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A RADICAL FOR GRADED LIE ALGEBRAS

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Abstract. For an arbitrary group G and a G-graded Lie algebra L over a field of characteristic zero we show that the Kostrikin radical of L is graded and coincides with the graded Kostrikin radical of L. As an important tool for our proof we show that the graded Kostrikin radical is the intersection of all graded-strongly prime ideals of L. In particular, graded-nondegenerate Lie algebras are subdirect products of graded-strongly prime Lie algebras.

1. Introduction

Graded structures have been a topic of interest of many authors. Authors like C. Nastasescu and F. Van Oystaeyen wrote in the late 1970s several papers and monographs containing results concerning the theory of filtered and graded rings and modules, which were motivated by applications to projective geometry but could be explained adopting a purely ring-theoretic point of view.

Within the setting of graded rings, many efforts have been directed towards the theory of radicals. The general theory of radicals of rings and algebras began in papers of A. G. Kurosh and S. A. Amitsur in the 1950s. The prime radical, also known as the Baer radical, is defined as the intersection of all prime ideals of a ring and is the least ideal giving semiprime quotient. Its graded version is the so called graded-prime radical of graded

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associative rings and algebras, and was investigated in works such as [1], [2], [8]. In [2] this radical is defined as the intersection of all graded-prime ideals, it is the least giving graded-semiprime quotient, and coincides with the largest graded ideal contained in the usual prime radical, see [2, 5.1].

For rings and associative algebras, the question whether the graded-prime and the prime radicals coincide have different answers depending on the characteristic of the base ring. In [2, 5.5], M. Cohen and S. Montgomery showed that a graded-semiprime graded ring A over a finite group G such that A had no $|G|$-torsion was always semiprime, so in this situation both radicals coincide. In Corollary 3.4 of this work we extended that result for associative algebras graded by not necessarily finite abelian groups G such that the base field has characteristic zero or p and with no elements of order p in G.

On the other hand, A. D. Sands and H. Yahya provided a counterexample (see [10, 2.6]) to the coincidence of the graded-prime radical and the prime radical when the ring had $|G|$-torsion: if R is a semiprime ring with p-torsion and $\mathbb{Z}_p$ is the cyclic group of p elements, then the group ring $R[\mathbb{Z}_p]$ is a $\mathbb{Z}_p$-graded ring which is graded-semiprime but not semiprime (for any $x \in R$, the element $\bar{x} = \sum_{\alpha \in \mathbb{Z}_p} x \cdot \alpha$ is an absolute zero divisor of $R[\mathbb{Z}_p]$, i.e., $\bar{x}R[\mathbb{Z}_p]\bar{x} = 0$).

The notion of semiprimeness in associative algebras can also be reproduced in the context of Lie algebras, but this analogue is not very useful in this setting. Nevertheless, a more appropriate notion for Lie algebras is that of nondegeneracy: a Lie algebra is nondegenerate if it has no nonzero absolute zero divisors, i.e., elements x such that $\mathrm{ad}_x^2 = 0$, with adjoint map defined by $\mathrm{ad}\, x(y) = [x, y]$. Semiprimeness and nondegeneracy are equivalent notions for associative rings and algebras, but have different meaning for Lie algebras (nondegenerate algebras are always semiprime but there exist semiprime – even simple – Lie algebras which are degenerate, for example Witt algebras over fields of prime characteristic). The prime radical has then its analogue in the Kostrikin radical for Lie algebras: the least ideal giving a nondegenerate quotient. For G-graded Lie algebras one can also define the graded Kostrikin radical, i.e., the least graded ideal giving graded-nondegenerate quotient. Its construction by transfinite induction is similar to its non-graded version (see [14]), simply starting with homogeneous absolute zero divisors.

This paper answers the following question about the coincidence of the graded Kostrikin radical and the Kostrikin radical: *Does the graded radical coincide with its non-graded version?*

The associative counterexample given a few lines above also serves as a counterexample of the non coincidence of the Kostrikin radicals for graded Lie algebras when the base field has prime characteristic: consider the Lie graded-nondegenerate algebra $R[\mathbb{Z}_p]^{(-)}$ over $\mathbb{Z}_p$; $\bar{x}$ defined as above is also an absolute zero divisor of $R[\mathbb{Z}_p]^{(-)}$. When the Lie algebra is graded by $\mathbb{Z}$ it is well known that both radicals coincide, see 3.1. The main result of the paper extends that result for arbitrary group-graded Lie algebras over fields of characteristic zero, i.e., we prove that nondegeneracy and graded-nondegeneracy are equivalent notions (Theorem 4.7).

The paper is organized as follows. Section 2 contains some necessary background material and definitions, together with two results: one characterizing graded-nondegeneracy in Lie algebras through the graded Jordan algebras associated to homogeneous ad-nilpotent elements of index less than or equal to 3, and a combinatorial result that relates non-commutative elements of a group G with orthogonal ideals of a G-graded Lie algebra L (in particular, it shows that if L is graded-prime and G is generated by the support of L, the group G must be abelian). In Section 3 we study Lie algebras graded by abelian groups, and prove that the Kostrikin radical and the graded Kostrikin radical coincide when working over fields of characteristic zero (in particular, the Kostrikin radical of such Lie algebras is G-graded and therefore the graded Kostrikin radical is the largest graded ideal contained in the Kostrikin radical, which is the usual definition of the nondegenerate – Baer or prime – radical in the associative case). In the last section we prove one of the main theorems of the paper: if G is an arbitrary group and L is a G-graded Lie algebra over a field of characteristic zero, the graded Kostrikin radical of L is the intersection of all graded-strongly prime graded ideals of L. Finally, this result together with the coincidence of the Kostrikin and the graded Kostrikin radicals for graded Lie algebras by abelian groups over fields of characteristic zero, and together with the combinatorial result that relates non-commutative elements of a group G with orthogonal ideals of a G-graded Lie algebra L, make possible to prove that the Kostrikin and the graded Kostrikin radicals coincide for arbitrary group-graded Lie algebras over fields of characteristic zero. Therefore, the notions of nondegeneracy and graded-nondegeneracy are equivalent.

2. Preliminaries

2.1. Throughout this paper and at least otherwise specified, we will be dealing with Lie algebras L and associative algebras R. As usual, given a Lie algebra L, $\mathrm{ad}_x\, y = [x, y]$ will denote the Lie bracket. An element $x \in L$ is an absolute zero divisor of L if $\mathrm{ad}_x^2 = 0$ and L is nondegenerate if it has no nonzero absolute zero divisors. A Lie algebra is prime if it has no nonzero orthogonal ideals (nonzero $I, J \lhd L$ such that $[I, J] = 0$), and it is strongly prime if it is prime and nondegenerate. For associative algebras, the notion of nondegeneracy is equivalent to semiprimeness: an associative algebra R is semiprime if it does not have nonzero nilpotent ideals, and it is nondegenerate if it has no nonzero absolute zero divisors (elements $x \in R$ such that $xRx = 0$).

2.2. Let G be a group. We say that a (non-necessarily associative) algebra A over the ring Φ is graded by G, and we denote it by A_G, if there exists a decomposition

$$A = \bigoplus_{g\in G} A_g$$

where each A_g is a Φ-submodule of A satisfying $A_g \cdot A_{g'} \subset A_{gg'}$ for every $g, g' \in G$. An element $0 \neq x \in A_g$ is called a homogeneous element of A. We recall that the support of an element $a = \sum_{g\in G} a_g \in A$ is the finite set $\operatorname{supp}(a) = \{g \in G \mid a_g \neq 0\}$ and the support of A as a G-graded algebra is the set $\operatorname{supp}(A_G) = \bigcup_{a\in A} \operatorname{supp}(a)$.

2.3. Note that if G is a group, N is a normal subgroup of G and A is a G-graded algebra, then A has a natural structure of G/N-graded algebra:

$$A = \bigoplus_{\overline{g}\in G/N} A_{\overline{g}} \quad \text{where} \quad A_{\overline{g}} = \bigoplus_{\{x\in G \mid \overline{x}=\overline{g}\}} A_x.$$

We have to be careful with graded homomorphisms. For example, the identity $i : A_G \to A_{G/N}$ is a graded homomorphism (isomorphism) of A but, in general, its inverse $i : A_{G/N} \to A_G$ might not be graded.

2.4. A graded Lie or associative algebra is graded-nondegenerate if it has no nonzero homogeneous absolute zero divisors (for associative algebras, this is equivalent to being graded-semiprime). It is clear that nondegeneracy implies graded-nondegeneracy. A graded Lie algebra is graded-prime if it has no nonzero graded-orthogonal ideals (nonzero graded $I, J \lhd L$ such that $[I, J] = 0$), and it is graded-strongly prime if it is graded-prime and graded-nondegenerate.

2.5. Similar to the (graded) prime or Baer radical in associative algebras, for a Lie algebra L there always exists the least ideal I of L whose associated quotient algebra L/I is nondegenerate (graded-nondegenerate). This ideal is radical in the sense of Amitsur–Kurosh, and can be constructed in the following way: let $I_0(L) = 0$, and let $I_1(L)$ be the ideal of L generated by all (homogeneous) absolute zero divisors of L. Using transfinite induction we define a chain of ideals $I_\alpha(L)$ by $I_\alpha(L) = \bigcup_{\beta<\alpha} I_\beta(L)$ for a limit ordinal α, and $I_\alpha(L)/I_{\alpha-1}(I) = I_1\big(L/I_{\alpha-1}(L)\big)$ otherwise. The ideal $K(L) = \bigcup_{|\alpha|\leqq|L|} I_\alpha(L)$ is called the Kostrikin radical of L (resp., the graded Kostrikin radical $K^G(L)$ of L). By construction, if $a \in L$ satisfies that $\big[a, [a, L]\big] \subset K(L)$, then $a \in K(L)$ since a is an absolute zero divisor in the quotient algebra $L/K(L)$, which is nondegenerate. Equivalently, if $a \in L$ is a homogeneous element of L such that $\big[a, [a, L]\big] \subset K^G(L)$, then $a \in K^G(L)$.

Although the (graded) prime radical for associative algebras is usually defined as the intersection of all (graded) prime ideals and can be characterized as the set of (homogeneous) elements whose associated m-sequence vanishes in a finite number of steps [7, §10] (a sequence $\{a_i\}_{i\in\mathbb{N}}$ of elements in an associative algebra A is called an m-sequence if $a_1 \in A$ and each $a_{i+1} \in a_i A a_i$), a transfinite construction analogue to that of the the Kostrikin radical can be done in the associative setting and ends up with the (graded) prime radical.

2.6. In this paper we will also deal with Jordan algebras as a tool. When $\frac{1}{2} \in \Phi$, Jordan algebras J have commutative bilinear product $a \bullet b$ with quadratic operator $U_a b = 2(a \bullet b) \bullet a - a^2 \bullet b$, and satisfy the Jordan identity $a \bullet (b \bullet a^2) = (a \bullet b) \bullet a^2$. A Jordan algebra J is nondegenerate if it has no nonzero absolute zero divisors, i.e., nonzero elements $a \in J$ such that $U_a J = 0$. The nondegenerate radical for Jordan algebras is called the McCrimmon radical and can be characterized as the set of elements $a \in J$ such that every m-sequence starting with a vanishes in a finite number of steps (an m-sequence in a Jordan algebra J is a sequence of elements $\{a_i\}_{i\in\mathbb{Z}}$ such that $a_1 \in J$ and $a_{i+1} \in U_{a_i} J$ for $i \geqq 1$), see [11, Theorem 2].

We say that an element x in a Lie algebra L is a Jordan element if $\mathrm{ad}_x^3 = 0$. When $\frac{1}{2}$, $\frac{1}{3}$ belong to Φ, every Jordan element gives rise to a Jordan algebra, called the Jordan algebra of L at x, see [4]: Let L be a Lie algebra and let $x \in L$ be a Jordan element. Then L with the new product given by $a \bullet b := \frac{1}{2}\big[[a,x],b\big]$ is an algebra such that

$$\ker(x) := \big\{z \in L \mid \big[x,[x,z]\big] = 0\big\}$$

is an ideal of $(L, \bullet)$. Moreover, $L_x := \big(L/\ker(x), \bullet\big)$ is a Jordan algebra. In this Jordan algebra the U-operator has this very nice expression:

$$U_{\overline{a}}\overline{b} = \frac{1}{8}\overline{\mathrm{ad}_a^2\,\mathrm{ad}_x^2\,b} \quad \text{for all} \quad a, b \in L.$$

A Lie algebra L is nondegenerate if and only if L_x is nonzero for every Jordan element $x \in L$. Moreover, in this case, L_x is a nondegenerate Jordan algebra [4, 2.15(i)].

If G is a group, L is a G-graded Lie algebra, and we consider a homogeneous Jordan element x of L, the associated Jordan algebra L_x is a G-graded Jordan algebra: Let us suppose that $x \in L_h$ with $h \in G$. It is clear that $\ker(x)$ is a graded submodule of L: Given $a = \sum_{g\in G} a_g \in \ker(x)$, $0 = \big[x,[x,a]\big] = \sum_{g\in G}\big[x,[x,a_g]\big]$ and, by grading, $\big[x,[x,a_g]\big] = 0$ for every $g \in G$. Now, the quotient module $L/\ker(x) = \oplus_{g\in G} M_g$ is graded by G (if we define $L_x^g := M_{gh^{-1}}$, the Jordan algebra L_x of L at x can be written as $L_x = \bigoplus_{g\in G} L_x^g$ where for homogeneous elements $a \in L_x^g$, $b \in L_x^{g'}$ we have $a \bullet b = \frac{1}{2}\big[[a,x],b\big] \in L_x^{gg'}$).

Note that every homogeneous element $\overline{a}$ of L_x can be represented by a homogeneous element of L. We will use it without mentioning it. Sometimes the grading of L_x inherited from the grading of L could be trivial. For example, if $L = \mathrm{TKK}\,(V)$ is the TKK-algebra of a Jordan pair V (see [9, 1.5(6)] or [3, 2.7]), L is a $\mathbb{Z}$-graded Lie algebra and given any element $x \in L_1$ we have that L_x only has one nontrivial component (labeled by 0), and therefore the $\mathbb{Z}$-grading associated to L_x is trivial.

We have the following characterization of graded-nondegeneracy in Lie algebras:

Lemma 2.7. *Let G be a group and L a G-graded Lie algebra. Then L is graded-nondegenerate if and only if L_x is nonzero for every homogeneous Jordan element $x \in L$. Moreover, in this case, L_x is a graded-nondegenerate Jordan algebra.*

Proof. The proof follows [4, 2.15(i)]. It is clear that a homogeneous element $x \in L$ is an absolute zero divisor of L if and only if L_x is zero. Moreover, if x is a homogeneous Jordan element of L and $\overline{a} \in L_x$ is a nonzero homogeneous absolute zero divisor of the Jordan algebra L_x, then $\big[x,[x,a]\big]$ is a homogeneous element of L such that

$$\mathrm{ad}^2_{[x,[x,a]]}(L) = \mathrm{ad}^2_x\,\mathrm{ad}^2_a\,\mathrm{ad}^2_x(L) = 0 \quad \text{because} \quad \mathrm{ad}^2_a\,\mathrm{ad}^2_x(L) = U_a L_x \subset \ker\,(x).$$

Therefore, L graded-nondegenerate implies that L_x is graded-nondegenerate. □

The next proposition relates noncommutative elements of G with orthogonal ideals of L for a G-graded Lie algebra L. In particular this result generalizes the well-known fact that the support of a graded-simple Lie algebra L generates an abelian subgroup of G.

Proposition 2.8. *Let L be a Lie algebra graded by a group G and let $g, g' \in G$ be such that $gg' \neq g'g$. Then $\big[\mathrm{id}_L(L_g), \mathrm{id}_L(L_{g'})\big] = 0$ where $\mathrm{id}_L(L_g)$ and $\mathrm{id}_L(L_{g'})$ denote the ideals of L generated by L_g and $L_{g'}$ respectively. In particular, if L is graded-prime and G is generated by the support of L, then G is an abelian group.*

Proof. Let us prove the following property: Let $g_1, g_2, \ldots, g_n \in G$

$$(*) \qquad \begin{cases} \text{if } \Big[L_{g_1}, \big[L_{g_2}, \big[\,\cdots, [L_{g_{n-1}}, L_{g_n}]\big]\big]\Big] \neq 0 \\ \text{then } \ g_i g_j = g_j g_i \ \ \forall\, i,j \in \{1,2,\ldots,n\}. \end{cases}$$

This property is true for $n = 2$:

$$L_{g_1g_2} \supset [L_{g_1}, L_{g_2}] = [L_{g_2}, L_{g_1}] \subset L_{g_2g_1}$$

which implies, if $0 \neq [L_{g_1}, L_{g_2}] \subset L_{g_1g_2} \cap L_{g_2g_1}$, that $g_1g_2 = g_2g_1$. Let us suppose that $(*)$ is true for every $n-1$ elements and let $0 \neq \big[L_{g_1}, \big[L_{g_2}, \big[\,\cdots, [L_{g_{n-1}}, L_{g_n}]\big]\big]\big]$. By hypothesis,

$$g_ig_j = g_jg_i \quad \text{if} \quad i, j \in \{2, 3, \dots, n\} \tag{1}$$

By the Jacobi identity we have two possibilities:

(a) If $0 \neq \big[[L_{g_1}, L_{g_2}], \big[\,\cdots, [L_{g_{n-1}}, L_{g_n}]\big]\big] \subset \big[L_{g_1g_2}, \big[\,\cdots, [L_{g_{n-1}}, L_{g_n}]\big]\big]$ then $g_1g_2 = g_2g_1$ and, by the induction hypothesis and (1), the elements g_1g_2 and g_2 commute with every g_k, $k \geqq 3$. Then $g_k(g_1g_2) = (g_1g_2)g_k = g_1g_kg_2$ and multiplying by g_2^{-1} on the right we get $g_1g_k = g_kg_1$ for every $k \geqq 3$.

(b) If $\big[L_{g_2}, \big[L_{g_1}, \big[\,\cdots, [L_{g_{n-1}}, L_{g_n}]\big]\big]\big] \neq 0$, take $h = g_3 \dots g_n$. By induction,

$$g_1h = hg_1, \quad g_2(g_1h) = (g_1h)g_2, \quad \text{and} \quad g_1g_k = g_k \quad \text{for every} \quad k \geqq 3,$$

so by (1), $g_2(g_1h) = (g_1h)g_2 = g_1g_2h$, which implies $g_1g_2 = g_2g_1$.

Now, if $g, g' \in G$ with $gg' \neq g'g$, then for every $g_1, g_2, \dots, g_n \in G$

$$\big[L_{g'}, \big[L_{g_1}, \big[\cdots [L_{g_n}, L_g]\big]\big]\big] = 0$$

which proves that $\big[\mathrm{id}_L(L_{g'}), \mathrm{id}_L(L_g)\big] = 0$. □

3. Nondegeneracy in Lie algebras graded by abelian groups

3.1. It is well known that a $\mathbb{Z}$-graded Lie algebra L is nondegenerate if and only if it is graded-nondegenerate. Indeed, if $x \in \bigoplus_{n\in\mathbb{Z}} L_n$ is an absolute zero divisor of L and k is the biggest (or the lowest) integer such that $0 \neq \pi_k(x) = x_k$, where $\pi_k : L \to L_k$ denotes the canonical projection onto L_k, then x_k is a nonzero homogeneous absolute zero divisor of L: let us suppose that $x = \sum_{i\leqq k} x_i$ and $a_s \in L_s$, then

$$0 = \big[x, [x, a_s]\big] = \Bigg[\sum_{i\leqq k} x_i, \Bigg[\sum_{j\leqq k} x_j, a_s\Bigg]\Bigg]$$

$$= \sum_{r<2k+s} \Bigg(\sum_{i+j=r} \big[x_i, [x_j, a_s]\big]\Bigg) + \big[x_k, [x_k, a_s]\big]$$

and, by grading, $\big[x_k,[x_k,a_s]\big]=0$ for every $a_s\in L_s$ with $s\in\mathbb{Z}$. Therefore, if $L/K^G(L)$ contains a nonzero absolute zero divisor, since it is $\mathbb{Z}$-graded, it also contains a nonzero homogeneous absolute zero divisor, which is not possible by definition of $K^G(L)$. Thus $K(L)=K^G(L)$ (which implies, in particular, that $K(L)$ is a $\mathbb{Z}$-graded ideal of L).

Let us extend this result to associative and Lie algebras which are graded by abelian groups over fields of characteristic zero. The next two lemmas are folklore.

LEMMA 3.2. *Let G be an abelian group and let A be a G-graded associative or Lie algebra over a ring of scalars Φ. Let $x=\sum_{g\in G}x_g\in A$ be an absolute zero divisor of A such that the subgroup H of G generated by* $\operatorname{supp}(x)$ *is either $\mathbb{Z}_n$ or $\mathbb{Z}$. Then, for any $\xi\in\Phi$ satisfying $\xi^n=1$ if $H=\mathbb{Z}_n$, or any $\xi\in\Phi$ if $H=\mathbb{Z}$, the element $x_\xi:=\sum_{k\in H}\xi^k x_k$ is an absolute zero divisor of A.*

PROOF. Since x is an absolute zero divisor of A, $L_xR_x=0$ if A is associative (here L_x stands for the left multiplication and R_x stands for the right multiplication in A), or $\operatorname{ad}_x^2=0$ if A is a Lie algebra. Therefore,

$$0=L_xR_x=\Big(\sum_{i\in H}L_{x_i}\Big)\Big(\sum_{j\in H}R_{x_j}\Big)=\sum_{r\in H}\Big(\sum_{i+j=r}L_{x_i}R_{x_j}\Big)$$

if A is associative,

$$0=\operatorname{ad}_x^2=\Big(\sum_{i\in H}\operatorname{ad}_{x_i}\Big)\Big(\sum_{j\in H}\operatorname{ad}_{x_j}\Big)=\sum_{r\in H}\Big(\sum_{i+j=r}\operatorname{ad}_{x_i}\operatorname{ad}_{x_j}\Big)$$

if A is Lie. Let us continue assuming that A is an associative algebra (the Lie case follows similarly). From the above formula, by grading

$$\sum_{i+j=r}L_{x_i}R_{x_j}=0\quad\text{for every}\quad r\in H.$$

Now,

$$L_{x_\xi}R_{x_\xi}=\Big(\sum_{i\in H}\xi^iL_{x_i}\Big)\Big(\sum_{j\in H}\xi^jR_{x_j}\Big)=\sum_{r\in H}\Big(\sum_{i+j=r}\xi^{i+j}L_{x_i}R_{x_j}\Big)$$

$$=\sum_{r\in H}\xi^{i+j}\Big(\sum_{i+j=r}L_{x_i}R_{x_j}\Big)=0,$$

which implies that x_ξ is an absolute zero divisor of A. $\square$

LEMMA 3.3. *Let G be an abelian group and let A be a G-graded associative or Lie algebra over an algebraically closed field Φ of characteristic zero or p such that there are no elements of order p in G. Let $x = \sum_{g\in G} x_g \in A$ be an absolute zero divisor of A. Then every homogeneous component of x can be written as a linear combination of absolute zero divisors of A.*

PROOF. Let H be the subgroup of G generated by $\operatorname{supp}(x)$. H is a finitely generated abelian group and therefore $H \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ where, by hypothesis, every n_i is coprime with p. Let us give a proof by induction on m, the number of cyclic groups that appear in the factorization of H. If $m = 0$ there is nothing to prove since $x = 0$. Let us suppose that our claim is true for $m - 1$ and suppose that H has $m > 0$ factors. We can decompose $H \cong K \times K'$ (with K equal to $\mathbb{Z}_{n_1}$ or $\mathbb{Z}$; if $m = 1$, $K' = 0$). Now $x = \sum_{(k,k')\in K\times K'} x_{(k,k')}$. We can see A as a G/K-graded and a G/K'-graded algebra, see 2.3. The subgroup of G/K generated by support of x with respect to the G/K-grading is isomorphic to K' while the subgroup of G/K' generated by support of x with respect to the G/K'-grading is isomorphic to K, so we can represent x as

$$x = \sum_{k\in K} \Big(\sum_{k'\in K'} x_{(k,k')} \Big).$$

Now, if $\xi \in \Phi$ satisfies $\xi^{n_1} = 1$ if $K = \mathbb{Z}_{n_1}$ (resp., $\xi \in \Phi$ if $K = \mathbb{Z}$), we have by 3.2 that

$$x_\xi = \sum_{k\in K} \xi^k \Big(\sum_{k'\in K'} x_{(k,k')} \Big) = \sum_{k'\in K'} \Big(\sum_{k\in K} \xi^k x_{(k,k')} \Big)$$

is an absolute zero divisor of L. Moreover, by the induction hypothesis, since K' is either 0 or a product of $m - 1$ cyclic groups, every component $z_{k'} = \sum_{k\in K} \xi^k x_{(k,k')}$ can be written as a linear combination of absolute zero divisors of A.

On the other hand, if we take n_1 different elements in Φ such that $\xi^{n_1} = 1$ if $K = \mathbb{Z}_{n_1}$, which is possible because n_1 is coprime with p (resp., take enough different elements of Φ if $K = \mathbb{Z}$), we can see the identities

$$z_{k'} = \sum_{k\in K} \xi^k x_{(k,k')}, \qquad k' \in K'$$

as a linear system of equations in the variables $x_{(k,k')}$. The matrix of this linear system is invertible since its determinant is a Vandermonde determinant and all the ξ's are different, so we can rewrite every $x_{(k,k')}$ as a linear combination of elements of $\{z_{k'} \mid k' \in K'\}$, and each of them is a sum of absolute zero divisors of A. Therefore, every homogeneous component $x_{(k,k')}$ of x belongs to the span of the absolute zero divisors of A. □

Corollary 3.4. *Let G be an abelian group and let A be a G-graded associative algebra over a field Φ of characteristic zero or p such that there are no elements of order p in G. Then the graded-prime radical $B^G(A)$ coincides with the prime radical $B(A)$.*

Proof. Let us consider the algebraic closure $\overline{\Phi}$ of Φ and the scalar extension $A_{\overline{\Phi}}$ of A over $\overline{\Phi}$. If $x = \sum_{g\in G} x_g \in A$ is a nonzero absolute zero divisor of A, it is an absolute zero divisor of $A_{\overline{\Phi}}$ and therefore, by 3.3 every x_g is a sum of absolute zero divisors of $A_{\overline{\Phi}}$. Then every x_g satisfies the finite m-sequence condition in $A_{\overline{\Phi}}$, so also in A and therefore x_g belongs to the prime radical $B(A)$, and x is contained in the largest graded ideal $B^G(A)$ contained in $B(A)$. If we continue the transfinite process of building the prime radical of A (see 2.5) we end up with the fact that $B(A)$ is contained in $B^G(A)$, so $B^G(A) = B(A)$. □

Theorem 3.5. *Let G be an abelian group and let L be a G-graded Lie algebra over a field Φ of characteristic zero. Then $K^G(L) = K(L)$. In particular, the Kostrikin radical of L is G-graded.*

Proof. It is clear that $K^G(L) \subset K(L)$. Let us prove the converse. We can work in $L/K^G(L)$ and suppose that L is graded-nondegenerate. Let us consider the algebraic closure $\overline{\Phi}$ of Φ and the scalar extension $L_{\overline{\Phi}}$ of L over $\overline{\Phi}$. Now, if $x = \sum_{g\in G} x_g$ is a nonzero absolute zero divisor of L, it is an absolute zero divisor of $L_{\overline{\Phi}}$ and therefore, by 3.3 every x_g is a sum of absolute zero divisors of $L_{\overline{\Phi}}$. By [14, Lemma 8] x_g is a strongly Engel element of $L_{\overline{\Phi}}$. In particular x_g is an ad-nilpotent element of $L_{\overline{\Phi}}$, and therefore an ad-nilpotent element of L of index n, i.e., there exists $n \in \mathbb{N}$ such that $\mathrm{ad}^n_{x_g}(L) = 0$ but $\mathrm{ad}^{n-1}_{x_g}(L) \neq 0$. By [5] let $0 \neq y \in \mathrm{ad}^{n-1}_{x_g}(L)$ be a homogeneous ad-nilpotent element of index 3 and let us consider the G-graded Jordan algebra L_y. In L_y every element is nilpotent of bounded index since y is still strongly Engel in L and we can argue as in [6, Proposition 3.2]. By [13, Lemma 17, p. 849] L_y is radical in the sense of McCrimmon, so starting with a nonzero homogeneous element of L_y we can construct an m-sequence of L_y consisting of homogeneous elements, which in its last nonzero step gives rise to a homogeneous absolute zero divisor of $\overline{a} \in L_y$. Therefore, $0 \neq [y,[y,a]]$ is a homogeneous absolute zero divisor of L, a contradiction because L is graded-nondegenerate. □

4. Nondegeneracy in Lie algebras graded by arbitrary groups

In this section we prove that for an arbitrary group G, the graded Kostrikin radical of a G-graded Lie algebra L over a field of characteristic zero is the intersection of all graded-strongly prime ideals of L. This is a classical result for associative algebras, and its version in the non-associative case has been very useful, see [11] or [15] for the Jordan case or more recently [6, Proposition 3.10] for the Lie case. As a consequence, we show that the notions of nondegeneracy and graded-nondegeneracy are equivalent for G-graded Lie algebras over fields of characteristic zero.

4.1. Given $n \in \mathbb{N}$ and a Lie algebra L, let

$$B_n(L) = \Big\{ \sum_{i=1}^{n} \big[[[a_i, b_{i_1}], \dots, b_{i_{k_i}}] \big] \mid 0 \leqq k_i \leqq n,\ b_{i_j} \in L,\ \mathrm{ad}^2_{a_i} = 0 \Big\}$$

be the set of sums of n monomials in L whose distance to an absolute zero divisor of L is less than or equal to n. Notice that $B_1 \subset B_2 \subset \ldots \subset B_n$ and $K_1(L) = \bigcup_n B_n$.

Lemma 4.2 [6, Lemma 3.4]. *For each $n, r \in \mathbb{N}$ there exists $f(n,r) \in \mathbb{N}$ with $f(n,r) \geqq 3$ such that for every Lie algebra L over a field of characteristic zero and for every $a \in B_n(L)$*

$$\mathrm{ad}^{f(n,r)}_{[[a,b_1],\dots,b_k]} = 0 \quad \text{for every} \quad b_1, \dots, b_k \in L, \quad 0 \leqq k \leqq r.$$

4.3. Given a Lie algebra L over a field of characteristic zero, we say that the sequence $\{c_i\}_{i\in\mathbb{N}}$ is

(i) an m-sequence of L if $c_1 \in L$ and each $c_{i+1} \in \big[c_i, [c_i, L]\big]$,

(ii) a generalized m-sequence of L if $c_1 \in L$ and each c_{i+1}, $i \geqq 1$, is an element of the form

$$\mathrm{ad}^{q_i}_{c_i} x_0, \quad \mathrm{ad}^{q_i}_{[c_i,x_1]} x_0, \quad \text{or} \quad \mathrm{ad}^{q_i}_{[[c_i,x_1],x_2]} x_0$$

for some $x_0, x_1, x_2 \in L$ and $q_i = f(i, 3i+2)$, see [6, 3.5].

Lemma 4.4. *Let L be a Lie algebra over a field of characteristic zero, and let $x \in K(L)$ be a Jordan element. Then the Jordan algebra L_x of L at x is radical in the sense of McCrimmon and therefore every m-sequence of L starting with x has finite length.*

Proof. Suppose first that $x \in K_1(L)$ and consider the Jordan algebra L_x of L at x. Since $x \in K_1(L)$, x is strongly Engel (see [14, Lemma 8]) and there exists $n \in \mathbb{N}$ such that for any $a \in L$, the element $[a,x]$ is ad-nilpotent of index less than or equal to n. Then the Jordan algebra L_x is nilpotent of index at most $n+1$: for any $a \in L$, $a^{(n+1,x)} = a \bullet a^{(n,x)} = \frac{1}{2^n} \mathrm{ad}^n_{[a,x]} a = 0$. Therefore L_x is radical in the sense of McCrimmon (see [12, Lemma 17, p. 849]) and every m-sequence in L_x has finite length. But then every m-sequence of L starting with x has also finite length by [6, Proposition 2.2].

Now if $x \in K(L) = \bigcup K_\beta(L)$, $x \in K_\alpha(L)$ for some α which is not a limit ordinal, and $x + K_{\alpha-1}(L) \in K_1\big(L/K_{\alpha-1}(L)\big)$ is also a Jordan element of $L/K_{\alpha-1}(L)$. By the previous paragraph every m-sequence of L starting with x stops in a finite number of steps in an element of $K_{\alpha-1}(L)$, and the result follows by induction. □

PROPOSITION 4.5. *Let G be a group and let L be a G-graded Lie algebra over a field of characteristic zero, let $\{c_i\}_{i\in\mathbb{N}}$ be an infinite generalized m-sequence of L of nonzero homogeneous elements or an infinite m-sequence of L of nonzero homogeneous Jordan elements, and let P be a graded ideal of L which is maximal among those graded ideals of L not containing any element of $\{c_i\}_{i\in\mathbb{N}}$. Then P is a graded-strongly prime ideal of L, i.e., L/P is a graded-strongly prime Lie algebra.*

PROOF. To see that L/P is graded-prime, if A/P and B/P are two nonzero graded ideals of L/P, there exist some $c_j \in A$, some $c_k \in B$, so $c_l \in A \cap B$ for every $l \geqq j, k$. Then, $c_{\max(j,k)+1} \in [A, B]$ so $[A/P, B/P] \neq \bar{0}$.

To see that L/P is graded-nondegenerate, suppose on the contrary that the graded Kostrikin radical $K^G(L/P) \neq 0$. Consider $\hat{K} = \pi^{-1}\big(K^G(L/P)\big)$, where $\pi : L \to L/P$ denotes the canonical projection, which is a graded ideal of L properly containing P, so there exists some $c_j \in \hat{K}$, hence $\bar{0} \neq c_j + P \in K^G(L/P) \subset K(L/P)$. By [6, Proposition 3.6] if $\{c_i\}_{i\in\mathbb{N}}$ is a generalized m-sequence of homogeneous elements, or by 4.4 if $\{c_i\}_{i\in\mathbb{N}}$ is an m-sequence of homogeneous Jordan elements, the sequence $\{c_i + P\}_{i\in\mathbb{N}}$ has finite length, so there exists some $c_k + P = \bar{0}$, i.e., $c_k \in P$, a contradiction. □

Now we state the two most important results of the paper: if G is an arbitrary group and L is a G-graded Lie algebra over a field of characteristic zero, firstly, the Kostrikin radical of L is the intersection of all graded-strongly prime ideals of L and, secondly, nondegeneracy and graded-nondegeneracy are equivalent notions for L.

THEOREM 4.6. *Let G be an (arbitrary) group and L a G-graded Lie algebra over a field of characteristic zero. The graded Kostrikin radical $K^G(L)$ of L is the intersection of all graded-strongly prime ideals of L and, as a consequence, $L/K^G(L)$ is isomorphic to a subdirect product of graded-strongly prime Lie algebras.*

PROOF. If $\{P_\alpha\}_{\alpha\in\Gamma}$ denotes the set of all graded-strongly prime ideals of L, it is clear that $K^G(L) \subset P_\alpha$ for each α since L/P_α is graded-nondegenerate, so $K^G(L) \subset \bigcap_{\alpha\in\Gamma} P_\alpha$. Conversely, let $a \in L$ be a homogeneous element that does not belong to $K^G(L)$. We can work in $L/K^G(L)$ and suppose that L is graded-nondegenerate.

If $\mathrm{id}_L(a)$ does not contain nonzero homogeneous ad-nilpotent elements we can construct an infinite generalized m-sequence starting with a and consisting of nonzero homogeneous elements and, therefore, by 4.5 there exists a graded-strongly prime ideal P not containing a.

On the other hand, if $\mathrm{id}_L(a)$ contains a nonzero homogeneous ad-nilpotent element b of index n, by [5, Corollary 2.4] every element of $\mathrm{ad}_b^{n-1}(L)$ is a Jordan element. Therefore there exists a nonzero homogeneous Jordan element $c \in \mathrm{id}_L(a)$, and we can construct an infinite m-sequence of nonzero homogeneous Jordan elements starting with c (by graded-nondegeneracy of L). Now, by 4.5 there exists a graded-strongly prime ideal P of L not containing c, and therefore not containing a. In any case $a \notin \bigcap_{\alpha\in\Gamma} P_\alpha$. □

THEOREM 4.7. *Let G be an (arbitrary) group and L a G-graded Lie algebra over a field of characteristic zero. Then L is nondegenerate if and only if it is graded-nondegenerate, i.e., $K^G(L) = K(L)$.*

PROOF. It is clear that nondegeneracy implies graded-nondegeneracy. Let us suppose that L is graded-nondegenerate. By 4.6 there exists a family $\{P_\alpha\}_{\alpha\in\Gamma}$ of graded-strongly prime ideals of L such that $\bigcap_{\alpha\in\Gamma} P_\alpha = 0$. For every $\alpha \in \Gamma$, L/P_α is a graded-strongly prime Lie algebra, so by 2.8 the support of L/P_α generates an abelian subgroup G_α of G and we can suppose that L/P_α is graded by G_α. Now, let us denote by $\pi_\alpha : L \to L/P_\alpha$ the canonical projection and let $x = \sum_{g\in G} x_g \in L$ be an absolute zero divisor of L. Then for every α, $\pi_\alpha(x)$ is an absolute zero divisor of L/P_α, which is graded-nondegenerate over an abelian group and therefore, by 3.5, L/P_α is nondegenerate. So for every $\alpha \in \Gamma$, $\pi_\alpha(x) = 0$ and therefore $x \in \bigcap_{\alpha\in\Gamma} P_\alpha = 0$, i.e., $x = 0$ and L is a nondegenerate Lie algebra. □

4.8. Remark. As a consequence of this theorem, if a G-graded Lie algebra L over a field of characteristic zero contains nonzero absolute zero divisors, it also contains nonzero homogeneous absolute zero divisors.

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