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The Inner Ideals of the Simple Finite Dimensional Lie Algebras

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Abstract. The inner ideals of the simple finite dimensional Lie algebras over an algebraically closed field of characteristic 0 are classified up to conjugation by automorphisms of the Lie algebra, and up to Jordan isomorphisms of their corresponding subquotients (any proper inner ideal of such an algebra is abelian and therefore it has a subquotient which is a simple Jordan pair). While the description of the inner ideals of the Lie algebras of types A_l, B_l, C_l and D_l can be obtained from the Lie inner ideal structure of the simple Artinian rings and simple Artinian rings with involution, the description of the inner ideals of the exceptional Lie algebras (types G_2, F_4, E_6, E_7 and E_8) remained open. The method we use here to classify inner ideals is based on the relationship between abelian inner ideals and $\mathbb{Z}$-gradings, obtained in a recent paper of the last three named authors with E. Neher. This reduces the question to deal with root systems.
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Introduction

Let L be a Lie algebra over a ring of scalars Φ. A Φ-submodule B of L is an inner ideal if $[B,[B,L]] \subset B$, and B is abelian if $[B,B] = 0$. The initial motivation to study inner ideals in Lie algebras was due to the fact that inner ideals are closely related to ad-nilpotent elements, and certain restrictions of these elements yield an elementary criterion for distinguishing the nonclassical from classical (finite dimensional) simple Lie algebras over an algebraically closed field of characteristic greater than 5 [2].

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Abelian inner ideals and their associated notions of kernel and subquotient became a key notion to develop a socle theory for nondegenerate Lie algebras [5], and were used in [7] to construct gradings of Lie algebras: it requires the existence of abelian inner ideals whose subquotient is a Jordan pair covered by a finite grid, and this produces a grading of the Lie algebra by the weight lattice of the root system associated to the covering grid.

In [1], G. Benkart examined the Lie inner ideal structure of semiprime associative rings, and of the skew elements of prime rings with involution. An extension of these results was carried out by the last three named authors in [6], where the inner ideals of infinite dimensional finitary simple Lie algebras were described. However, in both of these works, a type of inner ideals, the so-called point spaces, was omitted. This description has been recently completed by G. Benkart and A. Fernández López in [3].

In this paper we adopt a different approach to determine the inner ideals of the simple finite-dimensional Lie algebras over an algebraically closed field F of characteristic 0 based on the connection between abelian inner ideals and $\mathbb{Z}$-gradings mentioned above. For any proper inner ideal B of such an algebra L there exists a finite $\mathbb{Z}$-grading $L = L_{-n} \oplus \cdots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_n$ with $B = L_n$, also called the extreme of the grading. As the $\mathbb{Z}$-gradings are always compatible with a root decomposition, B can be expressed as sum of root spaces. More precisely, any $\mathbb{Z}$-grading is the diagonalization relative to ad_h, for a semisimple element h in a Cartan subalgebra such that the coordinates relative to a basis of the root system $\alpha_i(h)$ are nonnegative integers, and in such case the extreme is determined by the indices i such that $\alpha_i(h) \neq 0$. This provides us an easy procedure to determine the inner ideals, which produces a classification (called the Lie classification) of the inner ideals of L up to conjugation by automorphisms of L. On the other hand, the subquotient of any proper (equivalently, abelian) inner ideal of L is a classical Jordan pair [7]. This yields another classification (the Jordan classification) of the proper inner ideals of L up to Jordan pair isomorphisms of their subquotients. It must be noted that while two abelian inner ideals which are conjugate by an automorphism of L have necessarily isomorphic subquotients, the converse is not true, so the Lie classification is finer than the Jordan one.

Finally, by using methods of classical theory of Lie algebras, we give in the appendix an alternative proof to the fact that every abelian inner ideal coincides with the extreme of a finite $\mathbb{Z}$-grading.

1. Lie algebras and Jordan pairs

1.1. Throughout this paper we will deal with *finite dimensional* Lie algebras L [9], [10], with $[x, y]$ denoting the Lie bracket and ad_x the adjoint map determined by x, and *finite dimensional* Jordan pairs $V = (V^+, V^-)$ [11], with Jordan products $Q_x y$ and linearizations $\{x, y, z\} := Q_{x,z} y$, for $x, z \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$, over an algebraically closed field F of characteristic zero.

1.2. An *inner ideal* of a Jordan pair V is an F-subspace B of V^σ such that $\{B, V^{-\sigma}, B\} \subset B$. We say that two inner ideals B and B' of V are *conjugate*

if there exists an automorphism of V sending B to B'. An F-subspace B of a Lie algebra L is an *inner ideal* if $[B,[B,L]] \subset B$, and B is *abelian* if $[B,B]=0$. Two inner ideals B and B' of a Lie algebra L are *conjugate* if there exists an automorphism φ of L such that $\varphi(B)=B'$.

1.3. Let $B \subset V^+$ be an inner ideal of V. The *kernel* of B is the set $\operatorname{Ker}_V B = \{x \in V^- \mid Q_B x = 0\}$. Then $(0, \operatorname{Ker}_V B)$ is an ideal of the Jordan pair (B, V^-) and the quotient $\operatorname{Sub}_V B = (B, V^-)/(0, \operatorname{Ker}_V B) = (B, V^-/\operatorname{Ker}_V B)$ is called the *subquotient* of B [12]. The kernel and the corresponding subquotient of an inner ideal $B \subset V^-$ are defined similarly.

Let V and V' be two Jordan pairs over F, and let B and B' be inner ideals of V and V' respectively. We say that B is *isomorphic* to B' if $\operatorname{Sub}_V B \cong \operatorname{Sub}_{V'} B'$.

The analogous versions of all these results hold for abelian inner ideals of a Lie algebra, if we replace the Jordan triple product $\{x,y,z\}$ by the left double commutator $[[x,y],z]$: Any abelian inner ideal B of a Lie algebra L gives rise to a Jordan pair, which is called the subquotient of B in L [7]; the *kernel* of B is the set $\operatorname{Ker}_L B = \{x \in L \mid [B,[B,x]] = 0\}$, and the pair of F-vector spaces $\operatorname{Sub}_L B = (B, L/\operatorname{Ker}_L B)$ with the triple products given by

$$\{b, \overline{x}, b'\} := \overline{[[b,x],b']} \quad \text{for every } b, b' \in B \text{ and } x \in L,$$
$$\{\overline{x}, b, \overline{y}\} := \overline{[[x,b],y]} \quad \text{for every } b \in B \text{ and } x, y \in L,$$

where $\overline{a}$ denotes the coset of a relative to the subspace $\operatorname{Ker}_L B$, is a Jordan pair called the *subquotient* of B. Due to this notion, we can define a new relation between inner ideals of Lie algebras: if B and B' are abelian inner ideals of Lie algebras L and L' respectively, then B and B' are said to be *isomorphic* if $\operatorname{Sub}_L B \cong \operatorname{Sub}_{L'} B'$ as Jordan pairs. In the particular case of a simple finite dimensional Lie algebra, every proper inner ideal is abelian [2, 1.13], so it makes sense to associate a Jordan pair $\operatorname{Sub}_L B$ to any proper inner ideal B of L. Notice that in this case such a subquotient $\operatorname{Sub}_L B$ is always a simple Jordan pair, according to [7, 3.5(vi)].

It turns out that an F-subspace C of B is an inner ideal of L if and only if it is an inner ideal of $\operatorname{Sub}_L B$ [7, 3.5(i)].

1.4. An important class of inner ideals of Jordan pairs and Lie algebras are the so called point spaces. For a Jordan pair $V=(V^+,V^-)$, a subspace P of V^σ, $\sigma = \pm$, is called a *point space* if $Q_x V^{-\sigma} = Fx$ for any nonzero $x \in P$. A subspace P of a Lie algebra L is called a *point space* if $[P,P]=0$ and every nonzero element $x \in P$ is extremal, i.e., $\operatorname{ad}_x^2 L = Fx$. If P is a point space of L, then P is an abelian inner ideal, P is a point space of the Jordan pair $\operatorname{Sub}_L P$, and any subspace Q of P is also a point space. All point spaces of the same dimension are isomorphic [3, 4.6].

1.5. As a general rule, we will use the same symbol to denote inner ideals of Jordan pairs and abelian inner ideals of Lie algebras which belong to the same class of isomorphy. Thus, P_r will denote a point space (both of a Jordan pair

or a Lie algebra) of dimension r over F. When required, we will use accents to distinguish between inner ideals which are isomorphic but not conjugate.

2. The inner ideal structure of the classical Jordan pairs revisited

By a *classical Jordan pair* we mean a finite-dimensional simple Jordan pair over an algebraically closed field F. In this section we review the classification of the inner ideals of the classical Jordan pairs over F. By [11, 17.4], any classical Jordan pair is isomorphic to one of the following:

(I) *The Jordan pair* $\mathcal{M}_{p\times q} := (M_{p\times q}(F), M_{q\times p}(F))$, $Q_x y = xyx$, *of* $p\times q$ *and* $q\times p$ *matrices with entries in* F, *and where* $p \leq q$. The nonzero inner ideals of $\mathcal{M}_{p\times q}$ contained in $M_{p\times q}(F)$ are, up to conjugation, of the form

$$M_{r\times s} := \sum_{\substack{1\leq i\leq r \\ 1\leq j\leq s}} F[ij], \quad \text{with } r \leq p,\ s \leq q, \text{ and } r \leq s$$

where $[ij]$ denotes the (i,j)-matrix unit. Moreover, the subquotient of $M_{r\times s}$ is isomorphic to $\mathcal{M}_{r\times s}$. This can be obtained from the classification of inner ideals in Jordan pairs covered by grids [14, 3.2], or from the geometric description of the inner ideals of Jordan pairs of finite rank continuous operators [8, Prop. 2.4]. Note that for each positive integer r, $M_{1\times r}$ is a point space of dimension r, so, according to our notation criterion above, $\operatorname{Sub} P_r \cong \mathcal{M}_{1\times r}$.

(II) *The Jordan pair* $\mathcal{K}_n := (K_n(F), K_n(F))$, $Q_x y = -xyx$, *of skew-symmetric* $n\times n$ *matrices with entries in* F $(n \geq 4)$. It follows from [14, 3.2(e)] that $\mathcal{K}_n$ contains two types of nonzero inner ideals up to conjugation:

(i) $K_s := e_s K_n(F) e_s$, for $2 \leq s \leq n$, where $e_s = [11] + \cdots + [ss]$, with subquotient $\mathcal{K}_s$ and

(ii) the point spaces $P_r = \sum_{j=2}^{r+1} F([1j] - [j1])$ for $1 \leq r \leq n-1$.

Note that K_s is a point space if and only if $s \leq 3$ $(K_2 = P_1)$.

(III) *The Jordan pair* $\mathcal{S}_n := (S_n(F), S_n(F))$, $Q_x y = xyx$, *of symmetric* $n\times n$ *matrices with entries in* F $(n \geq 2)$. By [14, 3.2(c)] or [13, Theorem 3], every nonzero inner ideal of $\mathcal{S}_n$ is (up to conjugation) of the form $S_r := e_r S_n(F) e_r$, for $1 \leq r \leq n$, where $e_r = [11] + \cdots + [rr]$, with subquotient $\mathcal{S}_r$.

(IV) *The Clifford Jordan pair* $\mathcal{Q}_n := (X, X)$, $Q_x y = q(x,y)x - q(x)y$, *defined by a nondegenerate quadratic form* q *on an* n*-dimensional vector space* X *over* F. By [13, Theorem 6], the inner ideals of $\mathcal{Q}_n$ are $Q_n := X$ (with subquotient $\mathcal{Q}_n$) and the totally isotropic subspaces of X. Hence, if $n = 2m$ or $n = 2m+1$, $\mathcal{Q}_n$ contains a maximal point space of dimension m. Moreover, by Witt's Theorem, two inner ideals of $\mathcal{Q}_n$ are conjugate if and only if they have the same dimension.

(V) *The Albert pair* $\mathcal{A} := (H_3(\mathcal{C}), H_3(\mathcal{C}))$, *defined by the exceptional Jordan algebra* $H_3(\mathcal{C})$ *over* F. By [13, Main Theorem], $\mathcal{A}$ contains two maximal (proper) inner

ideals up to conjugation: the 6-dimensional point space $P_6 = F[11] + \epsilon C[12] + F\epsilon[13]$, where ϵ is a primitive idempotent of the Cayley algebra C (see also [13, p. 457]) and the Peirce-2-space determined by the Jordan algebra idempotent $e := [11] + [22]$, i.e, $Q_eH_3(C)$. Since $Q_eH_3(C)$ is a 10-dimensional simple Jordan algebra of capacity 2 over F, it is the Jordan algebra defined by a nondegenerate quadratic form on a 10-dimensional vector space over F, so $\mathrm{Sub}_{\mathcal{A}} Q_eH_3(C) \cong \mathcal{Q}_{10}$, and so we can put $Q_{10} = Q_eH_3(C)$ according to our notation criterion 1.5. Moreover, $\mathcal{A}$ contains two 5-dimensional point spaces which are not conjugate: $P_5 = F[11] + \epsilon C[12] \subset P_6 \cap Q_{10}$ and $P_5' = F[11] + C[12]\epsilon \subset Q_{10}$, which is also a maximal point space.

(VI) *The Bi-Cayley pair* $\mathcal{B} := (M_{1\times 2}(C), M_{2\times 1}(C))$, $Q_a b = (ab)a$, *where* C *is the Cayley algebra over* F. The inner ideals of $\mathcal{B}$ are, up to conjugation, $M_{1\times 2}(C)$, $C[11]$, and the linear spans of the $+$-parts of the families of collinear idempotents, following [14, 3.2] and the notations therein. In fact, the subquotient of $C[11]$ is isomorphic to $\mathcal{Q}_8$ and the inner ideals determined by the families of collinear idempotents are the point spaces of $\mathcal{B}$ [14, 3.3(1)]. By [11, 12.10], $\mathcal{B}$ is isomorphic to the Peirce-1-space of the Albert pair $\mathcal{A}$ with respect to the idempotent $e_1 = [11]$, hence the families of collinear idempotents of $\mathcal{B}$ are those of $\mathcal{A}$ contained in the Peirce-1-space with respect to e_1, so we can apply the results obtained for the Albert pair to get the point spaces of the Bi-Cayley pair. Thus $\mathcal{B}$ contains a maximal point space of dimension 5 (the one obtained by eliminating the [11]-part of the inner ideal P_6 of the Albert pair), and two point spaces of dimension 4 which are not conjugate (those obtained by eliminating the [11]-part of the inner ideals P_5 and P_5' of the Albert pair.)

3. The inner ideal structure of simple Lie algebras over algebraically closed fields of characteristic zero.

In this section we determine the inner ideal structure of the simple Lie algebras over algebraically closed fields of characteristic zero, both from the Lie and Jordan point of view.

3.1. $\mathbb{Z}$-*gradings.*

A $\mathbb{Z}$-*grading* of a Lie algebra L is a decomposition in vector subspaces

$$L = \bigoplus_{i=-n}^{n} L_i, \quad L_{-n} + L_n \neq 0,$$

such that $[L_i, L_j] \subset L_{i+j}$ for all i, j, with the understanding that $L_{i+j} = 0$ if $|i + j| > n$. This is called a $(2n + 1)$-grading, and it is said that L is $(2n + 1)$-*graded.*

A standard example of a Lie algebra with a 3-grading is that given by the TKK-algebra of a Jordan pair: For any Jordan pair V, there exists a Lie algebra with a 3-grading $\mathrm{TKK}(V) = L_{-1} \oplus L_0 \oplus L_1$, the *Tits-Kantor-Koecher algebra of* V, uniquely determined by the following conditions, cf. [15, 1.5(6)]:

(TKK1) The associated Jordan pair (L_1, L_{-1}) is isomorphic to V.

(TKK2) $[L_1, L_{-1}] = L_0$.

(TKK3) $[x_0, L_1 \oplus L_{-1}] = 0$ implies $x_0 = 0$, for any $x_0 \in L_0$.

In general, by a *TKK-algebra* we mean a Lie algebra of the form $\mathrm{TKK}(V)$ for some Jordan pair V.

Recall some basic facts about gradings. If we have a $\mathbb{Z}$-grading $L = L_{-n} \oplus \cdots \oplus L_n$, the map $D\colon L \to L$ such that $D(x) = kx$ for any $x \in L_k$, $k = -n, \ldots, n$, is a derivation of L. As any derivation is inner, $D = \mathrm{ad}_h$ for some semisimple h which belongs to a Cartan subalgebra H of L. Let $L = H \oplus (\bigoplus_{\alpha \in \Phi} L_\alpha)$ be the root decomposition of L relative to H. Note that $\alpha(h) \in \mathbb{Z}$ for any root α, because ad_h gives rise to the $\mathbb{Z}$-grading above. Take a basis $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ of the associated root system, and for each $i = 1, \ldots, l$, define $h_i \in H$ by $\alpha_j(h_i) = \delta_{ij}$. As $\{h_1, \ldots, h_l\}$ is a basis of H and $h = \sum \alpha_i(h) h_i$, there exist nonnegative integers $(p_1, \ldots, p_l)$ such that $h = \sum p_i h_i$. The root space L_α, for $\alpha = \sum m_i \alpha_i$, is contained in the homogeneous component $L_{\sum m_i p_i}$ of the $\mathbb{Z}$-grading of L, and the Cartan subalgebra H is contained in L_0. In particular, the $\mathbb{Z}$-gradings of L are in correspondence with the labels $(p_1, \ldots, p_l)$ of nonnegative integers. Moreover, two $\mathbb{Z}$-gradings can be taken into one another by an outer automorphism if and only if the corresponding sets of labels can be taken into one another by an automorphism of the Dynkin diagram, [16].

3.2. *Inner ideals and $\mathbb{Z}$-gradings.*

The $\mathbb{Z}$-gradings are closely related to abelian inner ideals: For any $\mathbb{Z}$-grading $L = \oplus_{i=-n}^{n} L_i$, L_n and L_{-n} (also called the *extremes* of the grading) are abelian inner ideals of L. Conversely, every abelian inner ideal whose subquotient is covered by a finite grid produces a grading of the Lie algebra by the weight lattice of the root system associated to the covering grid [7, 6.1]. As L is a finite dimensional simple Lie algebra, every proper inner ideal B of L is abelian and its associated subquotient is covered by a finite grid, so it gives rise to a grading $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ for which $B = L_n$. In the appendix, we prove that every abelian inner ideal is the extreme of a $\mathbb{Z}$-grading by using Lie techniques, that is, a proof independent of that of [7, 6.1].

Suppose we have a $\mathbb{Z}$-grading of L,

$$L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$$

determined by $(p_1, \ldots, p_l) \in \mathbb{N}^l$, as in 3.1. The extremes of this grading are easy to determine: If we denote by $\tilde{\alpha} = \sum_{i=1}^{l} n_i \alpha_i$ the maximal root relative to Δ, the root space associated to the maximal root $L_{\tilde{\alpha}}$ is contained in the extreme L_n, and $n = \sum n_i p_i$. Now note that for any root $\alpha = \sum m_i \alpha_i \in \Phi$, the root space L_α is contained in L_n if and only if $\sum m_i p_i = \sum n_i p_i$; that is, if and only if $m_j = n_j$ for all j such that $p_j \neq 0$. Therefore, denoting by $I = \{j \in \{1, \ldots, l\} \mid p_j \neq 0\}$, we have that $L_n = B_I$ for

$$B_I := \bigoplus_{\alpha \in \Phi} \{L_\alpha \mid \alpha = \sum_{1 \leq i \leq l} m_i \alpha_i \text{ with } m_j = n_j \text{ for all } j \in I\}.$$

To summarize, for H and Δ as above,

Theorem 3.1. *Let B be a nonzero abelian inner ideal of a simple Lie algebra L over an algebraically closed field of characteristic zero. Then there is a subset $I \subset \{1,\dots,l\}$ and an inner automorphism φ such that $\varphi(B) = B_I$.*

A straightforward observation is that, for $I \subset J$, the abelian inner ideal $B_J \subset B_I$. In particular, the maximal abelian inner ideals are conjugate to $B_{\{i\}}$ for some $i \in \{1,\dots,l\}$, although not conversely. Another interesting fact is that every chain of abelian inner ideals of L has length not greater than l. Moreover there is always a chain of abelian inner ideals of L with length just l. This is clear by recalling that for each $\alpha \in \Phi^+ \setminus \Delta$ there is $i \in \{1,\dots,l\}$ such that $\alpha - \alpha_i \in \Phi$.

3.3. For each simple finite-dimensional Lie algebra L over an algebraically closed field of characteristic zero, we classify its abelian (equivalently, proper) inner ideals: up to conjugation (Lie classification) and up to isomorphism (Jordan classification).

- *The Lie classification.* We will apply the method described in 3.2 to find the abelian inner ideals of L. After choosing a Cartan subalgebra and a basis of the related root system, each inner ideal will be conjugate to B_I for some nonempty subset $I \subset \{1,2,\cdots,l\}$. Further conjugations will be obtained by means of diagram automorphisms and some special cases will be dealt separately using techniques related to eigenvalues and traces of ad-semisimple elements.

- *The Jordan classification.* To determine the subquotient of an abelian inner ideal B of L, we compare the lattice of the inner ideals of L contained in B (provided by the Lie classification) with the lattices of the inner ideals of the classical Jordan pairs (cf. Section 2). While in most cases this information is enough to determine $\mathrm{Sub}_L B$, in others, as those of the Lie algebras of type E_7 and E_8, additional information about the inner ideal structure of L, as the eventual existence of outer automorphisms, is required.

3.4. *The inner ideal structure of A_n, $n \geq 1$.*
The Lie classification of the inner ideals of A_n. Choose the set of positive roots of A_n described in [4, Planche I (II)], that is,

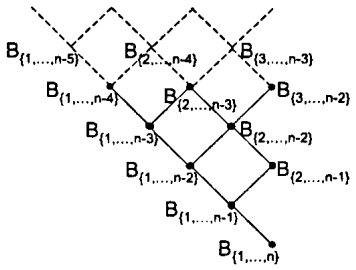
$$\Phi^+ = \{\alpha_r + \alpha_{r+1} + \cdots + \alpha_s \mid 1 \leq r \leq s \leq n\},$$

whose maximal root is $\tilde{\alpha} = \alpha_1 + \cdots + \alpha_n$. Following the process described in 3.2, take the nonzero abelian inner ideals B_I for $I \subset \{1,\dots,n\}$. Note that for $k = \min I$ and $j = (\max I) - k$, $B_I = B_{\{k,\dots,k+j\}}$, which coincides with the sum of the root spaces for the following roots

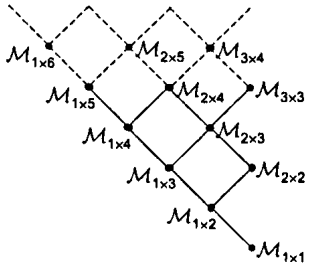
$$\{\alpha_r + \alpha_{r+1} + \cdots + \alpha_k + \cdots + \alpha_{k+j} + \cdots + \alpha_s \mid 1 \leq r \leq k \leq k+j \leq s \leq n\},$$

which is a set of cardinal $k(n+1-k-j)$.

Recall also that $B_{\{k,\dots,k+j\}}$ is conjugate to $B_{\{n+1-k-j,\dots,n+1-k\}}$, because there is a diagram automorphism interchanging the nodes s and $n+1-s$ in the Dynkin diagram of A_n. Hence any abelian inner ideal of A_n is conjugate to one of the B_I's in the following diagram:



The Jordan classification of the inner ideals of A_n. As above, every nonzero abelian inner ideal of A_n has the form $B_{\{k,\dots,k+j\}}$ for $1 \le k \le [(n+1)/2]$ and $j \le n-2k+1$, whose dimension is $k(n+1-k-j)$. The subquotient associated to each abelian inner ideal $B_{\{k,\dots,k+j\}}$ of A_n is a classical Jordan pair, hence, by comparing the dimensions of these abelian inner ideals with the dimensions of the inner ideals of the classical simple Jordan pairs, we conclude that the subquotient of $B_{\{k,\dots,k+j\}}$ is a Jordan pair of type $\mathcal{M}_{k\times(n+1-k-j)}$. In particular, the inner ideals in the set $\{B_{\{k,\dots,k+j\}} \mid 1 \le k \le [(n+1)/2],\ j \le n-2k+1\}$ are not isomorphic and therefore not conjugate, so we can assure the previous diagram covers the abelian inner ideals of A_n up to conjugation. We get the following diagram of subquotients of A_n:



This information about inner ideals and subquotients, together with their corresponding TKK-algebras, is collected in the next table, $1 \le k \le [(n+1)/2]$:

Abelian inner ideals	dimension	subquotients	TKK algebras
$B_{\{k\}}$	$k(n-k+1)$	$\mathcal{M}_{k\times(n-k+1)}$	A_n
$B_{\{k,k+1\}}$	$k(n-k)$	$\mathcal{M}_{k\times(n-k)}$	A_{n-1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$B_{\{k,\dots,k+j\}}$	$k(n+1-k-j)$	$\mathcal{M}_{k\times(n+1-k-j)}$	A_{n-j}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$B_{\{k,\dots,n-k+1\}}$	k^2	$\mathcal{M}_{k\times k}$	A_{2k-1}

3.5. *The inner ideal structure of* B_n, $n \ge 2$.

The Lie classification of the inner ideals of B_n. Consider the set of positive roots of B_n given in [4, Planche II (II)], that is,

$$\Phi^+ = \{\alpha_r + \alpha_{r+1} + \cdots + \alpha_s \mid 1 \le r \le s \le n\} \cup$$
$$\cup \{\alpha_r + \alpha_{r+1} + \cdots + \alpha_{t-1} + 2\alpha_t + \cdots + 2\alpha_n \mid 1 \le r < t \le n\},$$

whose maximal root is $\tilde{\alpha} = \alpha_1 + 2\alpha_2 \cdots + 2\alpha_n$. Following the process described in 3.2, take the nonzero abelian inner ideals B_I for $I \subset \{1,\dots,n\}$.

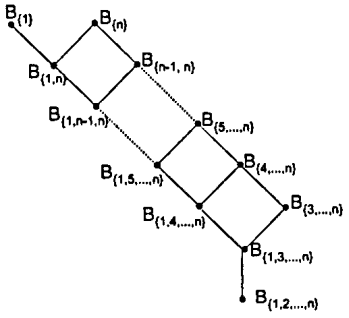
- If $I = \{1\}$, the roots related to B_I are those $\alpha = \sum m_i\alpha_i$ with $m_1 = 1$, that is $B_{\{1\}} \cong \{\alpha_1 + \cdots + \alpha_s \mid 1 \leq s \leq n\} \cup \{\alpha_1 + \cdots + \alpha_{t-1} + 2\alpha_t + \cdots + 2\alpha_n \mid 1 < t \leq n\}$ (identifying B_I with the related roots), of dimension $2n - 1$.

- If $I \supsetneq \{1\}$, take $s = \min(I \setminus \{1\})$, and then $B_I = B_{\{1,s,\ldots,n\}} \cong$
$$\{\alpha_1 + \cdots + \alpha_{t-1} + 2\alpha_t + \cdots + 2\alpha_n \mid 1 < t \leq s \leq n\},$$
that is, the roots verifying $m_1 = 1$ and $m_s = 2$. It has dimension $s - 1$.

- If $1 \notin I$, take $r = \min I$. Then $B_I = B_{\{r,\ldots,n\}} \cong$
$$\{\alpha_k + \cdots + \alpha_{t-1} + 2\alpha_t + \cdots + 2\alpha_n \mid 1 \leq k < t \leq r\},$$
that is, the roots verifying $m_r = 2$. It has dimension $\binom{r}{2}$. Note that $B_{\{2,\ldots,n\}} = B_{\{1,\ldots,n\}}$, so we can consider $r \geq 3$.

It is also worth noting that for $n \geq 4$ the inner ideals $B_{\{3,\ldots,n\}}$ and $B_{\{1,4,\ldots,n\}}$ are 3-dimensional point spaces, which are not conjugate under any automorphism of B_n. In fact, for $n > 4$, $B_{\{3,\ldots,n\}}$ is a maximal point space, while $B_{\{1,4,\ldots,n\}}$ is contained in the 4-dimensional point space $B_{\{1,5,\ldots,n\}}$ [3, Corollary 5.15].

However, for $n = 4$, both $B_{\{3,4\}}$ and $B_{\{1,4\}}$ are maximal point spaces, although yet they are not conjugate. In fact, while $B_{\{1,4\}}$ is the extreme of a 7-grading of $L = B_4$ (the one given by the label (1,0,0,1)), $B_{\{3,4\}}$ cannot be extreme of any 7-grading of L. Suppose on the contrary that $B_{\{3,4\}} = L_3$ for a 7-grading of L, and let $s \in L$ be an ad-semisimple element such that $L_n = \{x \in L : [s,x] = nx\}$ for $n = \pm 3, \pm 2, \pm 1, 0$. If $L = H \oplus (\bigoplus_{\alpha\in\Phi} L_\alpha)$ is the root decomposition of L relative to H (our fixed Cartan subalgebra), of course $s \notin H$, since the grading would correspond to a label $(0,0,p_3,p_4)$ but $2(p_3 + p_4) \neq 3$. To eliminate the possibility $s = h + \sum_{\alpha\in\Phi} w_\alpha$, $h \in H$ and $w_\alpha \in L_\alpha$ not all of them zero, consider U the sum of the root spaces related to the roots $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$, $\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4$, $\alpha_2 + 2\alpha_3 + 2\alpha_4$, $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4$, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ and $\alpha_1 + \alpha_2 + \alpha_3$. By using that w_α vanishes if $\alpha + \gamma \in \Phi$ for some $\gamma \in B_{\{3,4\}}$, and that $2(\alpha_3 + \alpha_4)(h) = 3$, it is routine to show that U is invariant under ad_s, and that the trace of the restriction of ad_s to U is non-integer, which is a contradiction. Therefore, any abelian inner ideal of B_n is conjugate to one (and only one) of the B_I's in the following diagram:

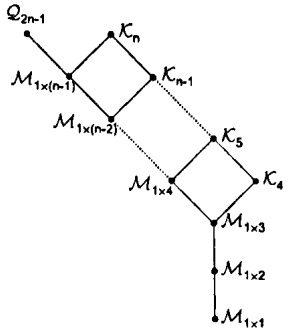


The Jordan classification of the inner ideals of B_n. The inner ideal structure of the classical Jordan pairs given in Section 2, together with the dimensions of the inner ideals of the Lie classification above, allow us to determine the subquotients of the abelian inner ideals of B_n. We get

- The subquotient of the abelian inner ideal $B_{\{1\}}$ is isomorphic to $\mathcal{Q}_{2n-1}$.

- The subquotient of an abelian inner ideal of the form $B_{\{r,\dots,n\}}$ is isomorphic to $\mathcal{K}_r$.
- The subquotient of an abelian inner ideal of the form $B_{\{1,s,\dots,n\}}$ is isomorphic to $\mathcal{M}_{1\times(s-1)}$.
- Both the inner ideals $B_{\{1,4,\dots,n\}}$ and $B_{\{3,\dots,n\}}$ are 3-dimensional point spaces, and therefore give rise to the same subquotient.

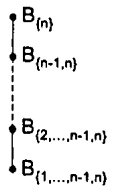
Therefore, the diagram of subquotients of B_n is:



This information about inner ideals and subquotients, together with their corresponding TKK-algebras, is collected in the next table, for $3 \le r \le n$, $2 \le s \le n$:

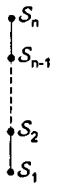
Abelian inner ideals	dimension	subquotients	TKK algebras
$B_{\{1\}}$	$2n-1$	$\mathcal{Q}_{2n-1}$	B_n
$B_{\{r,\dots,n\}}$	$\binom{r}{2}$	$\mathcal{K}_r$	D_r
$B_{\{1,s,\dots,n\}}$	$s-1$	$\mathcal{M}_{1\times s-1}$	A_{s-1}

3.6. *The inner ideal structure of C_n, $n \ge 3$.*
The Lie classification of the inner ideals of C_n. Consider the set of positive roots for C_n of [4, Planche III (II)] given by $\Phi^+ = \{\alpha_j+\cdots+\alpha_{i-1}+2\alpha_i+\cdots+2\alpha_{n-1}+\alpha_n \mid 1 \le j \le i \le n\} \cup \{\alpha_j+\cdots+\alpha_i \mid 1 \le j \le i \le n-1\}$, whose maximal root is $\tilde{\alpha} = 2\alpha_1+\cdots+2\alpha_{n-1}+\alpha_n$. For $I \subset \{1,\dots,n\}$, take $r = \min I$ and observe that $B_I = B_{\{r,\dots,n\}} = \{\alpha_j+\cdots+\alpha_{i-1}+2\alpha_i+\cdots+2\alpha_{n-1}+\alpha_n \mid 1 \le j \le i \le r\}$, with dimension $\binom{r+1}{2}$. Therefore any nonzero abelian inner ideal of C_n is conjugated to one of the B_I's in the following diagram:



Moreover, this is the diagram up to conjugation, since all the dimensions are different.

The Jordan classification of the inner ideals of C_n. Since the subquotient of each $B_{\{r,\dots,n\}}$ is a classical Jordan pair V whose inner ideals coincide with the abelian inner ideals of C_n contained in $B_{\{r,\dots,n\}}$, V is isomorphic to a Jordan pair $\mathcal{S}_r$ of symmetric $r \times r$ matrices over F. Thus the subquotients of C_n are:



This information about inner ideals and subquotients, together with their corresponding TKK-algebras, is collected in the next table, for $1 \leq r \leq n$,

Abelian inner ideals	dimension	subquotients	TKK algebras
$B_{\{r,\dots,n\}}$	$\binom{r+1}{2}$	$\mathcal{S}_r$	C_r

3.7. *The inner ideal structure of D_n, $n \geq 4$.*
The Lie classification of the inner ideals of D_n. Consider the set of n^2-n positive roots of D_n given in [4, VI.§4.8, Planche IV (II)], that is, $\Phi^+ =$

$$\begin{aligned}
&\{\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \mid i < j < n-1\} \\
\cup\ &\{\alpha_i + \cdots + \alpha_j \mid i \leq j \leq n-1\} \\
\cup\ &\{\alpha_i + \cdots + \alpha_n \mid i \leq n-2\} \\
\cup\ &\{\alpha_n\} \\
\cup\ &\{\alpha_i + \cdots + \alpha_{n-2} + \alpha_n \mid i \leq n-2\},
\end{aligned}$$

whose maximal root is $\tilde{\alpha} = \alpha_1 + 2\alpha_2 \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$. Following the process described in 3.2, take the nonzero abelian inner ideals B_I for $I \subset \{1, \dots, n\}$:

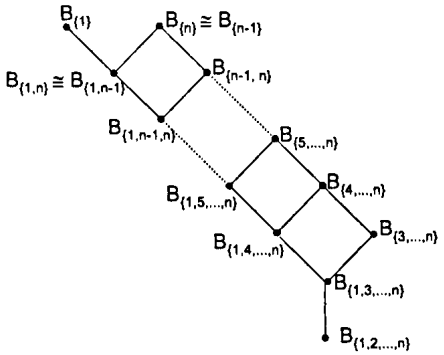
- If $I = \{1\}$, the related roots are $B_{\{1\}} \cong \{\alpha_1 + \cdots + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \mid 1 < j < n-1\} \cup \{\alpha_1 + \cdots + \alpha_j \mid 1 \leq j \leq n\} \cup \{\alpha_1 + \cdots + \alpha_{n-2} + \alpha_n\}$, of dimension $2n-2$.

- If $I \supsetneq \{1\}$, take $s = \min(I \setminus \{1\})$. If $s < n-1$, then $B_I = B_{\{1,s,\dots,n\}}$ and $B_{\{1,s,\dots,n\}} \cong \{\alpha_1 + \cdots + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \mid 1 < j \leq s\}$, which has dimension $s-1$. Moreover, $B_{\{1,n-1\}} \cong \{\alpha_1 + \cdots + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \mid 1 < j < n-1\} \cup \{\alpha_1 + \cdots + \alpha_{n-1}, \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n\}$, with dimension $n-1$; $B_{\{1,n\}} \cong \{\alpha_1 + \cdots + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \mid 1 < j < n-1\} \cup \{\alpha_1 + \cdots + \alpha_{n-2} + \alpha_n, \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n\}$, with dimension $n-1$; and $B_{\{1,n-1,n\}} \cong \{\alpha_1 + \cdots + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \mid 1 < j < n-1\} \cup \{\alpha_1 + \cdots + \alpha_{n-1} + \alpha_n\}$, with dimension $n-2$.

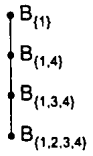
- If $1 \notin I$, take $r = \min I$. If $r < n-1$, then $B_I = B_{\{r,\dots,n\}}$, and $B_{\{r,\dots,n\}} \cong \{\alpha_i + \cdots + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \mid i < j \leq r\}$, which has dimension $\binom{r}{2}$. Moreover, $B_{\{n-1\}} = \{\alpha_i + \cdots + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \alpha_r + \cdots + \alpha_{n-1}, \alpha_s + \cdots + \alpha_{n-1} + \alpha_n \mid i < j < n-1, r \leq n-1, s \leq n-2\}$ has dimension $\binom{n}{2}$; $B_{\{n\}} = \{\alpha_i + \cdots + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \alpha_r + \cdots + \alpha_n, \alpha_s + \cdots + \alpha_{n-2} + \alpha_n, \alpha_n \mid i < j < n-1, r, s \leq n-2\}$ has dimension

$\binom{n}{2}$; and $B_{\{n-1,n\}} = \{\alpha_i + \cdots + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n,\ \alpha_r + \cdots + \alpha_n \mid i < j < n-1,\ r \leq n-2\}$ has dimension $\binom{n-1}{2}$.

Hence any abelian inner ideal of D_n is conjugate to one of the B_I's in the following list: $\{B_{\{1\}},\ B_{\{n-1\}},\ B_{\{n\}},\ B_{\{1,n\}},\ B_{\{1,n-1\}}, B_{\{1,n-1,n\}}, B_{\{n-1,n\}},\ B_{\{1,s,\ldots,n\}}, B_{\{r,\ldots,n\}} \mid 2 \leq r, s \leq n-2\}$. We can consider $r \geq 3$ because $B_{\{2,\ldots,n\}} = B_{\{1,\ldots,n\}} = \{\tilde{\alpha}\}$. Furthermore, note that we can fold the diagram by means of the order two outer automorphism which interchanges the nodes $n-1$ and n, so any abelian inner ideal is conjugate to one of the next diagram:



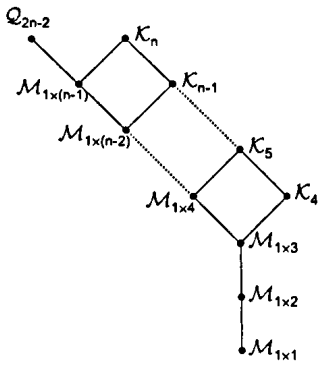
Besides, in the case $n = 4$ there is an order three automorphism mapping L_{α_1} to L_{α_3}, and L_{α_3} to L_{α_4}, so in this case not only $B_{\{3\}}$ and $B_{\{4\}}$ are conjugate, but also $B_{\{1\}}$, and the diagram becomes



The Jordan classification of the inner ideals of D_n. Again we can use the information on the inner ideal structure of the classical Jordan pairs provided in Section 2 to compute the subquotients of the abelian inner ideals of D_n. We obtain:

- The subquotient of the abelian inner ideal $B_{\{1\}}$ is isomorphic to $\mathcal{Q}_{2n-2}$.
- The subquotient of an abelian inner ideal of the form $B_{\{r,\ldots,n\}}$ is isomorphic to $\mathcal{K}_r$.
- The subquotient of an abelian inner ideal of the form $B_{\{1,s,\ldots,n\}}$ is isomorphic to $\mathcal{M}_{1\times(s-1)}$.

Hence, for $n > 4$, the inner ideals of the diagram above are not isomorphic, up to $B_{\{1,4,\ldots,n\}}$ and $B_{\{3,\ldots,n\}}$ which are isomorphic (both are point spaces of the same dimension), but not conjugate: $B_{\{3,\ldots,n\}}$ is a maximal point space, but $B_{\{1,4,\ldots,n\}}$ is contained in the 4-dimensional point space $B_{\{1,5,\ldots,n\}}$, [3, Corollary 5.15]. Therefore, the diagram of subquotients of D_n is



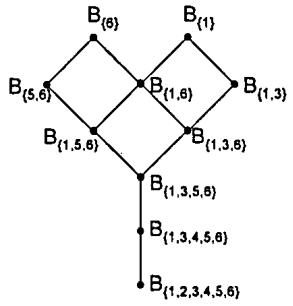
(For $n = 4$, the inner ideals $B_{\{1\}}$ and $B_{\{4\}}$ are conjugate in D_4, so they yield isomorphic subquotients. This fact also follows from the Jordan theory: the Jordan pairs $\mathcal{Q}_6$ and $\mathcal{K}_4$ are isomorphic [11, 17.11 (V)].)

All this information is collected in the next table, for $3 \leq r \leq n$, $2 \leq s \leq n$,

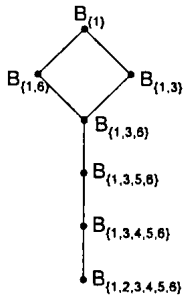
Abelian inner ideals	dimension	subquotients	TKK algebras
$B_{\{1\}}$	$2n-2$	Q_{2n-2}	D_n
$B_{\{r,\ldots,n\}}$	$\binom{r}{2}$	$\mathcal{K}_r$	D_r
$B_{\{1,s,\ldots,n\}}$	$s-1$	$\mathcal{M}_{1\times(s-1)}$	A_{s-1}

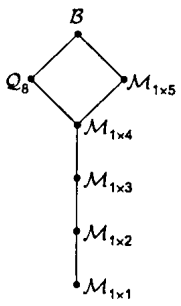
3.8. *The inner ideal structure of E_6.*

We choose the system of positive roots for E_6 given in [4, Planche V (II)], with maximal root $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6$. Following the process described in 3.2, every nonzero abelian inner ideal of E_6 is conjugate to one in the next diagram:



By folding the Dynkin diagram we get that $B_{\{6\}}$ is conjugate to $B_{\{1\}}$, as well as $B_{\{5,6\}}$ to $B_{\{1,3\}}$, and $B_{\{1,5,6\}}$ to $B_{\{1,3,6\}}$. The remaining cases correspond to not conjugate inner ideals because the dimensions are different. Therefore we conclude that the nonzero abelian inner ideals of E_6 up to conjugation and their corresponding subquotients are:





The Lie classification *The Jordan classification*

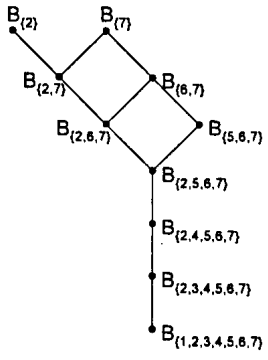
It is worth noting that the order two outer automorphism of E_6 connecting the inner ideals $B_{\{1\}}$ and $B_{\{6\}}$ yields two conjugate copies of the inner ideal structure of the Bi-Cayley pair $\mathcal{B}$ within E_6. This explains the apparently contradictory fact that while in $\mathcal{B}$ there are two 4-dimensional point spaces which are not conjugate, in E_6 there is a unique 4-dimensional inner ideal up to conjugation.

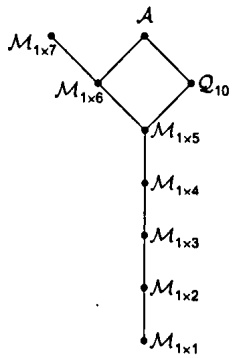
The information about the inner ideals of E_6 and of their subquotients, together with their corresponding TKK-algebras, is collected in the next table:

Abelian inner ideals	dimension	subquotients	TKK algebras
$B_{\{1\}}$	16	$\mathcal{B}$	E_6
$B_{\{1,6\}}$	8	$\mathcal{Q}_8$	D_5
$B_{\{1,3\}}$	5	$\mathcal{M}_{1\times5}$	A_5
$B_{\{1,3,6\}}$	4	$\mathcal{M}_{1\times4}$	A_4
$B_{\{1,3,5,6\}}$	3	$\mathcal{M}_{1\times3}$	A_3
$B_{\{1,3,4,5,6\}}$	2	$\mathcal{M}_{1\times2}$	A_2
$B_{\{1,2,3,4,5,6\}}$	1	$\mathcal{M}_{1\times1}$	A_1

3.9. *The inner ideal structure of E_7.*

We choose the system of positive roots for E_7 given in [4, Planche VI (II)]. Following the process described in 3.2 and 3.3, the nonzero abelian inner ideals of E_7 (up to conjugation), and their associated subquotients are:





The Lie classification *The Jordan classification*

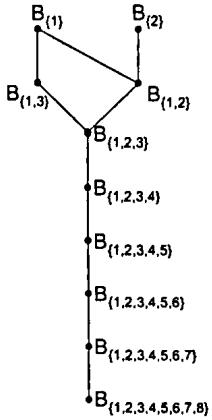
Notice that the 5-dimensional inner ideals $B_{\{2,6,7\}}$ and $B_{\{5,6,7\}}$ are not conjugate since $B_{\{5,6,7\}}$ is the extreme of a 7-grading of L, while $B_{\{2,6,7\}}$ cannot be expressed as L_3 for a 7-grading $L_{-3} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_3$. If fact, if this

was the case, this grading would be induced by an ad-semisimple element $s = h + \sum w_\alpha$, $h \in H$ and $w_\alpha \in L_\alpha$ for all $\alpha \in \Phi$ such that $(\alpha + B_{\{2,6,7\}}) \cap \Phi = \emptyset$, relative to the root space decomposition $L = H \oplus (\bigoplus_{\alpha\in\Phi} L_\alpha)$. Then the 5-dimensional invariant subspace of L generated by the root space corresponding to $\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$ would have an eigenvalue of the form $3 - \alpha_2(h)$, for $\alpha_2(h) \in \{1, \ldots, 6\}$, while the 7-dimensional invariant subspace of L generated by the root $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ would have two eigenvalues with value at most 2 whose sum would be $3 + 2\alpha_2(h)$, a contradiction. Nevertheless, both $B_{\{2,6,7\}}$ and $B_{\{5,6,7\}}$ yield the same subquotient, $\mathcal{M}_{1\times5}$, as point spaces of the same dimension 1.4.

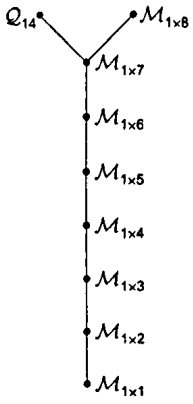
This information about the inner ideals of E_7 and of their subquotients, together with their corresponding TKK-algebras, is collected in the next table:

Abelian inner ideals	dimension	subquotients	TKK algebras
$B_{\{7\}}$	27	$\mathcal{A}$	E_7
$B_{\{2\}}$	7	$\mathcal{M}_{1\times7}$	A_7
$B_{\{6,7\}}$	10	$\mathcal{Q}_{10}$	D_6
$B_{\{2,7\}}$	6	$\mathcal{M}_{1\times6}$	A_6
$B_{\{2,6,7\}}$ $B_{\{5,6,7\}}$	5	$\mathcal{M}_{1\times5}$	A_5
$B_{\{2,5,6,7\}}$	4	$\mathcal{M}_{1\times4}$	A_4
$B_{\{2,4,5,6,7\}}$	3	$\mathcal{M}_{1\times3}$	A_3
$B_{\{2,3,4,5,6,7\}}$	2	$\mathcal{M}_{1\times2}$	A_2
$B_{\{1,2,3,4,5,6,7\}}$	1	$\mathcal{M}_{1\times1}$	A_1

3.10. *The inner ideal structure of E_8.*
We choose the system of positive roots for E_8 given in [4, Planche VII (II)]. Following the process described in3.2 and 3.3, the nonzero abelian inner ideals of E_8 (up to conjugation), together with their associated subquotients are:



The Lie classification



The Jordan classification

The inner ideals $B_{\{1,2\}}$ and $B_{\{1,3\}}$ are not conjugate. In fact, $B_{\{1,2\}}$ can be expressed as the extreme of a 11-grading, but if this were the case for $B_{\{1,3\}}$, this last grading would be induced by an ad-semisimple element $s = h + \sum w_\alpha$, $h \in H$ and $w_\alpha \in L_\alpha$ for all $\alpha \in \Phi$, relative to the root space decomposition

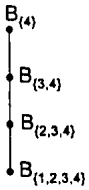
$L = H \oplus (\bigoplus_{\alpha \in \Phi} L_\alpha)$. Consider the invariant 8-dimensional subspace U of L generated by the root space related to $2\alpha_1 + 3\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$. The trace of $\operatorname{ad} s$ restricted to U would be $30 + 15/2$ and this is not possible since $\operatorname{ad} s$ is semisimple with eigenvalues $0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$. However, both $B_{\{1,2\}}$ and $B_{\{1,3\}}$ yield the same subquotient, $\mathcal{M}_{1\times 7}$.

This information about inner ideals and subquotients, together with their corresponding TKK-algebras, is collected in the next table:

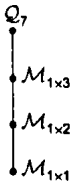
Abelian inner ideals	dimension	subquotients	TKK algebras
$B_{\{1\}}$	14	$\mathcal{Q}_{14}$	D_8
$B_{\{2\}}$	8	$\mathcal{M}_{1\times 8}$	A_8
$B_{\{1,2\}}$ $B_{\{1,3\}}$	7	$\mathcal{M}_{1\times 7}$	A_7
$B_{\{1,2,3\}}$	6	$\mathcal{M}_{1\times 6}$	A_6
$B_{\{1,2,3,4\}}$	5	$\mathcal{M}_{1\times 5}$	A_5
$B_{\{1,2,3,4,5\}}$	4	$\mathcal{M}_{1\times 4}$	A_4
$B_{\{1,2,3,4,5,6\}}$	3	$\mathcal{M}_{1\times 3}$	A_3
$B_{\{1,2,3,4,5,6,7\}}$	2	$\mathcal{M}_{1\times 2}$	A_2
$B_{\{1,2,3,4,5,6,7,8\}}$	1	$\mathcal{M}_{1\times 1}$	A_1

3.11. *The inner ideal structure of F_4.*

We choose the system of positive roots for F_4 described in [4, Planche VIII (II)]. Following the process described in 3.2, the nonzero abelian inner ideals of F_4, up to conjugation, jointly with their associated subquotients are:



The Lie classification



The Jordan classification

This information about inner ideals and subquotients, together with their corresponding TKK-algebras, is collected in the next table:

Abelian inner ideals	dimension	subquotients	TKK algebras
$B_{\{4\}}$	7	$\mathcal{Q}_7$	B_4
$B_{\{3,4\}}$	3	$\mathcal{M}_{1\times 3}$	A_3
$B_{\{2,3,4\}}$	2	$\mathcal{M}_{1\times 2}$	A_2
$B_{\{1,2,3,4\}}$	1	$\mathcal{M}_{1\times 2}$	A_1

3.12. *The inner ideal structure of G_2.*

We choose the system of positive roots for G_2 described in [4, Planche IX (II)]. Following the process described in 3.2 and 3.3, the nonzero abelian inner ideals of G_2 (up to conjugation), together with their associated subquotients are:

The Lie classification

The Jordan classification

This information about inner ideals and subquotients, together with their corresponding TKK-algebras, is collected in the next table:

Abelian inner ideals	dimension	subquotients	TKK algebras
$B_{\{1\}}$	2	$\mathcal{M}_{1\times 2}(F)$	A_2
$B_{\{1,2\}}$	1	$\mathcal{M}_{1\times 1}(F)$	A_1

4. Appendix: Structure of the inner ideals

In this section we prove that every abelian inner ideal can be obtained by the process described in 3.2. Our proof uses well-known notions of the classical theory of Lie algebras (see for instance [9, §9,§10 and §14]) and a graded version of the Jacobson-Morozov theorem [7, Proposition 5.2], which, in Seligman words "is really a summary of certain results of Jacobson". It provides an alternative proof to the fact that every abelian inner ideal B of a finite dimensional Lie algebra L coincides with L_n, for some $\mathbb{Z}$-grading $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$, c.f. [7].

4.1. Let L be a finite dimensional simple Lie algebra over an algebraically closed field F of characteristic zero. Let H be a Cartan subalgebra of L and let $L = H \oplus (\bigoplus_{\alpha\in\Phi} L_\alpha)$ be the root decomposition of L relative to H. Take a basis $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ of the root system Φ, and let $\tilde{\alpha} = \sum_{i=1}^{l} n_i\alpha_i$ be the maximal root relative to Δ. For each nonempty set $I \subset \{1, \ldots, l\}$, denote

$$P_{H,\Delta,I} = P_I = \{\alpha = \textstyle\sum_{i=1}^{l} k_i\alpha_i \in \Phi \mid k_i = n_i \ \forall i \in I\} \subset \Phi,$$
$$B_{H,\Delta,I} = B_I = \textstyle\bigoplus_{\alpha\in P_I} L_\alpha \leq L.$$

To show that B_I is an abelian inner ideal of L is an easy exercise left to the reader. Our goal is to prove that for any abelian inner ideal B there exist a Cartan subalgebra H, a basis Δ of the associated root system, and a set I in the above conditions such that B is equal to $B_{H,\Delta,I}$. We begin by finding the Cartan subalgebra.

Lemma 4.1. *If B is an abelian inner ideal, there exists a Cartan subalgebra H of L such that $B = \bigoplus_{\alpha\in\Phi} L_\alpha \cap B$, where $L = H \oplus (\bigoplus_{\alpha\in\Phi} L_\alpha)$ is the decomposition in root spaces relative to H.*

Proof. Assume $0 \neq B$ and take $0 \neq e_1 \in B$. Since e_1 is ad-nilpotent of index 3, by Jacobson-Morozov Theorem, see [17, V.8.3], there exists a standard triple $\{e_1, h_1, f_1\}$ of L ($[h_1, e_1] = 2e_1$, $[h_1, f_1] = -2f_1$ and $[e_1, f_1] = h_1$, with f_1 also ad-nilpotent of index 3) such that $\operatorname{ad} h_1$ diagonalizes L as

$$L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2,$$

that is, $[h_1, x_i] = ix_i$ for any $x_i \in L_i$, $i = 0, \pm 1, \pm 2$. Note that $e_1 \in [e_1, [e_1, L]]$ and that B is a homogeneous subspace, $B = \bigoplus_i (B \cap L_i)$, since $[h_1, B] = [[e_1, f_1], B] \subset [[B, L], B] \subset B$. Moreover, by [5, 1.18(iii)] or [2, Lemma 2.1(3)], $L_2 = [e_1, [e_1, L]] \subset B$.

Note that if $\{e_i, h_i, f_i\}_{i=1}^s$ is a family of standard triples such that $e_i \in B$ and $[h_i, h_j] = 0$ for all $i, j = 1, \ldots, s$, there exists a grading of L, which can be considered a grading by a subgroup Γ of $\mathbb{Z}^s$, $L = \oplus_{\alpha \in \Gamma} L_\alpha$ such that $B = \oplus_{\alpha \in \Gamma} (L_\alpha \cap B)$.

Let us consider $\{e_i, h_i, f_i\}_{i=1}^k$ a family of standard triples such that in the associated grading, $L = \oplus_{\alpha \in \Gamma'} L_\alpha$, the number of homogeneous components $L_\alpha \subset B$ is maximal and let us prove that if $b \in B$ is homogeneous with $b \in L_\gamma$, then $L_\gamma \subset B$: if there exists an homogeneous element $0 \neq e_{k+1} \in B \cap L_\beta$ for some homogeneous component L_β not contained in B, as above $e_{k+1} \in [e_{k+1}, [e_{k+1}, L]]$. An easy adaptation of [17, V.8.3] (details can be consulted in [7, Proposition 5.2]) shows that there is a standard triple $\{e_{k+1}, h_{k+1}, f_{k+1}\}$ of L such that $f_{k+1} \in L_{-\beta}$. Indeed, taking the grading into account, there exists a homogeneous element $f \in L_{-\beta}$ such that $[e_{k+1}, [e_{k+1}, f]] = -2e_{k+1}$. The endomorphism $E = \operatorname{ad} e_{k+1} \in \operatorname{End} L$ is homogeneous, so $\operatorname{Ker} E$ is a homogeneous submodule, invariant for $H = \operatorname{ad} h$ if $h = [e_{k+1}, f]$, and such that $H(H-1)(H-2)|_{\operatorname{Ker} E} = 0$. Hence $H|_{\operatorname{Ker} E \cap L_{-\beta}}$ is diagonalizable with eigenvalues 0, 1 and 2, and $(H+2)|_{\operatorname{Ker} E \cap L_{-\beta}}$ is invertible. As $[e_{k+1}, [h, f]] = [[e_{k+1}, h], f] + [h, h] = -2[e_{k+1}, f]$, then $(H+2)f \in \operatorname{Ker} E \cap L_{-\beta}$ and there is $v \in \operatorname{Ker} E \cap L_{-\beta}$ such that $(H+2)f = (H+2)v$. Now take $f_{k+1} = f - v$ and $h_{k+1} = [e_{k+1}, f_{k+1}]$.

Therefore $h_{k+1} \in [L_\beta, L_{-\beta}] \subset L_0$, so that $[h_{k+1}, h_i] = 0$ for $i = 1, \ldots, k$, and the homogeneous component L_γ of the simultaneous diagonalization of L relative to the family $\{h_i\}_{i=1}^{k+1}$ containing e_{k+1} is contained in B, because $\operatorname{ad} h_{k+1}$ acts in L_γ with eigenvalue 2 (and again by [2, Lemma 2.1(3)], $\{x \in L \mid [h_{k+1}, x] = 2x\} \subset B$), a contradiction with the choice of the family $\{e_i, h_i, f_i\}_{i=1}^k$. In particular, B is the sum of the homogeneous components L_α such that $B \cap L_\alpha \neq 0$. Note that $h_i \in L_0$ for $i = 1, \ldots, n$.

Under these conditions, take any Cartan subalgebra H containing $\sum_{i=1}^n F h_i$. Since $B \cap L_\alpha$ is either L_α or 0, B is still homogeneous for the simultaneous diagonalization relative to H. Besides $B \cap H = 0$, since, in other case, as $H \subset L_0$ (H is abelian), then $L_0 \subset B$ and $0 \neq -2e_1 = [h_1, e_1] \in [L_0, B] \subset [B, B] = 0$. ∎

4.2. For H and B as in the above lemma, we denote by $P := \{\alpha \in \Phi \mid L_\alpha \subset B\}$ the set of roots related to B. That B is an abelian inner ideal of L is equivalent to the following conditions for P:

(i) $(P + P) \cap (\Phi \cup \{0\}) = \emptyset$

(ii) $(P + ((P + \Phi) \cap \Phi)) \cap \Phi \subset P$

The length of the longest chain of nonempty subsets of Φ contained in P and verifying the conditions (i) and (ii) will be called the *rank* of P, and denoted by $\operatorname{rank} P$. The following theorem will show that the rank of P coincides with the length of the longest chain of nonzero abelian inner ideals contained in B and that

the length of the longest chain of nonzero abelian inner ideals contained in L is l, the rank of L as a Lie algebra.

If $B \neq 0$ (equivalently, $P \neq \emptyset$), P always contains long roots according to the following lemma:

Lemma 4.2. *If $\beta \in P$ is a short root, then P contains any long root $\alpha \in \Phi$ such that $(\alpha, \beta) > 0$. Moreover, $\|\alpha\|^2/\|\beta\|^2 = 2$ for any long root $\alpha \in \Phi$.*

Proof. If α is a long root with $(\alpha, \beta) > 0$, then $\langle \alpha, \beta \rangle = 2, 3$ by [9, §9] and $\{\alpha, \alpha - \beta, \alpha - 2\beta\} \subset \Phi$, so $\alpha = \beta + (\beta + (\alpha - 2\beta)) \in P$, by 4.2(ii). If $\|\alpha\|^2/\|\beta\|^2 = 3$, then $\alpha - 3\beta \in \Phi$, so $\alpha - \beta = \beta + (\beta + (\alpha - 3\beta)) \in P$ by 4.2(ii) and hence $\alpha = \beta + (\alpha - \beta) \in (P + P) \cap \Phi = \emptyset$ by 4.2(i), which is not possible. ∎

Proposition 4.3. *Suppose that there exist a basis $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ of the root system Φ relative to H and a permutation $\{i_1, \ldots, i_l\}$ of $\{1, \ldots, l\}$ such that for $1 \leq k \leq l$,*

$$B_{\{i_1,\ldots,i_l\}} \subsetneq B_{\{i_1,\ldots,i_{l-1}\}} \subsetneq \cdots \subsetneq B_{\{i_1,\ldots,i_k\}} \subsetneq B \subset B_{\{i_1,\ldots,i_{k-1}\}}.$$

Then

1. $L_{\pm\alpha_{i_k}} \oplus \cdots \oplus L_{\pm\alpha_{i_l}} \subset [B, L]$,
2. $B = B_{\{i_1,\ldots,i_{k-1}\}}$.

Proof. Denote by $\tilde{\alpha} = \sum_{i=1}^{l} n_i \alpha_i$ the maximal root relative to Δ. We proceed by induction on $n = l + 1 - k$. The case $n = 1$, i.e., $l = k$, is left to the reader. For the general case, applying the induction hypothesis to

$$B_{\{i_1,\ldots,i_l\}} \subsetneq B_{\{i_1,\ldots,i_{l-1}\}} \subsetneq \cdots \subsetneq B_{\{i_1,\ldots,i_{k+1}\}} \subsetneq B_{\{i_1,\ldots,i_k\}},$$

we get that $L_{\pm\alpha_{i_{k+1}}} \oplus \cdots \oplus L_{\pm\alpha_{i_l}} \subset [B_{\{i_1,\ldots,i_k\}}, L] \subset [B, L]$.

Firstly, let us see that also $L_{-\alpha_{i_k}} \subset [B, L]$, that is, that $-\alpha_{i_k} \in P + \Phi$: Take a maximal element β in $P \setminus P_{\{i_1,\ldots,i_k\}}$. Then $\beta = \sum k_i \alpha_i$ with $k_{i_j} = n_{i_j}$ if $j \leq k - 1$, $k_{i_k} \neq n_{i_k}$ ($k_i \in [-n_i, n_i]$ for all i). Since $\beta \neq \tilde{\alpha}$, we can take s such that $\beta + \alpha_s \in \Phi$ (necessarily $s \notin \{i_1, \ldots, i_{k-1}\}$). But if $s \in \{i_{k+1}, \ldots, i_l\}$ we would have $\alpha_s \in P + \Phi$, $\beta \in P$ and $\beta + \alpha_s \in \Phi$, consequently by 4.2(ii) $\beta + \alpha_s \in P$ and $\beta + \alpha_s \in P \setminus P_{\{i_1,\ldots,i_k\}}$ is $\succ \beta$, a contradiction with the choice of β. Therefore $\alpha_s = \alpha_{i_k}$ and we have proved $\beta + \alpha_{i_k} \in \Phi$. Then $-\alpha_{i_k} = \beta + (-\beta - \alpha_{i_k}) \in P + \Phi$.

Secondly, let us show that $B_{\{i_1,\ldots,i_{k-1}\}} \subset B$, or equivalently, that every $\gamma \in P_{\{i_1,\ldots,i_{k-1}\}}$ satisfies $\gamma \in P$: For such $\gamma \in P_{\{i_1,\ldots,i_{k-1}\}}$ we can choose indices $j_1, \ldots, j_s \in \{1, \ldots, l\}$ such that $\gamma + \alpha_{j_1} + \cdots + \alpha_{j_s} = \tilde{\alpha}$ and $\gamma + \alpha_{j_1} + \cdots + \alpha_{j_r} \in \Phi$ for all $r = 1, \ldots, s$. As the coordinates of γ corresponding to the indices $i_1, \ldots, i_{k-1}$ are maximum, we have that $\{j_1, \ldots, j_s\} \subset \{i_k, \ldots, i_l\}$, and according to the previous paragraph, $-\alpha_{j_1}, \ldots, -\alpha_{j_s} \in P + \Phi$. But $\tilde{\alpha} \in P$, so $\tilde{\alpha} - \alpha_{j_s} \in P$ (taking into account that $\tilde{\alpha} - \alpha_{j_s} \in \Phi$, $-\alpha_{j_s} \in P + \Phi$ and 4.2(ii)), and with the same argument $\tilde{\alpha} - \alpha_{j_s} - \alpha_{j_{s-1}} \in P$ and $\gamma = \tilde{\alpha} - \alpha_{j_s} - \cdots - \alpha_{j_1} \in P$.

Finally, let us prove that α_{i_k} also belongs to $P+\Phi$: Take $\gamma \in P \setminus P_{\{i_1,\dots,i_k\}}$, and as before choose $\{j_1,\dots,j_s\} \subset \{i_k,\dots,i_l\} \subset \{1,\dots l\}$ such that $\{\gamma, \gamma+\alpha_{j_1},\dots,\gamma+\alpha_{j_1}+\cdots+\alpha_{j_s}=\tilde{\alpha}\} \subset P$. Since the i_k'th coordinate of γ is not n_i, $i_k \in \{j_1,\dots,j_s\}$, and we have found $\gamma' \in P$ such that $\gamma'+\alpha_{i_k} \in P$. In particular $\alpha_{i_k} = (\gamma'+\alpha_{i_k})+(-\gamma') \in P+\Phi$, and this finishes our proof. ∎

Theorem 4.4. *Let B be a nonzero abelian inner ideal of a finite dimensional simple Lie algebra L over an algebraically closed field of characteristic zero. Then there exist a Cartan subalgebra H, a basis Δ of the root system Φ relative to H, and a nonempty subset $I \subset \{1,\dots,l\}$ such that $B = B_{H,\Delta,I}$.*

Proof. According to Lemma 4.1, we can take a Cartan subalgebra H of L such that $B = \bigoplus_{\alpha\in\Phi} L_\alpha \cap B$, where Φ denotes the root system relative to H. As before let $P = \{\alpha \in \Phi \mid L_\alpha \subset B\}$. Take an arbitrary basis $\Delta = \{\alpha_1,\dots,\alpha_l\}$ of Φ and let $\tilde{\alpha} = \sum n_i\alpha_i$ be the maximal root relative to Δ. We are going to prove by induction on $n = \operatorname{rank} P$ that there exist $\{i_1,\dots,i_{n-1}\} \subset \{1,\dots,l\}$ and an element σ in the Weyl group $\mathcal{W}$ of L such that

$$\emptyset \subsetneq P_{\{1,\dots,l\}} \subsetneq P_{\{1,\dots,l\}\setminus\{i_1\}} \subsetneq \cdots \subsetneq P_{\{1,\dots,l\}\setminus\{i_1,i_2,\dots,i_{n-1}\}} = \sigma(P).$$

In such case, for any automorphism $\hat{\sigma} \in \operatorname{aut} L$ satisfying $\hat{\sigma}(H) = H$ and $\hat{\sigma}(L_\alpha) = L_{\sigma(\alpha)}$ as in [9, §14], we get $\hat{\sigma}(B) = B_{H,\Delta,I}$ for $I = \{1,\dots,l\} \setminus \{i_1,i_2,\dots,i_{n-1}\}$ and hence $B = B_{H,\sigma^{-1}(\Delta),I}$.

We begin with the case $n = 1$. By Lemma 4.2 the set P contains some long root. Since all the long roots are conjugate, there exists $\sigma \in \mathcal{W}$ such that $\tilde{\alpha} \in \sigma(P)$, and so $\sigma(P) = P_{\{1,\dots,l\}}$.

Suppose now that P has rank $n > 1$ and that $n \le l$. Take $P' \subset P$ of rank $n-1$ satisfying 4.2(i)-(ii) (the $n-1$'th term of the chain of nonempty subsets of P satisfying 4.2(i)-(ii)). By the induction hypothesis there exist $\sigma \in \mathcal{W}$ and $\{i_1,\dots,i_{n-2}\}$ such that $\emptyset \subsetneq P_{\{1,\dots,l\}} \subsetneq P_{\{1,\dots,l\}\setminus\{i_1\}} \subsetneq \cdots \subsetneq P_{\{1,\dots,l\}\setminus\{i_1,\dots,i_{n-2}\}} = \sigma(P')$. Let us denote $J = \{1,\dots,l\} \setminus \{i_1,\dots,i_{n-2}\}$. Consider the sets

$$K := \{i \in \{1,\dots,l\} \mid \exists \gamma \in P_J \text{ such that } \gamma - \alpha_i \in \Phi\},$$
$$\mathcal{G} := \operatorname{gr}\langle\{\sigma_{\alpha_i} \mid i \notin K \cap J\}\rangle \le \mathcal{W}$$

where if $\alpha \in \Phi$, σ_α is the reflection given by $\sigma_\alpha(\gamma) := \gamma - \langle\gamma,\alpha\rangle\alpha$ for every $\gamma \in \Phi$. Note that $\mu(P_J) \subset P_J$ for all $\mu \in \mathcal{G}$: if $\gamma \in P_J$ and $i \notin J$, then $\sigma_{\alpha_i}(\gamma) = \gamma - \langle\gamma,\alpha_i\rangle\alpha_i \in P_J$ because σ_{α_i} does not move the coordinates of J; and if $i \in J \setminus K$, $\gamma - \alpha_i \notin \Phi$ by definition of K, and $\gamma + \alpha_i \notin \Phi$ by definition of J, hence $\langle\gamma,\alpha_i\rangle = 0$ and $\sigma_{\alpha_i}(\gamma) = \gamma \in P_J$.

We will find $\mu \in \mathcal{G}$ and $i \in K \cap J$ such that $P_{\{1,\dots,l\}\setminus\{i_1,\dots,i_{n-2}\}} \subsetneq P_{\{1,\dots,l\}\setminus\{i_1,\dots,i_{n-2},i\}} = \mu\sigma(P)$: If $\sigma(P) \setminus P_J$ contains some long root, take $\beta \in \sigma(P) \setminus P_J$ a long root, otherwise take $\beta \in \sigma(P) \setminus P_J$ arbitrarily. Take a maximal element $\tilde{\beta}$ in $\{\mu(\beta) \mid \mu \in \mathcal{G}\}$. Let $\mu \in \mathcal{G}$ be such that $\mu(\beta) = \tilde{\beta}$. Notice that $P_J \cup \{\tilde{\beta}\} \subset \mu\sigma(P)$. We claim that there exists $i \in K \cap J$ such that $\tilde{\beta} + \alpha_i \in \Phi$. Otherwise, $\tilde{\beta} + \alpha_i \notin \Phi$ for all $i \in K \cap J$, hence $(\tilde{\beta},\alpha_i) \ge 0$ for all $i \in K \cap J$. Besides $\sigma_{\alpha_i}(\tilde{\beta}) \not> \tilde{\beta}$ if $i \notin K \cap J$ (by the maximality of $\tilde{\beta}$), so we also get $(\tilde{\beta},\alpha_i) \ge 0$ when $i \notin K \cap J$. This means that

$\tilde{\beta} \in \{\delta \in \sum \mathbb{R}\alpha_i \mid (\delta, \alpha_i) \geq 0 \forall i = 1, \ldots, l\} =: \overline{C(\Delta)}$, the closure of the fundamental Weyl chamber relative to Δ. Hence $\eta(\tilde{\beta}) \prec \tilde{\beta}$ for all $\eta \in \mathcal{W}$ and $\tilde{\beta}$ is either the maximal (long) root $\tilde{\alpha}$ of Δ (which is not possible since $\tilde{\alpha} \in P_J$ but $\tilde{\beta} \notin P_J$) or the maximal short root of Δ (in particular, β is a short root). According to our choice of β (long if possible), $\mu\sigma(P) \setminus P_J$ does not contain long roots. Applying now Lemma 4.2 to $\mu\sigma(P)$, $\langle \tilde{\alpha}, \tilde{\beta} \rangle = 2$ so that $2\tilde{\beta} - \tilde{\alpha}$ is a long root belonging to $\mu\sigma(P) \setminus P_J$, which is a contradiction. Therefore, there exists $i \in K \cap J$ such that $\tilde{\beta} + \alpha_i \in \Phi$. Since $i \in K$, there is $\gamma \in P_J$ such that $\gamma - \alpha_i \in \Phi$. Note that $\gamma - \alpha_i = \gamma + \tilde{\beta} + (-\tilde{\beta} - \alpha_i) \in (\mu\sigma(P) + ((\mu\sigma(P) + \Phi) \cap \Phi)) \cap \Phi \subset \mu\sigma(P)$ by 4.2(ii). But $\gamma - \alpha_i \in P_{J \setminus \{i\}} \setminus P_J$, since $i \in J$. Consequently $\gamma - \alpha_i \in (\mu\sigma(P) \cap P_{J\setminus\{i\}}) \setminus P_J$ and $P_J \subsetneq \mu\sigma(P) \cap P_{J\setminus\{i\}} \subset P_{J\setminus\{i\}}$. By Proposition 4.3, $\mu\sigma(P) \cap P_{J\setminus\{i\}} = P_{J\setminus\{i\}}$, so that $P_{J\setminus\{i\}} \subset \mu\sigma(P)$. Moreover, $P_J \subsetneq P_{J\setminus\{i\}} \subset \mu\sigma(P)$, and again by Proposition 4.3, $\mu\sigma(P) = P_{J\setminus\{i\}} = P_{\{1,\ldots,l\}\setminus\{i_1,\ldots,i_{n-2},i\}}$, as searched.

Finally let us see that it is not possible that $\operatorname{rank} P > l$. Otherwise take $P' \subsetneq P$ verifying 4.2(i)-(ii) with $\operatorname{rank} P' = l$ (the l'th term of the chain of nonempty subsets of P satisfying 4.2(i)-(ii)). We have already proved that there is $\sigma \in \mathcal{W}$ and a permutation $\{i_1, \ldots, i_l\}$ such that

$$\{\tilde{\alpha}\} = P_{\{1,\ldots,l\}} = P_{\{i_1,\ldots,i_l\}} \subsetneq P_{\{i_1,\ldots,i_{l-1}\}} \subsetneq \cdots \subsetneq P_{\{i_1\}} = \sigma(P') \subsetneq \sigma(P).$$

According to Proposition 4.3, $\pm\alpha_j \in \sigma(P') + \Phi \subset \sigma(P) + \Phi$ if $j \neq i_1$. By taking a maximal element β in $\sigma(P) \setminus P_{\{i_1\}}$ and $s \in \{1, \ldots, l\}$ such that $\beta + \alpha_s \in \Phi$, it is not difficult to check that $s = i_1$, so in particular $-\alpha_{i_1} \in \sigma(P) + \Phi$. Take $\{j_1, \ldots, j_s\}$ such that $\{\beta, \beta + \alpha_{j_1} = \beta + \alpha_{i_1}, \ldots, \beta + \alpha_{j_1} + \cdots + \alpha_{j_s} = \tilde{\alpha}\} \subset \Phi$. Since $\tilde{\alpha} \in \sigma(P)$ and $-\alpha_j \in \sigma(P) + \Phi$ for any $j \in \{1, \ldots, l\}$, the above set of roots is contained in $\sigma(P)$, $\beta + \alpha_{i_1} \in \sigma(P)$ and $\alpha_{i_1} \in \sigma(P) + \Phi$. From here, $\Phi^+ \subset \sigma(P)$. This condition, jointly with 4.2(i), would force $\Phi^+ = \sigma(P) = \{\alpha_1\}$, but then $1 = \operatorname{rank} P > l = 1$, which is a contradiction. ∎

4.3. Remark: Theorem 4.4 provides an alternative proof to [7, Corollary 6.2] when L is a finite dimensional Lie algebra over an algebraically closed field because every nonzero abelian inner ideal B of L can be expressed as $B_{H,\Delta,I}$ for a certain Cartan subalgebra H, a basis $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ of the root system associated to H and a nonempty set $I \subset \{1, \ldots, l\}$, and we have already explained in 3.1 and 3.2 how any set of nonnegative integers $(p_1, \ldots, p_l)$ satisfying that $p_i = 0$ if $i \notin I$ induces a $\mathbb{Z}$-grading on $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ with $n = \sum n_i p_i$ (where n_i are the coordinates of the maximal root) and $B = L_n$.

4.4. Remark: Note that, as a consequence of the previous sections, we have proved that, for nonempty subsets I and J of $\{1, \ldots, l\}$ (H and Δ fixed as before) such that $B_I \neq B_J$, then B_I and B_J are nonconjugate by an inner automorphism of L.

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