ON QUOTIENT RINGS IN ALTERNATIVE RINGS

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Abstract. We introduce a notion of left nonsingularity for alternative rings and prove that an alternative ring is left nonsingular if and only if every essential left ideal is dense, if and only if its maximal left quotient ring is von Neumann regular (a Johnson-like Theorem). Finally, we obtain a Gabriel-like Theorem for alternative rings.

1. Introduction

The theory of rings of quotients has its origins between 1930 and 1940 in the works of O. Ore and K. Osano on the construction of the total ring of fractions. In that decade Ore proved a necessary and sufficient condition for a ring R to have a (left) classical ring of quotients (the left Ore condition). At the end of the 50's, Goldie, Lesieur and Croisot characterized those (associative) rings which are classical left orders in semiprime and Artinian rings [6, Chapter IV] (result known as Goldie's Theorem).

Later on in 1956, Y. Utumi introduced the notion of left quotient rings [10] and proved that the rings without absolute right zero divisors are precisely those which have a maximal left quotient ring. Following Goldie's idea of characterizing certain types of rings via a suitable envelope, R. E. Johnson characterized left nonsingular rings via their maximal left quotient rings [6, 13.36], and P. Gabriel specialized it further by giving similar characterizations for left nonsingular rings with finite left Goldie dimension [6, 13.40]. In the setting of Jordan algebras, F. Montaner and I. Paniello introduce an analogue of Johnsons associative algebra of quotients and they proved the existence and described the maximal algebras of quotients of prime strongly nonsingular Jordan algebras, see [9]. Recently, F. Montaner constructed a maximal algebras of quotients for any nondegenerate Jordan algebra, see [8].

It is natural to ask whether similar notions (and results) can be obtained for alternative rings. The question of Goldie's Theorems for alternative algebras was posed by H. Essannouni and A. Kaidi for Noetherian alternative rings, see [5]. Later, in 1994, the same authors established a Goldie-like theorem for alternative rings without elements of order three in their associator ideal. In [7], The second author

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of this paper together with M. Siles Molina introduced the notion of Fountain-Gould left orders in alternative rings and gave a Goldie-like characterization of alternative rings which are Fountain-Gould left orders in nondegenerate alternative rings which coincide with their socle (this result generalizes the classical Goldie's Theorems for alternative rings without additional conditions). In this work the authors introduced, as a tool, the notion of general left quotient rings and related properties of a ring with properties of its general ring of quotients. They also defined the left singular ideal of an alternative ring and gave a notion of left nonsingularity for nondegenerate alternative rings. In [1] the authors proved the existence of the maximal left quotient ring of an alternative ring which is a left quotient ring of itself.

In this paper we introduce the notion of left nonsingularity for (general) alternative rings. We characterize left nonsingular alternative rings as those for which every essential left ideal is dense. Moreover, we give both a Johnson and a Gabriel Theorem: an alternative ring is left nonsingular if and only if its maximal left quotient ring is von Neumann regular (Johnson's Theorem), and an alternative ring is left nonsingular with finite left Goldie dimension if and only if its maximal left quotient ring is nondegenerate and Artinian (Gabriel's Theorem).

2. Preliminaries

2.1. The following three basic central subsets can be considered in a non-associative ring R: the associative center $N(R)$, the commutative center $K(R)$, and the center $Z(R)$, defined by:

$$\begin{aligned} N(R) &= \{x \in R \mid (x,R,R) = (R,x,R) = (R,R,x) = 0\}, \\ K(R) &= \{x \in R \mid [x,R] = 0\}, \\ Z(R) &= N(R) \cap K(R) \end{aligned}$$

where $[x,y] = xy - yx$ denotes the commutator of two elements $x, y \in R$ and $(x,y,z) = (xy)z - x(yz)$ is the associator of three elements $x, y, z \in R$.

2.2. The defining axioms for an alternative ring R are the left and the right alternative laws:

$$(x,x,y) = 0 = (y,x,x)$$

for every $x, y \in R$. The standard reference for alternative rings is [11].

2.3. From now on, for a ring R, R^1 denotes its unitization: R if it is unital, or $R \times \mathbb{Z}$ with its abelian group structure and product given by $(x,m)(y,n) := (xy + nx + my, mn)$ if R is not unital.

2.4. Let R be an alternative ring. Every ideal contained in the associative center of R is called a nuclear ideal. The largest nuclear ideal of R is the associative nucleus, denoted by $U(R)$. The elements of $U(R)$ can be characterized as the elements $x \in R$ such that $(R^1 x, R, R) = 0$, see [11, Proposition 8.3.9]. We recall that R is called purely alternative if $U(R) = 0$. By $D(R)$ we denote the associator ideal: the ideal of R generated by $\{(x,y,z) \mid x,y,z \in R\}$, the set of all associators.

2.5. A ring without nonzero nilpotent ideals is called semiprime. By [11, Exercise 9.1.8], every semiprime alternative ring does not contain nonzero nilpotent left (right) ideals. An element a of an alternative ring R is called an absolute zero divisor if $aRa = \{0\}$. The ring R is called nondegenerate (or strongly semiprime) if R does not contain nonzero absolute zero divisors.

2.6. We recall that for every nonempty subset X of an alternative ring R, the left annihilator of X is defined as $lan_R(X) := \{a \in R \mid ax = 0 \text{ for all } x \in X\}$. Similarly the right annihilator of X is $ran_R(X) := \{a \in R \mid xa = 0 \text{ for all } x \in X\}$. We denote by $ann_R(X) := lan_R(X) \cap ran_R(X)$, the annihilator of X. In general the left (right) annihilator of a subset X of an alternative ring R does not have to be a left (right) ideal, however, it is true if X is a right (left) ideal of R or if $X \subset N(R)$.

Let R be an alternative ring and consider $X \subset R$. By $_R(X]$ we mean the left ideal of R generated by X. Similarly, $[X)_R$ denotes the right ideal of R generated by X.

2.7. The notion of left quotient ring of an alternative ring was introduced in [7], where the relationship among classical, Fountain-Gould and this type of rings of quotients was established.

Let R be a subring of an alternative ring Q. We recall that Q is a left quotient ring of R, denoted by $R \leq_q Q$, if:

(1) $N(R) \subset N(Q)$ and
(2) for every $p, q \in Q$, with $p \neq 0$, there exists $r \in N(R)$ such that $rp \neq 0$ and $rq \in R$.

Note that if Q is a left quotient ring of R, then R is a left quotient ring of itself, R and Q can be seen as left $N(R)$-modules, and R is a dense left $N(R)$-submodule of Q, see [6, 8.2].

2.8. We say that an alternative ring R has a maximal left quotient ring if there exists a ring Q such that:

(i) Q is a left quotient ring of R and
(ii) if S is a left quotient ring of R there exists a unique monomorphism of rings $f : S \to Q$ with $f(r) = r$ for every $r \in R$.

In [1] the authors proved that an alternative ring R has a unique, up isomorphism, maximal left quotient ring if and only if R is a left quotient ring of itself.

2.9. In [4] K. I. Beĭdar and A. V. Mikhalëv introduced what was called the nearly classical localization of an algebra. This construction will be useful for us in the case of a nondegenerate and purely alternative ring R: Let us consider $\phi := Z(D(R))$ the center of $D(R)$ and let us denoted by $\mathcal{F}$ the set of all essential ideals of ϕ. We can define the nearly classical localization of $D(R)$, $D(R)_{\mathcal{F}} := \varinjlim\{Hom_\phi(I, D(R)) \mid I \in \mathcal{F}\}$, the direct limit of the direct system $\{Hom_\phi(I, D(R)) \mid I \in \mathcal{F}\}$. The elements of $D(R)_{\mathcal{F}}$ can be represented as classes $[I, f]$ of pairs (I, f) where $I \in \mathcal{F}$ and $f \in Hom_\phi(I, D(R))$, modulo the equivalence relation $(I, f) \sim (I', f')$ if and only if f and f' agree on $I \cap I'$. Moreover

(i) By [2, 1.2], $D(R)_{\mathcal{F}}$ is a Cayley-Dickson algebra over its center, its center is von Neumann regular, see [4, Theorem 2.12(1)], which implies, by [2, Remark 1.8], that $D(R)_{\mathcal{F}}$ is a von Neumann regular ring.
(ii) By [1, 2.15 (5)], $D(R)_{\mathcal{F}}$ coincides with the maximal left quotient ring of $D(R)$ and therefore with the maximal left quotient ring of R (because $D(R)$ is a dense ideal of R).

3. The Singular Ideal of an Alternative Ring

In this section we study the notions of left singular ideal (and the underlying idea of left nonsingular ring), left quotient rings and essential or dense left ideals. In the setting of associative rings these notions are closely related: A left ideal I of a ring R is dense if and only if R is a left quotient ring of I, a ring R is left nonsingular if and only if every essential left ideal is dense and, when R is commutative, R is left (right) nonsingular if and only if it is semiprime, see [6].

The notion of left nonsingularity for nondegenerate alternative rings was introduced in [7], where it plays a fundamental role in the study of Fountain-Gould left orders. In this section we extend this notion to general alternative rings (without the extra hypothesis of nondegeneracy) and prove that a ring R is left nonsingular if and only if every essential left ideal is dense.

We recall that a left ideal I of an alternative ring R is essential if it has nonzero intersection with every nonzero left ideal of R. The left singular ideal of an alternative ring R is the set:

$$\mathcal{Z}_l(R) = \{x \in U(R) \mid lan_R(x) \text{ is an essential left ideal of } R\},$$

see [7]. A left ideal I of an alternative ring R is dense if for every $p, q \in R$, with $p \neq 0$, there exists $a \in N(R)$ such that $ap \neq 0$ and $aq \in I$. We want to point out that a left ideal I of an alternative ring R is dense if and only if R is a left quotient ring of I, see [1, 2.3] (note that a can be taken in $N(I)$).

Definition 3.1. *We say that an alternative ring R is left nonsingular if $D(R)$ is nondegenerate and $\mathcal{Z}_l(R) = 0$.*

It is clear that this definition agrees with the classical one in the setting of associative rings and with the definition given in [7] for nondegenerate alternative rings (because every ideal of a nondegenerate alternative ring is nondegenerate). Moreover, if R is a purely alternative ring, R is left nonsingular if and only if it is nondegenerate (a ring with a nondegenerate essential ideal is nondegenerate), which agrees with the fact that usually purely alternative rings behave like commutative associative rings.

The following three results are technical lemmas that will be used in the principal theorems of this section. Note that the first and second lemmas are trivial in the semiprime case.

Lemma 3.2. *Let R be an alternative ring, K an ideal of R and I a left ideal of R such that $K \cap I = 0$. Then $(I, R, R) \subset ann_R(K)$. Moreover, if I is not contained in $U(R)$ or if K is contained in $D(R)$ and $I \neq 0$, then $I \cap ann_R(K) \neq 0$.*

Proof. Since, $KI \subset K \cap I = 0$, for every $y \in I$, $z \in K$ and $a \in R$

$$(y, z, a) = -(a, z, y) = -(az)y + a(zy) = 0. \tag{1}$$

Now, the formula $(yz, a, b) - (y, za, b) + (y, z, ab) = y(z, a, b) + (y, z, a)b$, which holds in any ring and (1) imply

$$(yz, a, b) = y(z, a, b), \tag{i}$$

and the Kleinfeld function [11, Corollary of Lemma 7.1.2], $([z, y], a, b) + (z, y, [a, b]) = (zy, a, b) - y(z, a, b) - (y, a, b)z$ and (1) imply

$$(yz, a, b) = y(z, a, b) + (y, a, b)z. \tag{ii}$$

Finally, (i) and (ii) prove that $(y,a,b)z = 0$ for every $y \in I$, $z \in K$ and $a, b \in R$, i.e., $(I,R,R) \subset ann_R(K)$.

Now, if I is not contained in $U(R)$, by 2.4, $0 \neq (I,R,R) \subset ann_R(K) \cap I$, and if K is contained in $D(R)$ (and I is contained in $U(R)$) [11, Proposition 8.3.10] implies $I \subset ann_R(K)$. □

Lemma 3.3. *Let R be a purely alternative ring and let K be an ideal of R. Then there exists a left ideal L of R contained in $ann_R(K)$ such that $K \oplus L$ is an essential left ideal of R.*

Proof. By Zorn's Lemma, there exists a left ideal I of R which is maximal with respect to the property $K \cap I = 0$. By 3.2, ${}_R((I,R,R)] \subset ann_R(K)$. Let us show that $K \oplus {}_R((I,R,R)]$ is an essential left ideal of R: Let J be a nonzero left ideal of R. If $J \cap K \neq 0$ we have finished, otherwise by maximality of I, $J \cap I \neq 0$ and therefore, by 2.4, $0 \neq (J \cap I, R, R) \subset J \cap {}_R((I,R,R)]$. □

Lemma 3.4. *Let R be an alternative ring. Then:*

(i) *For every $0 \neq a \in R$, $lan_R([a)_R)$ is not a dense left ideal of R.*

(ii) *$U(R) \cap D(R) \subset \mathcal{Z}_l(R)$.*

Proof. (i). Suppose $0 \neq a \in R$ such that $lan_R([a)_R)$ is a dense left ideal of R. So there exists $r \in N(R)$ such that $0 \neq ra$ and $r' \in N(R)$ such that $0 \neq r'ra$ with $r'r \in lan_R([a)_R)$, a contradiction.

(ii). By [1, 1.10], $U(R) + D(R)$ is an essential left ideal of R which is contained in $lan_R(z)$ for every $z \in U(R) \cap D(R)$, see [11, Proposition 8.3.10]. □

Theorem 3.5. *An alternative ring R such that every essential left ideal is dense, is left nonsingular.*

Proof. Given any $0 \neq a \in \mathcal{Z}_l(R)$, $lan_R(a) = lan_R([a)_R)$, because $a \in U(R)$, is an essential left ideal of R. Now, by hypothesis it is dense which is not posible by 3.4(i), therefore $\mathcal{Z}_l(R) = \{0\}$

Now, we have to prove that $D(R)$ is nondegenerate. Let us suppose first that R is purely alternative: If K is a nonzero ideal of R such that $K^2 = 0$, then by 3.3 there exists a left ideal L of R contained in $ann_R(K)$ such that $K \oplus L$ is an essential left ideal of R, and by hypotheses, if $0 \neq x \in K$, there exists $n_1 + n_2 \in N(K \oplus L)$ (because, by [1, 2.3], R is a left quotient ring of $K \oplus L$), which implies $n_1 \in N(K)$ and $n_2 \in N(L)$, such that

$$0 \neq (n_1 + n_2)x \in (K)^2 \oplus LK = 0,$$

a contradiction. So R is semiprime. Now, by [1, 1.8] the linear span of all elements $\{x \in D(R) \mid xRx = 0\}$ is an ideal of R (denote it by $\mathcal{R}$) such that $N(\mathcal{R}) = 0$. And $\mathcal{R} \oplus ann_R(\mathcal{R})$ is an essential (and therefore dense) left ideal of R. So if $0 \neq x \in \mathcal{R}$, there exists $n_1 + n_2 \in N(\mathcal{R} \oplus ann_R(\mathcal{R}))$ (because, by [1, 2.3], R is a left quotient ring of $\mathcal{R} \oplus ann_R(\mathcal{R})$), which implies $n_1 \in N(\mathcal{R})$ and $n_2 \in N(ann_R(\mathcal{R}))$, such that

$$0 \neq (n_1 + n_2)x = n_2 x \in ann_R(\mathcal{R})\mathcal{R} = 0$$

a contradiction. So $\mathcal{R} = 0$ and $D(R)$ is a nondegenerate ideal of R which implies, since it is an essential ideal of R, that R is a nondegenerate ring.

Finally let us prove the general case. Let us prove that $\overline{R} := R/U(R)$ is a purely alternative ring such that every essential left ideal is dense: By 3.4(ii)

$$U(R) \cap D(R) \subset \mathcal{Z}_l(R) = \{0\}. \tag{1}$$

Now, if $\overline{x} \in U(\overline{R})$, then $\overline{(R^1x, R, R)} = \overline{0}$, i.e., $(R^1x, R, R) \subset U(R) \cap D(R) = 0$, by (1). So by 2.4, $x \in U(R)$ and therefore $\overline{x} = \overline{0}$. So $\overline{R}$ is purely alternative.

Let $\overline{K}$ be an essential left ideal of $\overline{R}$. Then $K = \pi^{-1}(\overline{K})$, where π denotes the canonical projection from R onto $R/U(R)$, is an essential left ideal of R. Now, given $\overline{p}, \overline{q} \in \overline{R}$, with $\overline{p} \neq \overline{0}$, we have that $p \notin U(R)$, so by 2.4, there exist $r \in R^1, s, t \in R$ such that $(pr, s, t) \neq 0$, and there exists $n \in N(R)$ such that $0 \neq n(pr, s, t) = ((np)r, s, t)$ and $nq \in K$. So $\overline{np} \neq 0$ and $\overline{nq} \in \overline{K}$, i.e., $\overline{K}$ is a dense ideal of $\overline{R}$. So $\overline{R}$ is a nondegenerate alternative ring and since $D(R)$ is isomorphic to an ideal of $\overline{R}$, $D(R)$ is nondegenerate. □

Proposition 3.6. *Let R be an alternative ring. Then:*

(i) *In general $\mathcal{Z}_l(U(R)) \subset \mathcal{Z}_l(R)$.*
(ii) *If $U(R)$ has no total right zero divisors (in particular if $U(R)$ is semiprime or left nonsingular), then $\mathcal{Z}_l(R) = \mathcal{Z}_l(U(R))$.*

So in any case, $\mathcal{Z}_l(U(R)) = 0$ if and only if $\mathcal{Z}_l(R) = 0$

Proof. (i). If $a \in \mathcal{Z}_l(U(R))$, then $lan_{U(R)}(a)$ is an essential left ideal of $U(R)$. Observe that $lan_{U(R)}(a) = lan_R(a) \cap U(R)$ is a left ideal of R. Now, $lan_{U(R)}(a) + D(R)$ is an essential left ideal of R contained in $lan_R(a)$: Let I be a nonzero left ideal of R. If $(I, R, R) = 0$, then $I \subset U(R)$, and therefore $I \cap lan_{U(R)}(a) \neq 0$. Otherwise, $0 \neq (I, R, R) \subset I \cap D(R)$. So $a \in \mathcal{Z}_l(R)$.

(ii). Suppose now that $U(R)$ has no total right zero divisors and take $a \in \mathcal{Z}_l(R)$. Then given a nonzero left ideal I of $U(R)$, we have that $U(R)I$ is a nonzero left ideal of R contained in I. So $0 \neq U(R)I \cap lan_R(a) \subset I \cap lan_{U(R)}(a)$, which implies that $a \in \mathcal{Z}_l(U(R))$. □

Theorem 3.7. *Every essential left ideal of a left nonsingular alternative ring is dense.*

Proof. Let I be an essential left ideal of R. Given $p, q \in R$, with $p \neq 0$:
If $0 \neq (R^1p, R, R)$, by [7, 5.2], there exists $a \in Z(D(R)) \subset Z(R)$ with $a(R^1p, R, R) \neq 0$. In particular $ap \neq 0$ and $aq \in D(R)$. Now, by [7, 1.2(iv)], there exists $0 \neq b \in Z(I \cap {}_R(ap])$ which implies that $bap \neq 0$ (if $bap = 0$, since $b \in {}_R(ap] \cap Z(R)$, $b = \sum(a_{i1}(\dots(a_{in_i}ap)\dots)$ and

$$b^2 = b\sum(a_{i1}(\dots(a_{in_i}ap)\dots) = \sum(a_{i1}(\dots(a_{in_i}bap)\dots) = 0,$$

a contradiction because there are no nonzero nilpotent elements in the center of the nondegenerate alternative algebra $D(R)$) and $baq = aqb \in I$.
If $(R^1p, R, R) = 0$, then $p \in U(R)$. Since $U(R)$ is left nonsingular, by 3.6, there exists $a \in U(R)$ such that $ap \neq 0$ (and $aq \in U(R)$). Now, since $I \cap U(R)$ is an essential left ideal of $U(R)$ (every nonzero left ideal J of $U(R)$ contains the nonzero left ideal $U(R)J$), by [6, 8.7 (3)], $I \cap U(R)$ is a dense left ideal of $U(R)$. So there exists $b \in I \cap U(R)$ such that $bap \neq 0$ and $baq \in I \cap U(R)$. □

Proposition 3.8. *Let R be a left nonsingular alternative ring.*

(i) *Every nonzero left ideal of R has nonzero intersection with $N(R)$.*
(ii) *The associative ring $N(R)$ is left nonsingular.*

Proof. (i). Let I be a nonzero left ideal of R. If $0 \neq I \cap D(R)$, there exists $0 \neq a \in Z(I \cap D(R)) \subset N(R) \cap I$, see [7, 5.2]. If $I \cap D(R) = 0$, then $(I, R, R) = 0$ which implies, by 2.4, that $I \subset U(R) \subset N(R)$.

(ii). Let $a \in \mathcal{Z}_l(N(R))$. Notice that $lan_{N(R)}(a) \subset lan_R(a)$. So, if I is a nonzero left ideal of R, by (i), $I \cap N(R)$ is a nonzero ideal of $N(R)$, therefore $0 \neq lan_{N(R)}(a) \cap I \cap N(R) \subset lan_R(a) \cap I$ and $lan_R(a)$ has nonzero intersection with every nonzero left ideal of R. So $lan_R(a)(= lan_R([a)_R))$ is an essential left ideal of R which implies by 3.7 it is dense and therefore, 3.4(i) proves that $a = 0$. □

4. Johnson and Gabriel Theorems for Alternative Rings

In associative rings two of the main results concerning the maximal left quotient ring are the Johnson's and the Gabriel's Theorems, see [6, 13.36 and 13.40]. In this section we obtain similar results for alternative rings.

Proposition 4.1. *Let R be a left nonsingular alternative ring and let us denote by Q the maximal left quotient ring of R. Then*

(i) $Q = Q^l_{max}(U(R)) \oplus Q^l_{max}(D(R))$.
(ii) $U(Q) = Q^l_{max}(U(R))$ *and* $D(Q) = Q^l_{max}(D(R))$.
(iii) $N(Q) = Q^l_{max}(N(R))$.

Proof. (i). By [1, 1.10], 3.4(ii) and 3.7, $U(R) \oplus D(R)$ is a dense left ideal of R, so $Q = Q^l_{max}(U(R) \oplus D(R)) = Q^l_{max}(U(R)) \oplus Q^l_{max}(D(R))$, see [1, 1.12 (ii) and 2.15 (4)].

(ii). Since $D(Q)$ is an ideal of Q which contains $D(R)$, it is a dense (left) ideal of $Q^l_{max}(D(R))$, so we have $Q^l_{max}(D(R)) = Q^l_{max}(D(Q))$. Now let $C = Z(Q^l_{max}(D(R)))$, by [3, 2.12], $C = Q_{max}(Z(D(R)))$. Then $Q(D)$ is a C-submodule of $Q^l_{max}(D(R))$, and it is orthogonally complete by [3, 3.1.18]. Now, since $Q(D)$ is a nonsingular C-module, it is an injective C-module by [3, 3.1.6], and since it is an essential C-submodule of $Q^l_{max}(D(R))$ one gets $Q(D) = Q^l_{max}(D(R))$.

Moreover, $Q^l_{max}(U(R))$ is an associative ideal of Q, so it is contained in $U(Q)$. Now, if $U(Q)$ is not contained in $Q^l_{max}(U(R))$, it has nonzero intersection with $Q^l_{max}(D(R))$, a contradiction because $Q^l_{max}(D(R))$ is a purely alternative ring (since every nonzero ideal of $Q^l_{max}(D(R))$ has nonzero intersection with $D(R)$), i.e., $U(Q) = Q^l_{max}(U(R))$.

(iii). By (i), $Q = Q^l_{max}(U(R)) \oplus Q^l_{max}(D(R))$ and therefore

$$\begin{aligned} N(Q) &= N(Q^l_{max}(U(R)) \oplus Q^l_{max}(D(R))) = N(Q^l_{max}(U(R))) \oplus N(Q^l_{max}(D(R))) \\ &=^{(1)} Q^l_{max}(U(R)) \oplus Q^l_{max}(N(D(R))) =^{(2)} Q^l_{max}(U(R) \oplus N(D(R))) \\ &=^{(3)} Q^l_{max}(N(R)) \end{aligned}$$

(1) since $D(R)$ is a nondegenerate and purely alternative ring its maximal left quotient ring coincides with the nearly classical localization, see 2.9(ii). Now, $N(Q^l_{max}(D(R))) = Q^l_{max}(N(D(R)))$ follows from [4, 2.10 (iii)],
(2) follows from [1, Example 2.15 (4)]
(3) because $U(R) \oplus N(D(R))$ is a dense ideal of $N(R)$: If I is a nonzero left ideal of $N(R)$ and $0 \neq y \in I$, $R^1 y$ is a nonzero left ideal of R. Now, $(R^1 y, R, R) = 0$ implies $0 \neq y \in U(R) \cap I$ and $(R^1 y, R, R) \neq 0$ implies, since $D(R)$ is nondegenerate, that $R^1 y \cap D(R) \neq 0$. So there exists $0 \neq ay \in Z(D(R))$. Therefore, $(ay)^2 \neq 0$ which implies $0 \neq ayy \in I \cap D(R)$. So $U(R) \oplus N(D(R))$ is an essential left ideal of $N(R)$ and, since $N(R)$ is left nonsingular by 3.8, $U(R) \oplus N(D(R))$ is a dense left ideal of $N(R)$. □

4.2. We recall that an element x of an alternative ring R is von Neumann regular if there exists $y \in R$ such that $xyx = x$ (note that R satisfies the Flexible Law). A ring R is von Neumann regular if every element of R is von Neumann regular.

Theorem 4.3 (Johnson's Theorem)**.** *Let R be an alternative ring. Then the following conditions are equivalent:*

(i) *R is left nonsingular.*
(ii) *The maximal left quotient ring of R is von Neumann regular.*

Proof. Suppose that R is left nonsingular. By 4.1(i), we have that $Q := Q^l_{max}(R) = Q^l_{max}(U(R)) \oplus Q^l_{max}(D(R))$. On the one hand, by 3.8, $N(R)$ is left nonsingular and, by 4.1(iii), $N(Q) = Q^l_{max}(N(R))$ is von Neumann regular, see Johnson's Theorem for associative rings, [6, 13.36]. So $U(Q) = Q^l_{max}(U(R))$ is von Neumann regular. On the other hand, since $D(R)$ is nondegenerate and purely alternative, $Q^l_{max}(D(R))$ is von Neumann regular by 2.9(i).

Now, suppose that Q is von Neumann regular. On the one hand, $U(Q)$ is von Neumann regular (because it is an ideal of Q), which implies that $U(R)$ is left nonsingular and, by 3.6, $\mathcal{Z}_l(R) = \mathcal{Z}_l(U(R)) = 0$. On the other hand, $Q^l_{max}(D(R))$ is von Neumann regular, in particular it is nondegenerate, and therefore $D(R)$ is nondegenerate: if $x \in R$ is such that $xRx = 0$, then for every $q \in Q^l_{max}(D(R))$ if $xqx \neq 0$ there exists $n \in N(D(R)) = Z(D(R))$ such that $n(xqx) \neq 0$ and $nq \in R$ but $n(xqx) = x(nq)x \in xRx = 0$, a contradiction. So $xqx = 0$ for every $q \in Q^l_{max}(D(R))$ and $x = 0$. □

Proposition 4.4. *Let R be an alternative ring with finite left Goldie dimension. Then:*

(i) *If $D(R)$ is nondegenerate then $D(R)$ has finite left Goldie dimension.*
(ii) *If $U(R)$ has not total right zero divisors (in particular, if R is left nonsingular) then $U(R)$ has finite left Goldie dimension.*

Proof. It follows because in both cases every nonzero left ideal I of $U(R)$ or $D(R)$ contains a nonzero left ideal of R: If I is contained in $D(R)$, there exists $0 \neq a \in Z(R) \cap I$ and therefore $0 \neq R^1 a \subset I$. If $I \subset U(R)$, I contains $U(R)I$ which is a nonzero left ideal of R. □

Theorem 4.5 (Gabriel's Theorem)**.** *Let R be an alternative ring. Then the following conditions are equivalent.*

(i) *R is left nonsingular and has finite left Goldie dimension.*
(ii) *The maximal left quotient ring of R is nondegenerate and Artinian.*

Proof. (i) $\Longrightarrow$ (ii). By 4.1(i), $Q := Q^l_{max}(R) = Q^l_{max}(U(R)) \oplus Q^l_{max}(D(R))$.

On the one hand, by 4.4(ii), $U(R)$ has finite left Goldie dimension and, by 3.6(i), it is left nonsingular. So by Gabriel's Theorem for associative rings, $Q^l_{max}(U(R))$ is nondegenerate and Artinian.

On the other hand, $D(R)$ is nondegenerate, purely alternative and by 4.4(i) it has finite left Goldie dimension. So by Goldie Theorem for alternative rings, see [7], there exists the classical left quotient ring of $D(R)$, denote it by $Q^l_{cl}(D(R))$, which is nondegenerate and Artinian. Moreover, by [1, Proposition 3.5], $Q^l_{max}(D(R)) = Q^l_{cl}(D(R))$. So $Q = Q^l_{max}(U(R)) \oplus Q^l_{max}(D(R))$ is a sum of nondegenerate and Artinian alternative rings, which implies that Q is nondegenerate and Artinian.

(ii) $\Longrightarrow$ (i). Q is Artinian so it has finite left Goldie dimension. Then following the proof of [7, 4.5(vi)] we have that R has finite left Goldie dimension (we do not need the semiprimeness of R since Q is unitary). Moreover, since Q is von Neumann regular, R is left nonsingular, see 4.3. □

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