplus I^*)$  is an essential ideal of  $R/I \times R/I^*$  and

$$I \cong \text{Skew}(f(I \oplus I^*)) \triangleleft \text{Skew}(R/I \times R/I^*) \cong R/I^*,$$

which implies that  $I/(Z(R) \cap I) \cong \text{Skew}(f(I \oplus I^*)) / (\text{Skew}(f(I \oplus I^*)) \cap Z(f(R)))$  is a strongly prime algebra and, since it is essential in  $L$ ,  $L/(Z(R) \cap L)$  is strongly prime.

Suppose that  $R$  is  $*$ -prime with involution of the first kind and  $R$  is an order in a simple algebra  $Q$  of dimension at most 16 over its center. Since  $Q$  is simple and finite dimensional,  $Q$  is a PI algebra, so  $R$  is a PI algebra and it is a central order in  $Q$ : for every  $q \in Q$  there exist  $\alpha \in Z(R)$  and  $x \in R$  such that  $q = \alpha^{-1}x$ . Now, we can extend the involution to  $Q$  and since the center of  $Q$  is the extended centroid of  $R$ , we have that the involution on  $Q$  is of the first kind. If  $\dim_{Z(Q)} Q = 16$  or 9, by [2, Theorem 2.10],  $\text{Skew}(Q, *)$  is nondegenerate, and if  $\dim_{Z(Q)} Q = 4$  or 1, by Lemma 4.5,  $\text{Skew}(Q, *)$  is either central or strongly prime. In any case,  $L$  is abelian if  $\text{Skew}(Q, *)$  is abelian and  $L$  is strongly prime (nondegenerate) if  $\text{Skew}(Q, *)$  is so: Let us show that  $L$  is strongly prime when  $\text{Skew}(Q, *)$  is strongly prime (the inheritance of nondegeneracy follows analogously). Given  $x, y \in L$  such that  $[x, [y, L]] = 0$  we have that for every  $q \in \text{Skew}(Q, *)$  there exist  $\alpha \in Z(R)$  and  $z \in R$  such that  $q = \alpha^{-1}z$  and therefore  $[x, [y, q]] = [x, [y, \alpha z]] = \alpha[x, [y, z]] = 0$  which implies that  $x = 0$  or  $y = 0$  and  $L$  is strongly prime, see [11, Theorem 1.6]. Finally, if  $\dim_{Z(Q)} Q = 16$  or 9,  $Z(R) \cap L \subset Z(L) \subset K(L) = 0$  and by Corollary 5.4 the intersection of all strongly prime ideals of  $L$  is zero.  $\square$

**Theorem 4.7.** *Let  $R$  be an associative algebra with involution  $*$  over a ring of scalars  $\Phi$  with no 2-torsion, let  $L = \text{Skew}(R, *)$ , and denote by  $K(L)$  its Kostrikin radical. Then:*

- (1)  $K(L)$  coincides with the intersection of all strongly prime ideals of  $L$ .
- (2)  $K(L) = \pi^{-1}(Z(L/(r(R) \cap L)))$  where  $r(R)$  is the Baer radical of  $R$  and  $\pi : L \rightarrow L/(r(R) \cap L)$  denotes the canonical projection.
- (3)  $K(L) = \{x \in L \mid \text{every } m\text{-sequence starting with } x \text{ has finite length}\}$ .

**Proof.** The intersection of all  $*$ -prime ideals of  $R$ ,  $\{I_i\}_{i \in \Delta}$ , is equal to the Baer radical  $r(R)$ . Now, for every  $*$ -prime ideal  $I_i$  of  $R$ , let us consider the epimorphism of Lie algebras

$$\Psi_i : \text{Skew}(R, *) \rightarrow \text{Skew}(R/I_i, *) / Z(\text{Skew}(R/I_i, *)).$$

By Proposition 4.6,  $\text{Ker } \Psi_i$  is either a strongly prime ideal of  $L$ , or it is the intersection of strongly prime Lie algebras, or it is the whole algebra  $L$ . Therefore, if  $x \in \bigcap \text{Ker } \Psi_i$  (which is an intersection of strongly prime ideals of  $L$ ) and  $a \in L$ , we have that  $[x, a] \in I_i$  for all  $i \in \Delta$  and therefore,  $[x, a] \in r(R)$ ,



which implies that  $x \in \pi^{-1}(Z(L/(r(R) \cap L)))$ . So if  $\{\mathcal{I}_i\}$  denotes the family of all strongly prime ideals of  $L$ ,

$$K(L) \subset \bigcap \mathcal{I}_i \subset \pi^{-1}(Z(L/(r(R) \cap L))).$$

Finally, if  $x \in \pi^{-1}(Z(L/(r(R) \cap L)))$ ,  $[x, [x, a]] \in r(R)$  for every element  $a \in L$  and therefore, every  $m$ -sequence of  $L$  starting with  $[x, [x, a]]$  has finite length, Lemma 4.1, which implies that  $x \in K(L)$  by 1.8.  $\square$

**Corollary 4.8.** *Let  $R$  be a semiprime associative algebra with involution  $*$  over a ring of scalars  $\Phi$  with no 2-torsion and let  $L = \text{Skew}(R, *)$ . Then  $K(L) = Z(L)$  and it coincides with the intersection of all strongly prime ideals of  $L$ .*

**Remark 4.9.** Since the Kostrikin radical of any ideal  $I$  of a Lie algebra  $L$  coincides with  $K(L) \cap I$ , see [23, Corollary 1, p. 543], Theorems 4.3, 4.7 and Corollaries 4.4, 4.8 also hold for ideals of the Lie algebras mentioned there. In particular, they hold for the Lie algebras  $[R, R]$  and  $[R, R]/(Z(R) \cap [R, R])$ , and for  $[\text{Skew}(R, *), \text{Skew}(R, *)]$ ,  $[\text{Skew}(R, *), \text{Skew}(R, *)]/(Z(R) \cap [\text{Skew}(R, *), \text{Skew}(R, *)])$ .

### 5. Lie algebras with descending chain conditions

**5.1.** Recall that the annihilator of an ideal  $I$  in a Lie algebra  $L$  is defined as  $\text{Ann}_L(I) = \{x \in L \mid [x, I] = 0\}$ . If  $I$  is an ideal of  $L$  which is nondegenerate as a Lie algebra (in particular if  $L$  is nondegenerate), then  $\text{Ann}_L(I) = \{x \in L \mid [x, [x, I]] = 0\}$  and  $I \cap \text{Ann}_L(I) = 0$ . Moreover, if  $I$  is nondegenerate and  $\text{Ann}_L(I) = 0$ ,  $L$  is a nondegenerate Lie algebra, see [6, 2.5].

If  $L$  is nondegenerate,  $\text{Ann}_L(I)$  is a nondegenerate ideal of  $L$  for every ideal  $I$  of  $L$ : let  $\bar{x} \in L/\text{Ann}_L(I)$  such that  $[\bar{x}, [\bar{x}, L/\text{Ann}_L(I)]] = \bar{0}$ . Then  $[x, [x, L]] \subset \text{Ann}_L(I)$ , so  $[x, [x, I]] \subset I \cap \text{Ann}_L(I) = 0$ , hence  $x \in \text{Ann}_L(I)$ .

**5.2.** We say that a Lie algebra  $L$  satisfies the descending chain condition for annihilator ideals if every descending chain of annihilator ideals  $\{\text{Ann}_L(I_i)\}_i$ ,  $\text{Ann}_L(I_i) \supset \text{Ann}_L(I_{i+1})$ , reaches zero in a finite number of steps. Since  $\text{Ann}_L(\text{Ann}_L(\text{Ann}_L(I))) = \text{Ann}_L(I)$  for every ideal  $I$  of  $L$ , we have that  $L$  satisfies the descending chain condition for annihilator ideals if and only if it satisfies the ascending one.

A nonzero ideal  $I$  of  $L$  is said to be uniform if for every two nonzero ideals  $J, J'$  of  $L$  such that  $J, J' \subset I$  we have that  $J \cap J' \neq 0$ . If  $L$  is semiprime, by [9, Proposition 3.1(i)]  $I$  is a uniform ideal of  $L$  if and only if  $\text{Ann}_L(I)$  is maximal among all annihilator ideals of nonzero ideals of  $L$ . The next proposition can be deduced from [9, Theorem 4.1].

**Proposition 5.3.** *If  $L$  is nondegenerate and every annihilator ideal of  $L$  is contained in a maximal annihilator ideal, then the intersection of all strongly prime ideals of  $L$  is zero. Moreover, if  $L$  satisfies the chain condition for annihilator ideals, then  $L$  is an essential subdirect product of finitely many strongly prime Lie algebras.*

**Proof.** Let  $0 \neq x \in L$ , consider the ideal  $J$  of  $L$  generated by  $x$  and its annihilator  $\text{Ann}_L(J)$ . By hypothesis, there exists a nonzero ideal  $I$  of  $L$  such that  $\text{Ann}_L(I)$  is a maximal annihilator ideal with  $\text{Ann}_L(J) \subset \text{Ann}_L(I)$ . Now, if  $x \in \text{Ann}_L(I)$ ,  $J \oplus \text{Ann}_L(J) \subset \text{Ann}_L(I)$ , a contradiction because  $J \oplus \text{Ann}_L(J)$  is an essential ideal of  $L$ . Therefore, the intersection of all maximal annihilator ideals of  $L$ , which are strongly prime ideals of  $L$  by 5.1 and [9, Proposition 3.1(ii)], is zero.

Now suppose that  $L$  satisfies the chain condition for annihilator ideals and consider the set of all uniform ideals  $\{I_i\}_i$  of  $L$ . By 5.1 and [9, Proposition 3.1(ii)],  $L/\text{Ann}_L(I_i)$  is strongly prime. Moreover, since  $\bigcap_i \text{Ann}_L(I_i) = \text{Ann}_L(\sum_i I_i)$  and every descending chain of annihilator ideals reaches zero, there exists a finite number of uniform ideals  $\{I_i\}_{i=1}^n$  such that  $\bigcap_{i=1}^n \text{Ann}_L(I_i) = 0$ . Finally, every  $I_i \cong (I_i + \text{Ann}_L(I_i))/\text{Ann}_L(I_i)$  is an essential ideal of the strongly prime Lie algebra  $L/\text{Ann}_L(I_i)$ , hence  $L$  is an essential subdirect product of the strongly prime Lie algebras  $\{L_i = L/\text{Ann}_L(I_i)\}_{i=1}^n$ .  $\square$

The following corollary shows that the characterization of the Kostrikin radical of a Lie algebra  $L$  as the intersection of all strongly prime ideals of  $L$  holds for Artinian Lie algebras, hence in particular for finite dimensional Lie algebras.

**Corollary 5.4.** *If  $L$  is an Artinian Lie algebra, the Kostrikin radical of  $L$  coincides with the intersection of all strongly prime ideals of  $L$ , and  $L/K(L)$  is an essential subdirect product of finitely many strongly prime Lie algebras.*

**Proof.** The nondegenerate Lie algebra  $L/K(L)$  remains Artinian and satisfies the chain condition for annihilator ideals, so the intersection of all strongly prime ideals of  $L$  is  $K(L)$  and  $L/K(L)$  is an essential subdirect product of finitely many strongly prime Lie algebras.  $\square$

**5.5.** An inner ideal of a Lie algebra  $L$  is a  $\Phi$ -submodule  $B$  of  $L$  such that  $[B, [B, L]] \subset B$ . An abelian inner ideal is an inner ideal  $B$  which is also an abelian subalgebra, i.e.,  $[B, B] = 0$ . If  $L$  is defined over a field of scalars with  $1/2$ ,  $1/3$  and  $1/5$ , the socle of a nondegenerate Lie algebra  $L$  is an ideal  $\text{Soc}(L)$  defined as the sum of all minimal inner ideals of  $L$ , and it is a direct sum of simple ideals [4, 2.4, 2.5].

**Proposition 5.6.** *If  $L$  is defined over a field of scalars with  $1/2$ ,  $1/3$  and  $1/5$ , and  $L$  is nondegenerate and has essential socle, then the intersection of all strongly prime ideals of  $L$  is zero and, therefore,  $L$  is an essential subdirect product of strongly prime Lie algebras.*

**Proof.** Let  $\text{Soc}(L) = \bigoplus_i L_i$  be the decomposition of the socle of  $L$  into simple ideals, see [4, 2.5(i)]. It is easy to see that  $\bigcap_i (\text{Ann}_L(L_i)) = \text{Ann}(\bigoplus_i L_i) = \text{Ann}(\text{Soc}(L)) = 0$  because  $\text{Soc}(L)$  is essential, so the intersection of all strongly prime ideals of  $L$  is zero and  $L$  is an essential subdirect product of the strongly prime Lie algebras  $\{L/\text{Ann}_L(L_i)\}_i$ .  $\square$

## Acknowledgments

The authors are grateful to Prof. E. Zelmanov for allowing them to include part of his notes in the final version of this paper. They would also like to thank Prof. J.A. Anquela, Prof. T. Cortés, Prof. A. Fernández López, and Prof. A. Golubkov for their careful reading of this manuscript and their valuable suggestions and comments.

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