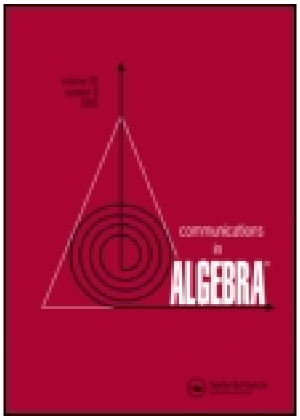

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PRINCIPAL FILTRATIONS OF LIE ALGEBRAS

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We show that any ad-nilpotent element x of index less than or equal to three of a Lie algebra L defines a 5-filtration in L. When L is nondegenerate, the $\mathbb{Z}$-graded Lie algebra $\widehat{L}$ associated to this filtration is isomorphic to L if and only x is von Neumann regular, if and only if $\widehat{L}$ is nondegenerate.

Key Words: Filtration; Graded Lie algebra.

2000 Mathematics Subject Classification: 17B60; 17B70.

Filtrations have recently been a topic of interest for Lie algebraists. In a series of works [2, 3, 8], Barnea and Passman have studied and classified maximal bounded $\mathbb{Z}$-filtrations of semisimple finite dimensional Lie algebras over $\mathbb{C}$. Maximal bounded filtrations were introduced in the context of affine Kac-Moody algebras, and their description was fundamental for classifying the maximal graded subalgebras of affine Kac-Moody algebras. In the context of infinite-dimensional Lie algebras we can highlight the work of Bahturin and Olshanksii [1] studying the so called tame filtrations, i.e., filtrations all of whose terms are finite-dimensional and their growth is majorated by an exponential function.

Let L be a Lie algebra over a ring of scalars Φ. A *$\mathbb{Z}$-filtration* $\{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ of L is a collection of Φ-modules

$$\cdots \subset \mathcal{F}_{-2} \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$$

indexed by $\mathbb{Z}$ such that $[\mathcal{F}_i, \mathcal{F}_j] \subset \mathcal{F}_{i+j}$. Associated to any $\mathbb{Z}$-filtration of L we have the $\mathbb{Z}$-graded Lie algebra

$$\widehat{L} = \bigoplus \mathcal{F}_i/\mathcal{F}_{i-1}.$$

We say that the filtration $\{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ is *bounded* if there exist $m, n \in \mathbb{Z}$ such that $\mathcal{F}_m = 0$ and $\mathcal{F}_n = L$. In this case it is clear that every element in $\mathcal{F}_j$, $j < 0$, is ad-nilpotent.

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The aim of this work is to describe some bounded filtrations related to ad-nilpotent elements of index less than or equal to three in non-necessarily finite dimensional Lie algebras. Similar filtrations in the context of Jordan pairs were given by Loos in [7]. As in Loos' work, we characterize when the associated $\mathbb{Z}$-graded Lie algebra is isomorphic to the original algebra.

Let L be a non-necessarily finite dimensional Lie algebra over a ring of scalars Φ. We say that a pair of elements $(x, y) \in L \times L$ is an *idempotent of* L if $\mathrm{ad}_x^3 L = 0 = \mathrm{ad}_y^3 L$ and $(x, [x, y], y)$ is an sl_2-triple of L. If 2, 3 and 5 are invertible in Φ, $[x, y]$ is ad-semisimple with minimum polynomial $t(t \pm 1)(t \pm 2)$ and induces a 5-grading $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ (see the well-known Jacobson-Morozov Lemma [9, V.8.2] and its extension to algebras over rings of scalars [5, 1.18(ii)]). As usual, this grading gives a filtration $\{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ in L with nonzero Φ-modules given by $\mathcal{F}_i = L_{-2} \oplus \cdots \oplus L_i$, $i = -2, -1, 0, 1$ and $\mathcal{F}_i = L$ for $i \geq 2$; viceversa, the associated Lie algebra determined by this filtration is isomorphic to L. Moreover, if we define $\mathrm{Ker}\, x = \{a \in L \mid [x, [x, a]] = 0\}$, one can observe that the nonzero Φ-modules $\mathcal{F}_i$ of this filtration are exactly

$$\mathcal{F}_{-2} = [x, [x, L]], \qquad \mathcal{F}_{-1} = [x, \mathrm{Ker}\, x], \qquad \mathcal{F}_1 = \mathrm{Ker}\, x,$$
$$\mathcal{F}_0 = \{a \in \mathrm{Ker}\, x \mid [a, \mathcal{F}_i] \subset \mathcal{F}_i, i = -2, \pm 1\}, \qquad \mathcal{F}_i = L, \quad i \geq 2. \tag{$*$}$$

Our goal is to generalize the filtration above if we only start with an element $x \in L$ such that $\mathrm{ad}_x^3 L = 0$, i.e., the element x is not necessarily part of an idempotent but still satisfies the property of being ad-nilpotent of index less than or equal to three. These elements are useful for the study of Lie algebras since one can associate a Jordan algebra to any of them and properties flow from the original Lie algebra to the associated Jordan algebra and viceversa, see [6]. Two natural questions arise:

1. Given such $x \in L$, *is it true that* $\{\mathcal{F}_i\}$ *is a filtration of* L?
2. If the answer to question 1 is YES, *when does that filtration coincide with the filtration given by an idempotent of* L?

In Theorem 1.2 we will show that formulae $(*)$ define a filtration in L, answering positively to question 1. Moreover, when L is nondegenerate, the filtration is given by an idempotent if and only if x is part of an idempotent (x is von Neumann regular), if and only if the Lie algebra associated to the filtration is nondegenerate, Proposition 1.5.

1. MAIN

Throughout this note and at least otherwise specified, we will be dealing with Lie algebras L over a ring of scalars Φ with 2 and 3 invertible in Φ. As usual, $[x, y]$ will denote the Lie bracket of two elements x, y of L, with ad_x the adjoint map determined by x.

We say that $x \in L$ is a *Jordan element if* $\mathrm{ad}_x^3 L = 0$, and we define $\mathrm{Ker}\, x = \{a \in L \mid \mathrm{ad}_x^2 a = 0\}$. Jordan elements satisfy several identities, see for example [4, 1.7] and [6, 2.3]. For the sake of completeness, in the following lemma we include the identities we are going to use in this article.

Lemma 1.1. *If $x \in L$ is a Jordan element and a, b are arbitrary elements of L, then*:

(i) $\mathrm{ad}_x^2\mathrm{ad}_a\mathrm{ad}_x = \mathrm{ad}_x\mathrm{ad}_a\mathrm{ad}_x^2$;
(ii) $[\mathrm{ad}_x^2 a, \mathrm{ad}_x b] = [\mathrm{ad}_x^2 b, \mathrm{ad}_x a] = -\mathrm{ad}_x^2[a, [x, b]]$.

Proof. (i) is [4, 1.7(i)], the first equality of (ii) is [6, 2.3(iv)], and the second part of (ii) follows since

$$\begin{aligned}[\mathrm{ad}_x^2 a, \mathrm{ad}_x b] &= \mathrm{ad}_{[x,[x,a]]}\mathrm{ad}_x b \\ &= (\mathrm{ad}_x^2\mathrm{ad}_a + \mathrm{ad}_a\mathrm{ad}_x^2 - 2\mathrm{ad}_x\mathrm{ad}_a\mathrm{ad}_x)\mathrm{ad}_x b \\ &= -\mathrm{ad}_x^2\mathrm{ad}_a\mathrm{ad}_x b \qquad \text{(using that } \mathrm{ad}_x^3 = 0 \text{ and (i))}.\end{aligned}$$
□

Theorem 1.2. *Let L be a Lie algebra over a ring of scalars Φ with $\frac{1}{2}, \frac{1}{3} \in \Phi$, let $x \in L$ a Jordan element, and define*

$$\mathcal{F}_i = 0, \quad i \leq -3, \qquad \mathcal{F}_{-2} = [x, [x, L]] \qquad \mathcal{F}_{-1} = [x, \mathrm{Ker}\, x]$$
$$\mathcal{F}_0 = \{a \in \mathrm{Ker}\, x \mid [a, x] \in [x, [x, L]]\} \qquad \mathcal{F}_1 = \mathrm{Ker}\, x \qquad \mathcal{F}_j = L, \quad j \geq 2.$$

Then $\{\mathcal{F}_i\}_i$ is a bounded filtration of L. Moreover, if we replace $\mathcal{F}_{-2}$ by $\mathcal{F}_{-2} + \Phi x$ and $\mathcal{F}_{-1}$ by $\mathcal{F}_{-1} + \Phi x$, we also obtain another bounded filtration of L.

Proof. It is easy to check that $\mathcal{F}_{-2} \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2$ since $[x, L] \subset \mathrm{Ker}\, x$ and $[[x, \mathrm{Ker}\, x], x] = 0$. Also $x \in \mathcal{F}_0$, so $\mathcal{F}_{-2} + \Phi x \subset \mathcal{F}_{-1} + \Phi x \subset \mathcal{F}_0$.

Let us prove that $[\mathcal{F}_i, \mathcal{F}_j] \subset \mathcal{F}_{i+j}$ for all $i, j \in \mathbb{Z}$:

(i) $[\mathcal{F}_{-2}, \mathcal{F}_{-1}] = 0$ and $[\mathcal{F}_{-2}, \mathcal{F}_{-2}] = 0$: if $a \in L$ and $z \in \mathrm{Ker}\, x$ we have by 1.1(ii) that

$$[[x, [x, a]], [x, z]] = [[x, [x, z]], [x, a]] = 0.$$

Since $\mathcal{F}_{-2} \subset \mathcal{F}_{-1}$, we also get that $[\mathcal{F}_{-2}, \mathcal{F}_{-2}] = 0$.

(ii) $[\mathcal{F}_{-2}, \mathcal{F}_0] \subset \mathcal{F}_{-2}$: if we take $a \in \mathcal{F}_0$, there exits $c \in L$ such that $[x, a] = [x, [x, c]]$ so for any $b \in L$

$$\begin{aligned}[[x, [x, b]], a] &= [[x, [x, c]], [x, b]] + [x, [[x, [x, c]], b]] + [x, [x, [b, a]]] \\ &= 2[[x, [x, c]], [x, b]] + [x, [x, [b, a]]] \in [x, [x, L]] \qquad \text{(by 1.1(ii))}.\end{aligned}$$

(iii) $[\mathcal{F}_{-2}, \mathcal{F}_1] \subset \mathcal{F}_{-1}$: for any $a \in L$ and any $z \in \mathrm{Ker}\, x$

$$\begin{aligned}[[x, [x, a]], z] &= [[x, z], [x, a]] + [x, [[x, z], a]] + [x, [x, [a, z]]] \\ &= 2[x, [[x, z], a]] + [x, [x, [a, z]]] \in [x, \mathrm{Ker}\, x] + [x, [x, L]] \subset [x, \mathrm{Ker}\, x]\end{aligned}$$

since $[[x, z], a] \in \mathrm{Ker}\, x$ because $\mathrm{ad}_x^2[[x, z], a] = [\mathrm{ad}_x z, \mathrm{ad}_x^2 a] = [\mathrm{ad}_x a, \mathrm{ad}_x^2 z] = 0$ by 1.1(ii).

(iv) $[\mathcal{F}_{-2}, \mathcal{F}_j] \subset \mathcal{F}_0$, $j \geq 2$: for any $a, b \in L$

$$[[x,[x,a]],b] \in \operatorname{Ker} x \text{ since } \operatorname{ad}_x^2[[x,[x,a]],b] = [\operatorname{ad}_x^2 a, \operatorname{ad}_x^2 b] = 0,$$
$$[[[x,[x,a]],b],x] = [[x,[x,a]],[b,x]] \in [x,[x,L]] \text{ (by 1.1(ii))}.$$

(v) $[\mathcal{F}_{-1}, \mathcal{F}_{-1}] \subset \mathcal{F}_{-2}$: for any $z, z' \in \operatorname{Ker} x$

$$[[x,z],[x,z']] = \frac{1}{2}[x[x,[z,z']]] \in [x,[x,L]].$$

(vi) $[\mathcal{F}_{-1}, \mathcal{F}_0] \subset \mathcal{F}_{-1}$: if we take $a \in \mathcal{F}_0$ there exits $c \in L$ such that $[x,a] = [x,[x,c]]$ so for any $z \in \operatorname{Ker} x$

$$\begin{aligned}[[x,z],a] &= [[x,[x,c]],z] + [x,[z,a]] = 2[[x,z],[x,c]] + [x,[z,a]] \\ &= 2[x,[[x,z],c]] + [x,[z,a]] \in [x, \operatorname{Ker} x]\end{aligned}$$

since $[[x,z],c] \in \operatorname{Ker} x$ and $[z,a] \in \operatorname{Ker} x$ by 1.1(ii) and (v) of this proof.

(vii) $[\mathcal{F}_{-1}, \mathcal{F}_1] \subset \mathcal{F}_0$: for any $z, z' \in \operatorname{Ker} x$

$$[[x,z],z'] \in \operatorname{Ker} x \text{ since } \operatorname{ad}_x^2[[x,z],z'] = 0,$$
$$[[[x,z],z'],x] = [[x,z],[z',x]] = -\frac{1}{2}[x,[x,[z,z']]] \in [x,[x,L]].$$

(viii) $[\mathcal{F}_{-1}, \mathcal{F}_j] \subset \mathcal{F}_1$, $j \geq 2$: for any $z \in \operatorname{Ker} x$ and any $a \in L$, $[[x,z],a] \in \operatorname{Ker} x$ since $\operatorname{ad}_x^2[[x,z],a] = [[x,z], \operatorname{ad}_x^2 a] = [\operatorname{ad}_x a, \operatorname{ad}_x^2 z] = 0$ by 1.1(ii).

(ix) $[\mathcal{F}_0, \mathcal{F}_0] \subset \mathcal{F}_0$: if $a, b \in \mathcal{F}_0$, there exist $c, d \in L$ such that $[x,a] = [x,[x,c]]$ and $[x,b] = [x,[x,d]]$ so

$$[a,b] \in \operatorname{Ker} x \text{ since } \operatorname{ad}_x^2[a,b] = 2[[x,a],[x,b]] = 2[[x,[x,c]],[x,[x,d]]] = 0,$$
$$[[a,b],x] = -[[x,[x,c]],b] - [a,[x,[x,d]]] \in [\mathcal{F}_{-2}, \mathcal{F}_0] \subset [x,[x,L]] \text{ by (ii) above.}$$

(x) $[\mathcal{F}_0, \mathcal{F}_1] \subset \mathcal{F}_1$: if $a \in \mathcal{F}_0$ there exists $c \in L$ such that $[x,a] = [x,[x,c]]$, so if we also take $z \in \operatorname{Ker} x$, $[a,z] \in \operatorname{Ker} x$ since $\operatorname{ad}_x^2[a,z] = 2[[x,a],[x,z]] = 2[[x,[x,c]],[x,z]] = 0$ by 1.1(ii).

The rest of the containments are trivial since $\mathcal{F}_2 = L$.

Finally, if we replace $\mathcal{F}_{-2}$ by $\mathcal{F}'_{-2} = \mathcal{F}_{-2} + \Phi x$ and $\mathcal{F}_{-1}$ by $\mathcal{F}'_{-1} = \mathcal{F}_{-1} + \Phi x$ we have

$$\begin{aligned}
&[\mathcal{F}'_{-2}, \mathcal{F}'_{-1}] = [\mathcal{F}_{-2}, \mathcal{F}_{-1}] + [x, \mathcal{F}_{-1}] + [\mathcal{F}_{-2}, x] + \Phi[x,x] = 0, \\
&[\mathcal{F}'_{-2}, \mathcal{F}_0] = [\mathcal{F}_{-2}, \mathcal{F}_0] + [x, \mathcal{F}_0] \subset \mathcal{F}_{-2}, \\
&[\mathcal{F}'_{-2}, \mathcal{F}_1] = [\mathcal{F}_{-2}, \mathcal{F}_1] + [x, \mathcal{F}_1] \subset \mathcal{F}_{-1}, \\
&[\mathcal{F}'_{-2}, \mathcal{F}_j] = [\mathcal{F}_{-2}, \mathcal{F}_j] + [x, \mathcal{F}_j] \subset \mathcal{F}_0 \quad \text{for every } j \geq 2, \\
&[\mathcal{F}'_{-1}, \mathcal{F}'_{-1}] = [\mathcal{F}_{-1}, \mathcal{F}_{-1}] + [x, \mathcal{F}_{-1}] + \Phi[x,x] \subset \mathcal{F}_{-2},
\end{aligned}$$

$$[\mathcal{F}'_{-1}, \mathcal{F}_0] = [\mathcal{F}_{-1}, \mathcal{F}_0] + [x, \mathcal{F}_0] \subset \mathcal{F}_{-1},$$
$$[\mathcal{F}'_{-1}, \mathcal{F}_1] = [\mathcal{F}_{-1}, \mathcal{F}_1] + [x, \mathcal{F}_1] \subset \mathcal{F}_0,$$
$$[\mathcal{F}'_{-1}, \mathcal{F}_j] = [\mathcal{F}_{-1}, \mathcal{F}_j] + [x, \mathcal{F}_j] \subset \mathcal{F}_1 \qquad \text{for every } j \geq 2,$$

and also

$$\cdots \subset \mathcal{F}'_{-2} \subset \mathcal{F}'_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$$

is a filtration of L. □

1.3. The first filtration given in 1.2 will be called the *filtration of L defined by x*. If we replace $\mathcal{F}_{-2}$ by $\mathcal{F}_{-2} + \Phi x$ and $\mathcal{F}_{-1}$ by $\mathcal{F}_{-1} + \Phi x$ this second filtration will be called the *principal filtration* given by x.

We say that an element $x \in L$ is *Von Neumann regular* if x is a Jordan element and satisfies $x \in \mathrm{ad}_x^2 L$. If x is Von Neumann regular, the two filtrations of 1.2 coincide.

1.4. Let L be a Lie algebra and let $x \in L$ be a Jordan element. If $\{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ is the filtration defined by x, then

$$\widehat{L} = \mathcal{F}_{-2} \oplus \mathcal{F}_{-1}/\mathcal{F}_{-2} \oplus \mathcal{F}_0/\mathcal{F}_{-1} \oplus \mathcal{F}_1/\mathcal{F}_0 \oplus \mathcal{F}_2/\mathcal{F}_1$$

is a 5-graded Lie algebra. In particular, $V := (\mathcal{F}_{-2}, \mathcal{F}_2/\mathcal{F}_1)$ is a Jordan pair associated to x and $W := (\mathcal{F}_{-1}/\mathcal{F}_{-2}, \mathcal{F}_1/\mathcal{F}_0)$ is a Kantor pair associated to x.

Proposition 1.5. *Let L be a nondegenerate Lie algebra over a ring of scalars Φ with $\frac{1}{2}, \frac{1}{3}, \frac{1}{5} \in \Phi$, and let $x \in L$ be a Jordan element. Let $\widehat{L}$ denote the 5-graded Lie algebra associated to the filtration of L defined by x. The following conditions are equivalent*:

(i) *x is von Neumann regular*;
(ii) *There exists $y \in L$ such that (x, y) is an idempotent*;
(iii) *$\widehat{L}$ is isomorphic to $L = L_{-2}^{(x,y)} \oplus L_{-1}^{(x,y)} \oplus L_0^{(x,y)} \oplus L_1^{(x,y)} \oplus L_2^{(x,y)}$ as a 5-graded Lie algebra*;
(iv) *$\widehat{L}$ is nondegenerate.*

Proof. (i) $\Longrightarrow$ (ii). If x is von Neumann regular, then there exists $y \in L$ such that (x, y) is an idempotent [5, 1.18(i)].

(ii) $\Longrightarrow$ (iii). Any idempotent (x, y) of L induces a grading on L given by the semisimple operator ad_h where $h = [x, y]$ (see [5, 1.18(ii)]):

$$L = L_{-2}^{(x,y)} \oplus L_{-1}^{(x,y)} \oplus L_0^{(x,y)} \oplus L_1^{(x,y)} \oplus L_2^{(x,y)}.$$

By definition, we have that $\mathcal{F}_{-2} = L_{-2}^{(x,y)}$. By grading $L_{-2}^{(x,y)} \oplus L_{-1}^{(x,y)} \oplus L_0^{(x,y)} \oplus L_1^{(x,y)} \subseteq \mathrm{Ker}\, x$ and, conversely, by [4, Lemma 2.1] if $0 \neq a \in L_2^{(x,y)}$, $[x, [x, a]] \neq 0$, so also $L_{-2}^{(x,y)} \oplus L_{-1}^{(x,y)} \oplus L_0^{(x,y)} \oplus L_1^{(x,y)} \supseteq \mathrm{Ker}\, x$, i.e., $\mathcal{F}_1 = \mathrm{Ker}\, x = L_{-2}^{(x,y)} \oplus L_{-1}^{(x,y)} \oplus L_0^{(x,y)} \oplus L_1^{(x,y)}$. By [4, Lemma 2.1], given $a_{-1} \in L_{-1}^{(x,y)}$, $-a_{-1} = \mathrm{ad}_h(a_{-1}) = [[x, y], a_{-1}] =$

$[x, [y, a_{-1}]]$ with $[y, a_{-1}] \in L_1^{(x,y)} \subseteq \operatorname{Ker} x$; therefore $\mathscr{F}_{-1} = [x, \operatorname{Ker} x] \supseteq L_{-2}^{(x,y)} \oplus L_{-1}^{(x,y)}$, and conversely, by grading, since $x \in L_{-2}^{(x,y)}$ and $\operatorname{Ker} x = L_{-2}^{(x,y)} \oplus L_{-1}^{(x,y)} \oplus L_0^{(x,y)} \oplus L_1^{(x,y)}$, $\mathscr{F}_{-1} = [x, \operatorname{Ker} x] \subseteq L_{-2}^{(x,y)} \oplus L_{-1}^{(x,y)}$. Finally, by grading $L_{-2}^{(x,y)} \oplus L_{-1}^{(x,y)} \oplus L_0^{(x,y)} \subset \mathscr{F}_0$; conversely, if some $a \in \mathscr{F}_0 \subset \operatorname{Ker} x$ has $0 \neq \pi_1(a) \in L_1^{(x,y)}$, $[x, a] \in L_{-2}^{(x,y)}$ together with $[x, \pi_1(a)] \in L_{-1}^{(x,y)}$ implies $[x, a] = 0$ and therefore by [4, Lemma 2.1] $a = 0$. Therefore,

$$\begin{aligned}
\mathscr{F}_{-2} &= L_{-2}^{(x,y)},\\
\mathscr{F}_{-1} &= L_{-2}^{(x,y)} \oplus L_{-1}^{(x,y)},\\
\mathscr{F}_0 &= L_{-2} \oplus L_{-1}^{(x,y)} \oplus L_0^{(x,y)},\\
\mathscr{F}_1 &= L_{-2} \oplus L_{-1}^{(x,y)} \oplus L_0^{(x,y)} \oplus L_1^{(x,y)},\\
\mathscr{F}_2 &= L = L_{-2} \oplus L_{-1}^{(x,y)} \oplus L_0^{(x,y)} \oplus L_1^{(x,y)} \oplus L_2^{(x,y)},
\end{aligned}$$

so $\widehat{L} = \mathscr{F}_{-2} \oplus \mathscr{F}_{-1}/\mathscr{F}_{-2} \oplus \mathscr{F}_0/\mathscr{F}_{-1} \oplus \mathscr{F}_1/\mathscr{F}_0 \oplus \mathscr{F}_2/\mathscr{F}_1 \cong L_{-2}^{(x,y)} \oplus L_{-1}^{(x,y)} \oplus L_0^{(x,y)} \oplus L_1^{(x,y)} \oplus L_2^{(x,y)} = L$ as a 5-graded Lie algebra.

(iii) $\Longrightarrow$ (iv). L is nondegenerate, so $L \cong \widehat{L}$ implies that $\widehat{L}$ is also nondegenerate.

(iv) $\Longrightarrow$ (i). Let us show that $x \in [x, [x, L]]$. Otherwise, x belongs to some $\mathscr{F}_i$, $i \geq -1$, so $0_{\widehat{L}} \neq x + \mathscr{F}_{i-1} \in \widehat{L}_i$. But for any $m = 0, \pm 1, \pm 2$:

(I) If $i = -1$ and $m = 0$, $[x + \mathscr{F}_{-2}, [x + \mathscr{F}_{-2}, \widehat{L}_0]] = [x, [x, \mathscr{F}_0]] = 0$ by 1.2(i);
(II) If $i = 0$ and $m = -2$, $[x + \mathscr{F}_{-1}, [x + \mathscr{F}_{-1}, \widehat{L}_{-2}]] = [x, [x, [x, [x, L]]]] = 0$; and
(III) If $2i + m \geq -1$: $[x + \mathscr{F}_{i-1}, [x + \mathscr{F}_{i-1}, \widehat{L}_m]] = [x, [x, \mathscr{F}_m]] + \mathscr{F}_{2i+m-1} = 0_{\widehat{L}_{2i+m}}$ since $[x, [x, \mathscr{F}_m]] \subset [x, [x, L]] = \mathscr{F}_{-2}$ is contained in any $\mathscr{F}_{-1} \subset \mathscr{F}_0 \subset \mathscr{F}_1 \subset \mathscr{F}_2$.

In any case, $[x + \mathscr{F}_{i-1}, [x + \mathscr{F}_{i-1}, \widehat{L}]] = 0_{\widehat{L}}$, which is not possible since $\widehat{L}$ is nondegenerate. □

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