Mediterr. J. Math. 13 (2016), 29–52
DOI 10.1007/s00009-014-0488-9
0378-620X/16/010029-24
published online July 2, 2015
© Springer Basel 2015

Mediterranean Journal of Mathematics

CrossMark

Prime Quotients of Jordan Systems and Lie Algebras

José A. Anquela, Teresa Cortés, Esther García and Miguel Gómez Lozano

Abstract. We show that, unlike alternative algebras, prime quotients of a nondegenerate Jordan system or a Lie algebra need not be nondegenerate, even if the original Jordan system is primitive, or the Lie algebra is strongly prime, both with nonzero simple hearts. Nevertheless, for Jordan systems and Lie algebras directly linked to associative systems, we prove that even semiprime quotients are necessarily nondegenerate.

Mathematics Subject Classification. 16W10, 17C10, 17B05.

Keywords. Jordan system, Lie algebra, associative system, semiprime quotient, nondegenerate.

1. Introduction

Absolute zero divisors in associative algebras generate nilpotent ideals: if $x \in R$ satisfies $xRx = 0$, then the ideal I generated by x has zero cube. As a consequence, every semiprime associative algebra is nondegenerate (does not have nonzero absolute zero divisors). These elementary assertions, which fail when we drop the associativity condition, have been traditionally used to measure the distance between a given variety of algebras and the variety of associative algebras. As examples:

(i) Shestakov [28] proves that, for finitely generated alternative algebras, absolute zero divisors generate nilpotent ideals, while McCrimmon [21] establishes the local nilpotency of alternative algebras generated by absolute zero divisors.

J. A. Anquela, T. Cortés, E. García: Partly supported by the Ministerio de Ciencia e Innovacion and Fondos FEDER, MTM2010-16153, the Ministerio de Economia y Competitividad and Fondos FEDER, MTM2014-52470-P, and the Junta de Andalucia FQM-264. M. Gómez Lozano: Partly supported by the Ministerio de Ciencia e Innovacion and Fondos FEDER, MTM2010-19482, the Ministerio de Economia y Competitividad and Fondos FEDER, MTM2014-52470-P, and the Junta de Andalucia FQM-264.

(ii) In [11], Beidar, Mikhalev and Shestakov prove that any prime quotient of a nondegenerate alternative algebra is nondegenerate, extending Kleinfeld's result [33, Ch. 9, Sect. 2, Th. 5].

Similar results for Jordan systems as those mentioned in (i) are due to Zelmanov [32] and Medevedev [25], while for Lie algebras, one can mention the papers by Kostrikin [18] and Zelmanov [31]. However, no analogue of (ii) for Jordan systems or Lie algebras was known.

This paper is devoted to settling this question: It will be shown that an analogue of (ii) for Jordan systems or Lie algebras is false in general, but it turns out to be true if we restrict to cases when the Jordan system or the Lie algebra is directly linked to an associative system.

The paper starts with a preliminary section devoted to recalling basic facts and terminology. After that, we show in the first section that there are nondegenerate Jordan systems and Lie algebras that have prime degenerate quotients. Using free special Jordan systems and special Pchelintsev monsters [27,29] yields examples of strongly prime Jordan systems having prime degenerate quotients. Similar examples of Lie algebras can be obtained using the free Lie algebra. The constructions given in [6,9] are used to show the existence of primitive Jordan systems and strongly prime Lie algebras, both with simple nondegenerate hearts, that have prime degenerate quotients nevertheless.

The second section is devoted to studying Jordan systems which are ample subsystems $H_0(R, *)$ of an associative system R with involution and even quotients of those. Using Herstein's constructions [7], we can give precise descriptions of semiprime and prime ideals of $H_0(R, *)$, showing that, in particular, they are nondegenerate and strongly prime, respectively. As a consequence, we obtain similar results for Jordan systems obtained by symmetrization $R^{(+)}$ of an associative system R. In this section, no assumption is made on the associative systems under consideration, nor on the rings of scalars.

The third section deals with Lie algebras of skew-symmetric elements of an associative algebra R with involution over a ring of scalars Φ, with $\frac{1}{2} \in \Phi$, and, more generally, with their quotients. In the particular case when the Lie algebra is of the form $R^{(-)}$, for an associative algebra R, a description of semiprime ideals is provided. The general case is far more involved. The study of semiprime ideals is reduced to that of prime ideals, where another reduction can be made. Indeed, R can be assumed to be $*$-prime, so that we can apply Martindale and Miers' Herstein's Lie theory if the $*$-central closure of R is not of type A_2 or BD_4. The remaining cases require different techniques, but, finally, full Lie analogues of the main results of the previous section are obtained.

2. Preliminaries

2.1 We will deal with associative, Jordan systems (algebras, triple systems and pairs) and Lie algebras over an arbitrary ring of scalars Φ. The reader is referred to [3,15–17,19,22,23] for basic facts and notions not explicitly mentioned in this section.

- Given a Jordan algebra J, its products will be denoted by x^2, U_xy, for $x, y \in J$. They are quadratic in x and linear in y and have linearizations denoted $x \circ y$, $U_{x,z}y = \{x, y, z\} = V_{x,y}z$, respectively.
- For a Jordan pair $V = (V^+, V^-)$, we have products $Q_xy \in V^\varepsilon$, for any $x \in V^\varepsilon$, $y \in V^{-\varepsilon}$, $\varepsilon = \pm$, with linearizations $Q_{x,z}y = \{x, y, z\} = D_{x,y}z$.
- A Jordan triple system J is given by its products P_xy, for any $x, y \in J$, with linearizations denoted by $P_{x,z}y = \{x, y, z\} = L_{x,y}z$.
- For a Lie algebra L, the (bilinear) product of the elements $x, y \in L$ will be denoted $[x, y]$. The map $\mathrm{ad}_x : L \longrightarrow L$ is given by $\mathrm{ad}_x(y) = [x, y]$.

2.2 A Jordan algebra gives rise to a Jordan triple system by simply forgetting the squaring and letting $P = U$. By doubling any Jordan triple system T, one obtains the *double Jordan pair* $V(T) = (T, T)$ with products $Q_xy = P_xy$, for any $x, y \in T$. From a Jordan pair $V = (V^+, V^-)$, one can get a (*polarized*) Jordan triple system $T(V) = V^+ \oplus V^-$ by defining $P_{x^+ \oplus x^-}(y^+ \oplus y^-) = Q_{x^+}y^- \oplus Q_{x^-}y^+$ [19, 1.13, 1.14].

2.3 An *ideal* of a Jordan triple system J is a Φ-submodule I of J such that it is both an *inner ideal* ($P_IJ \subseteq I$) and an *outer ideal* ($P_JI + \{J, J, I\} \subseteq I$). Similar notions are defined for Jordan algebras and pairs. An *ideal* of a Lie algebra L is a Φ-submodule I of L such that $[I, L] \subseteq I$.

2.4 Given an associative or Jordan system, or a Lie algebra M, the *heart* $\mathrm{Heart}(M)$ of M is the intersection of all nonzero ideals of M.

2.5 An associative system R gives rise to a Jordan system $R^{(+)}$ by *symmetrization*: over the same Φ-module (the same pair of Φ-modules, in the pair case), we define $x^2 = xx$, $U_xy = xyx$, for any $x, y \in R$ in the case of algebras, $P_xy = xyx$ in the case of triple systems, and $Q_{x^\sigma}y^{-\sigma} = x^\sigma y^{-\sigma}x^\sigma$, $\sigma = \pm$ in the pair case, where juxtaposition denotes the associative product in R.

Similarly, given an associative algebra R, we can build its *antisymmetrization* $R^{(-)}$, which turns out to be a Lie algebra: it is the same Φ-module, with a new product given by $[x, y] = xy - yx$.

2.6 The *center* $Z(R)$ of an associative algebra R is the set $Z(R) = \{z \in R \mid zx = xz, \text{ for any } x \in R\}$, which turns out to be subalgebra of R, and an ideal of $R^{(-)}$. The *center* $Z(L)$ of any Lie algebra L is $Z(L) = \{z \in L \mid [z, x] = 0, \text{ for any } x \in L\}$ which is always an ideal of L. Clearly, $Z(R) = Z(R^{(-)})$.

2.7 A Jordan system J is said to be *special* if there exists an associative system R such that J is a subsystem of $R^{(+)}$. A Jordan system which is not special is called *exceptional*. For a Lie algebra L, over a field Φ of characteristic not two, the Poincare–Birkhoff–Witt theorem [15, Cor. 17.3 B; 17, Cor. 1, p. 160] shows that there exists an associative Φ-algebra R such that L is a subalgebra of $R^{(-)}$.

2.8 A particularly important family of special Jordan systems is that of ample subsystems of associative systems with involution:

- If R is an associative algebra with involution $*$, a Φ-submodule $H_0(R, *)$ contained in the set of symmetric elements $H(R, *)$ is said to be an *ample subspace* of R if it contains all traces and norms of elements of R

($x+x^*, xx^* \in H_0(R,*)$ for any $x \in R$) and $xH_0(R,*)x^* \subseteq H_0(R,*)$ for any $x \in R$ [20, p. 387; 23, Sect. 0.8′].

- If $R = (R^+, R^-)$ is an associative pair with polarized involution $*$, an *ample subpair* $H_0(R,*) = (H_0^+, H_0^-)$ is a pair of Φ-submodules of symmetric elements ($H_0^\sigma \subseteq H(R^\sigma, *)$) containing all traces ($x + x^* \in H_0^\sigma$ for any $x \in R^\sigma$) and satisfying $xH_0^{-\sigma}x^* \subseteq H_0^\sigma$ for any $x \in R^\sigma$, $\sigma = \pm$ [8, Sect. 1.7].
- If R is an associative triple system with involution $*$, a Φ-submodule $H_0(R,*)$ contained in the set of symmetric elements $H(R,*)$ is said to be an *ample subsystem* of R if $V(H_0(R,*))$ is an ample subpair of $V(R)$ equipped with the polarized involution induced by $*$ [2, pp. 209–210].

2.9 An important subalgebra of the Lie algebra $R^{(-)}$, when R is an associative algebra with involution $*$, is given by the set $\mathrm{Skew}(R,*)$ of skew-symmetric elements of R with respect to $*$.

2.10 A Jordan system or a Lie algebra M is said to be *nondegenerate* if zero is the only *absolute zero divisor*. An absolute zero divisor of a Jordan algebra J is an element $x \in J$ such that $U_x = 0$ (similar definitions are given for Jordan pairs and triple systems), while an absolute zero divisor $x \in L$, where L is a Lie algebra, is defined by $\mathrm{ad}_x^2 = 0$.

2.11 We say that a Jordan algebra J is *semiprime* if $I^3 \neq 0$, for any nonzero ideal I of J, and say that J is *prime* if $U_I K \neq 0$, for any nonzero ideals I, K of J. Similarly, we can define semiprime and prime Jordan pairs and triple systems.

A Lie algebra L is said to be *semiprime* if $[I,I] \neq 0$ for any nonzero ideal I of L, and is said to be *prime* if $[I,K] \neq 0$ for any nonzero ideals I, K of L.

A Jordan system or a Lie algebra is said to be *strongly prime* if it is prime and nondegenerate.

An ideal I of a Jordan or associative system or a Lie algebra M is said to be *semiprime*, *prime*, *nondegenerate*, or *strongly prime* if the quotient M/I is semiprime, prime, nondegenerate, or strongly prime, respectively. For an associative system with involution $*$, we also have the notion of $*$-*prime* $*$-ideal of R, with obvious meaning.

3. Counterexamples

3.1 We can find prime quotients of strongly prime Jordan systems that are degenerate.

(i) Indeed, examples of prime degenerate Jordan algebras can be found in [26,27,29]. The examples given in [27] are special, over a field Φ of characteristic zero. Let J be such an algebra. In particular, J is the quotient of the free special Jordan Φ-algebra $\widetilde{J} = \mathrm{FSJ}_{\mathrm{alg}}[X]$ on an infinite set of variables X.

(ii) Let $\mathrm{FAss}_{\mathrm{alg}}[X]$ be the free associative Φ-algebra on X. Recall that $\mathrm{FSJ}_{\mathrm{alg}}[X]$ is the subalgebra of $\mathrm{FAss}_{\mathrm{alg}}[X]^{(+)}$ generated by X. If Φ is an integral domain (for example, when Φ is a field), it is readily seen that

$\mathrm{FSJ}_{\mathrm{alg}}[X]$ is even strongly prime since, for elements $a, b \in \mathrm{FSJ}_{\mathrm{alg}}[X]$, $U_a b = aba = 0$ implies that either $a = 0$ or $b = 0$ because $\mathrm{FAss}_{\mathrm{alg}}[X]$ does not have nonzero zero divisors.

(iii) By applying the functors given in (2.2) to the algebras given in (i) and (ii), together with the transfer of regularity [4, 0.3(ii)(iii)], one can get special Jordan pairs and triple systems $\widetilde{J}$ over a field Φ that are strongly prime but have prime degenerate quotients J.

3.2 There are also examples of prime quotients of strongly prime Lie algebras that are degenerate.

(i) In fact, there exist Lie algebras of Cartan type over fields of prime characteristic that are simple and degenerate (cf. [30, page 1]). If we want examples over fields of characteristic zero, we can take a prime degenerate algebra J as in [27], which is a Jordan algebra over a field Φ of characteristic zero so that the duplicated Jordan pair $V := (J, J)$ is also prime and degenerate [4, 0.3(iii)], hence the Lie algebra $TKK(V)$ is prime and degenerate by [12, 1.2, 2.2, 2.6]. Let us take one of those Lie algebras L over a field Φ which are prime and degenerate. Such an algebra is a quotient of the free Lie algebra $\mathrm{FLie}_{\mathrm{alg}}[X]$ on a sufficiently big set of variables X, and we can always take X to be infinite.

(ii) If we work over fields, $\mathrm{FLie}_{\mathrm{alg}}[X]$ is just the subalgebra of $\mathrm{FAss}_{\mathrm{alg}}[X]^{(-)}$ generated by X [17, Th. 7, p. 168]. We claim that

(a) $\mathrm{FLie}_{\mathrm{alg}}[X]$ is prime if X is infinite: if I, K were nonzero ideals of zero product, we could take nonzero elements $a \in I$, $b \in K$ and a variable $x \in X$ not involved either in a or in b. Now $[[a, x], b] \in [[I, L], K] \subseteq [I, K] = 0$, but $[[a, x], b] = axb - xab - bax + bxa$, which implies that $xab = 0$ since xab is just the sum of all associative monomials of $[[a, x], b]$ starting with the variable x. However, $xab = 0$ is impossible since $\mathrm{FAss}_{\mathrm{alg}}[X]$ does not have nonzero zero divisors if Φ is a field.

(b) $\mathrm{FLie}_{\mathrm{alg}}[X]$ is nondegenerate if X is infinite: if $a \in \mathrm{FLie}_{\mathrm{alg}}[X]$ is an absolute zero divisor, we can take a variable $x \in X$ not involved in a, and $0 = [a, [a, x]] = aax - 2axa + xaa$ implies $xaa = 0$, hence $a = 0$, due to the absence of nonzero zero divisors in $\mathrm{FAss}_{\mathrm{alg}}[X]$.

3.3 We can get wilder examples using the following results:

(I) [9, 1.4] Let J be a special Jordan system over a field Φ. There exists a special Jordan system $\widetilde{J}$ over Φ such that:
- (i) J is isomorphic to a subsystem M of $\widetilde{J}$,
- (ii) $\widetilde{J}$ is a primitive system, hence it is strongly prime,
- (iii) $\mathrm{Heart}(\widetilde{J})$ is simple and primitive,
- (iv) $\widetilde{J} = M \oplus \mathrm{Heart}(\widetilde{J})$, hence $\widetilde{J}/\mathrm{Heart}(\widetilde{J}) \cong J$.

(II) [6, 3.2] Let L be a Lie algebra over a field Φ of characteristic not two. There exists a Lie algebra $\widetilde{L}$ over Φ such that:
- (i) L is isomorphic to a subalgebra M of $\widetilde{L}$,
- (ii) $\widetilde{L}$ is strongly prime,
- (iii) $\mathrm{Heart}(\widetilde{L})$ is simple and nondegenerate,

(iv) $\widetilde{L} = M \oplus \mathrm{Heart}(\widetilde{L})$, hence $\widetilde{L}/\mathrm{Heart}(\widetilde{L}) \cong L$.

Indeed, if we take a prime degenerate special Jordan system over a field as in (3.1), we can apply (I) to obtain a primitive Jordan system $\widetilde{J}$ with nonzero simple primitive heart, such that J is a quotient of $\widetilde{J}$.

Similarly, if we take a prime degenerate Lie algebra L over a field of characteristic not two, as in (3.2), we can apply (II) to obtain a strongly prime Lie algebra $\widetilde{L}$ with nonzero simple nondegenerate heart, such that L is a quotient of $\widetilde{L}$.

4. Jordan Systems Linked to Associative Systems

This section is devoted to showing that prime quotients of a Jordan system J are automatically nondegenerate when J is an ample subsystem of an associative system with involution (2.8). This is based on Herstein's second construction [7].

4.1 Let $(R, *)$ be an associative system (algebra, pair, or triple system) with involution, $H_0 := H_0(R, *)$ an ample subsystem of R, and B be a $*$-ideal of R.

- If R is an algebra, we define

$$K(B, H_0) = \left\{ b + b^* + \sum_i \lambda_i b_i b_i^* + \sum_j b_j h_j b_j^* \mid b, b_i, b_j \in B, h_j \in H_0, \quad \lambda_i \in \Phi \right\},$$

  which turns out to be an ideal of H_0 contained in $B \cap H_0$ [7, 2.2].
- If $R = (R^+, R^-)$ is a pair, we define

$$K(B^\sigma, H_0) = \left\{ b + b^* + \sum_i b_i h_i b_i^* \mid b, b_i \in B^\sigma, \quad h_i \in H_0^{-\sigma} \right\},$$

  which is a semi-ideal of H_0 contained in $B^\sigma \cap H_0^\sigma$, $\sigma = \pm$ [7, 3.2]. We will write $K(B, H_0) = (K(B^+, H_0), K(B^-, H_0))$.
- If R is a triple system, we define

$$K(B, H_0) = \left\{ b + b^* + \sum_i b_i h_i b_i^* \mid b, b_i, \in B, \quad h_i \in H_0 \right\},$$

  which is a semi-ideal of H_0 contained in $B \cap H_0$ [7, 3.10].

We have adopted a uniform notation $K(B, H_0)$ for algebras, pairs, and triple systems, unlike in [7], to simplify the phrasing of the next results.

Lemma 4.1. *Let $(R, *)$ be an associative system with involution, $H_0 := H_0$ $(R, *)$ an ample subsystem of R, and P an ideal of H_0. Let $\mathcal{I}$ be the set of $*$-ideals I of R such that $K(I, H_0) \subseteq P$. Then, $\mathcal{I}$ is closed under sums ($I_1 + I_2 \in \mathcal{I}$ for any $I_1, I_2 \in \mathcal{I}$) so that the maximum of the elements of $\mathcal{I}$ exist.*

Proof. Let us assume first that we are dealing with algebras.

Let $a \in I_1$ and $b \in I_2$, $I_1, I_2 \in \mathcal{I}$,

(1) $(a+b)+(a+b)^* = a+a^*+b+b^* \in K(I_1,H_0)+K(I_2,H_0) \subseteq P$,
(2) $(a+b)(a+b)^* = aa^*+bb^*+ab^*+ba^* = aa^*+bb^*+ab^*+(ab^*)^* \in K(I_1,H_0)+K(I_2,H_0) \subseteq P$ since $ab^* \in I_1$,
(3) for any $h \in H_0$, $(a+b)h(a+b)^* = aha^*+bhb^*+ahb^*+bha^* = aha^*+bhb^*+ahb^*+(ahb^*)^* \in K(I_1,H_0)+K(I_2,H_0) \subseteq P$ since $ahb^* \in I_1$.

The above assertions (1–3) show that $K(I_1+I_2,H_0) \subseteq P$, i.e. $I_1+I_2 \in \mathcal{I}$. The above work applies verbatim to triple systems just forgetting (2), and it can be easily adapted to pairs too.

Now since a sum of ideals is just a union of finite sums, $\mathcal{I}$ is closed for arbitrary (not necessarily finite) sums of ideals. In particular, the sum of all ideals of $\mathcal{I}$ is the maximum in $\mathcal{I}$ we were looking for. □

When P is a semiprime ideal of R, we have an alternative description of the elements of $\mathcal{I}$ in Lemma 4.1.

Lemma 4.2. *Let $(R,*)$ be an associative system with involution, $H_0 := H_0$ $(R,*)$ an ample subsystem of R, P a semiprime ideal of H_0, and I a $*$-ideal of R. Then, $K(I,H_0) \subseteq P$ if and only if $I \cap H_0 \subseteq P$.*

Proof. Clearly $I \cap H_0 \subseteq P$ implies $K(I,H_0) \subseteq P$, since $K(I,H_0) \subseteq I \cap H_0$ (4.1).

If, conversely, $K(I,H_0) \subseteq P$, and we are dealing with algebras, we have

$$U_{I\cap H_0}(I \cap H_0) \subseteq K(I,H_0) \subseteq P \tag{1}$$

since, for any $a,b \in I \cap H_0$, $U_a b = aba = aba^* \in aH_0a^* \subseteq K(I,H_0)$.

Now (1) implies that the ideal $((I\cap H_0)+P)/P$ of H_0/P has zero cube, which implies $((I\cap H_0)+P)/P = 0$, i.e. $I\cap H_0 \subseteq P$ since H_0/P is semiprime.

The above argument can be easily adapted to the cases of pairs and triple systems. □

Theorem 4.3. *If $(R,*)$ is an associative system with involution, $H_0 := H_0$ $(R,*)$ is an ample subsystem of R, and P is a semiprime (resp. prime) ideal of H_0, then there exists a semiprime (resp. $*$-prime) $*$-ideal I of R such that $P = I \cap H_0$. Moreover, P is a nondegenerate (resp. strongly prime) ideal of H_0.*

Proof. Let us start with the case when P is a semiprime ideal of H_0.

By Lemmas 4.1 and 4.2,

$$\mathcal{I} = \{B \ *\text{-ideal of } R | K(B,H_0) \subseteq P\} = \{B\, *\text{-ideal of } R | B \cap H_0 \subseteq P\}$$

and there exists a maximum I of the elements of $\mathcal{I}$.

We claim that

(1) I is a semiprime ideal of R, i.e. $\widetilde{R} := R/I$ is semiprime,

which is well known to be equivalent to R/I being $*$-semiprime. Indeed, if N is a $*$-ideal of R such that $N^3 \subseteq I$, then the cube of N in $R^{(+)}$, which is spanned by elements of the form aba, for $a, b \in N$ (for $a \in N^\sigma$, $b \in N^{-\sigma}$, $\sigma = \pm$, in the pair case), is also contained in I. In particular, we have

$$U_{N\cap H_0}(N \cap H_0) \subseteq I \cap H_0 \subseteq P$$

in the algebra case, and similar properties in the pair and triple system cases, which yields, as in the proof of Lemma 4.2, that $((N\cap H_0)+P)/P$ is an ideal of zero cube in the semiprime system H_0/P, hence $N \cap H_0 \subseteq P$, i.e. $N \in \mathcal{I}$, which implies $N \subseteq I$.

Let φ: $R \longrightarrow R/I = \widetilde{R}$ be the natural projection.

It is straightforward to check that $\varphi(H_0)$ is an ample subsystem of $\widetilde{R}$. By (1), $\widetilde{R}$ is a semiprime associative system with involution, hence any of its ample subsystems is nondegenerate [7, 0.7(ii)]. In particular,

(2) $\varphi(H_0)$ is nondegenerate.

Now, $\varphi(P)$ is an ideal of $\varphi(H_0)$, and we claim that

(3) $\varphi(H_0)/\varphi(P) \cong H_0/P$, so that $\varphi(P)$ is a semiprime ideal of $\varphi(H_0)$.

Indeed the composition ψ: $H_0 \overset{\varphi}{\longrightarrow} \varphi(H_0) \longrightarrow \varphi(H_0)/\varphi(P)$ is surjective and

$$\begin{aligned} \operatorname{Ker}\psi &= \{x \in H_0 | x + I \in \varphi(P)\} \\ &= \{x \in H_0 | x - p \in I, \quad \text{for some } p \in P\} = P \end{aligned}$$

since $x - p = y \in I$ implies $y \in I \cap H_0 \subseteq P$, hence $x \in P$, and, conversely, $x \in P$ implies $x - x = 0 \in I$.

If $\varphi(P) \neq 0$, then we can use [7, 2.6, 3.6, 3.14], and there exists a nonzero $*$-ideal $\widetilde{B}$ of $\widetilde{R}$ such that $K(\widetilde{B}, \varphi(H_0)) \subseteq \varphi(P)$. Now Lemma 4.2 yields

(4) $\widetilde{B} \cap \varphi(H_0) \subseteq \varphi(P)$.

But $\widetilde{B} = B/I$ for some $*$-ideal B of R strictly containing I, hence (4) and the fact that $I \cap H_0 \subseteq P$ imply $B \cap H_0 \subseteq P$ [for example, in the case of algebras or triple systems, for any $b \in B \cap H_0$, $b + I \in \widetilde{B} \cap \varphi(H_0) \subseteq \varphi(P)$, which implies $b - p \in I$ for some $p \in P$, hence $b - p \in I \cap H_0 \subseteq P$, and $b \in P$], i.e. $B \in \mathcal{I}$, and $B \subseteq I$, which is a contradiction.

We have shown that $\varphi(P) = 0$, which implies $P \subseteq I$, hence $P \subseteq H_0 \cap I \subseteq P$.

Moreover, (3) reads $H_0/P \cong \varphi(H_0)$, hence H_0/P is nondegenerate by (2), i.e. P is a nondegenerate ideal.

If P is a prime ideal of H_0, the above is still valid, and we just need to show that I is a $*$-prime $*$-ideal of R. Indeed, if A and B are $*$-ideals of R such that $ABA \subseteq I$ ($AB \subseteq I$ in the algebra case), then $aba \in I$, for any $a \in A$, $b \in B$ ($aba \in I^\sigma$, for any $a \in A^\sigma$, $b \in B^{-\sigma}$, $\sigma = \pm$, in the pair case). In particular, we have

$$U_{A\cap H_0}(B \cap H_0) \subseteq I \cap H_0 \subseteq P$$

in the algebra case, and similar properties in the pair and triple system cases, which yields, as above, that $((A \cap H_0) + P)/P$ and $((B \cap H_0) + P)/P$ are orthogonal ideals of the prime system H_0/P, hence either $A \cap H_0 \subseteq P$, or $B \cap H_0 \subseteq P$, i.e. either $A \in \mathcal{I}$, or $B \in \mathcal{I}$, which implies $A \subseteq I$ or $B \subseteq I$. □

As a consequence, we can get a similar result for Jordan systems obtained from associative systems by symmetrization.

Corollary 4.4. *If R is an associative system, and P is a semiprime (resp. prime) ideal of $R^{(+)}$, then P is a semiprime (resp. prime) ideal of R. Moreover, P is a nondegenerate (resp. strongly prime) ideal of $R^{(+)}$.*

Proof. Take the associative system $S = R \oplus R^{op}$, equipped with the exchange involution $*$ given by $(x, y)^* = (y, x)$. It is well known and straightforward that

(1) $\psi : R^{(+)} \longrightarrow H(S, *)$, given by $\psi(x) = (x, x)$ is an isomorphism of Jordan systems.

Thus, $R^{(+)}/P \cong H(S, *)/\psi(P)$, and $\psi(P)$ is a semiprime (resp. prime) ideal of $H(S, *)$.

By Theorem 4.3, $\psi(P) = M \cap H(S, *)$, for some semiprime (resp. $*$-prime) $*$-ideal M of S, and $\psi(P)$ is a nondegenerate (resp. strongly prime) ideal, which implies that P is a nondegenerate (resp. strongly prime) ideal of $R^{(+)}$.

Moreover, a semiprime $*$-ideal M of S has necessarily the form $M = I \oplus I$, for some ideal I of R

[$I = \pi_1(M) = \pi_2(M)$, where π_i: $S \longrightarrow R$, $i = 1, 2$, is the natural projection: Clearly $M \subseteq I \oplus I$, but we also claim that

$$S(I \oplus I)S \subseteq M. \tag{2}$$

Indeed, for any $a \in I$, $x, y \in R$, there exists $b \in I$ with $(a, b) \in M$, hence $(x, 0)(a, b)(y, 0) = (xay, 0) \in M$, and we have $RIR \oplus 0 \subseteq M$. Similarly, $0 \oplus RIR \subseteq M$, which implies (1) because $S(I \oplus I)S = RIR \oplus RIR$. On the other hand, (2) implies $(I \oplus I)^3 \subseteq M$, and hence $I \oplus I \subseteq M$ since M is a semiprime ideal of S].

Now, the equality $\psi(P) = M \cap H(S, *)$ readily implies $P = I$, and semiprimeness (resp. $*$-primeness) of M as a $*$-ideal of S is easily seen to be equivalent to semiprimeness (resp. primeness) of I as an ideal of R [$S/M \cong R/I \oplus (R/I)^{op}$, hence we can use [5, 3.6]]. □

The third isomorphism theorem applied to Theorem 4.3 produces the following corollary, which, by Corollary 4.4(1), also applies to Jordan systems J which are quotients of $R^{(+)}$, where R is an associative system.

Corollary 4.5. *Let J be a Jordan system which is a quotient of $H_0(R, *)$, where $(R, *)$ is an associative system with involution. If P is a semiprime (resp. prime) ideal of J, then P is a nondegenerate (resp. strongly prime) ideal of J.*

5. Lie Algebras Linked to Associative Algebras

5.1 We will start this section with the study of Lie algebras of the form $R^{(-)}$, for an associative algebra R (2.5). Every such an algebra is isomorphic to $\mathrm{Skew}(S, *)$ (2.9), for the associative algebra $S = R \oplus R^{op}$ and the exchange involution $*$, and we will study algebras of the form $\mathrm{Skew}(S,*)$ later on in the section. However, we have decided to deal with algebras of the form $R^{(-)}$ independently, because a much more accurate description of semiprime ideals can be obtained in this case.

5.2 Given an associative Φ-algebra R, $\widehat{R}$ will denote the *unital hull* of R. We will make extensive use of $\widehat{R}$ to abbreviate the description of the ideal I of R generated by a subset $S \subseteq R$: $I = \widehat{R}S\widehat{R}$.

Lemma 5.1. *If R is an associative algebra over a ring of scalars Φ with $\frac{1}{2} \in \Phi$, and P is a semiprime ideal of $R^{(-)}$, then $\widehat{R}[P,P]\widehat{R} \subset P$.*

Proof. For any $p \in P$ and $a \in R$,

$$[p,a^2] = [p,a]a + a[p,a] = 2a[p,a] + [[p,a],a],$$

hence

$$2a[p,a] = [p,a^2] - [[p,a],a] \in P.$$

Therefore, since $\frac{1}{2} \in \Phi$, $a[p,a] \in P$, and

$$a[p,b] + b[p,a] \in P, \quad \text{for any } p \in P,\ a,b \in R \tag{1}$$

by linearization. Moreover, if "$\equiv$" denotes congruence modulo P,

$$\begin{aligned}[p^2,a] &= [p,a]p + p[p,a] = [[p,a],p] + 2p[p,a] \equiv_{(1)} [[p,a],p] - 2a[p,p] \\ &= [[p,a],p] \in P.\end{aligned}$$

We have shown $p^2 + P \in Z(R^{(-)}/P)$, but $Z(R^{(-)}/P) = 0$ since $R^{(-)}/P$ is a semiprime Lie algebra and, therefore, $p^2 \in P$ for every $p \in P$. By linearization, $pq + qp \in P$, for any $p,q \in P$. Since we also have $[p,q] \in P$, we have $2pq \in P$, which implies

$$pq \in P, \quad \text{for any } p,q \in P. \tag{2}$$

Now, for any $a \in \widehat{R}$, $[ap,q] = [a,q]p + a[p,q]$ implies that

$$a[p,q] = [ap,q] - [a,q]p \in P + PP \subseteq_{(2)} P,$$

and we have shown that $\widehat{R}[P,P] \subseteq P$. Finally, for any $a,b \in \widehat{R}$, $a[p,q]b = a[[p,q],b] + ab[p,q] = a[[p,b],q] + a[p,[q,b]] + ab[p,q] \in \widehat{R}[P,P] \subset P$. □

5.3 Let R be an associative algebra, and P be an ideal of $R^{(-)}$. The set $\mathcal{I}$ of the ideals of R contained in P is closed under the sum, so that $I := \sum_{M\in\mathcal{I}} M$ is the maximum of the elements in $\mathcal{I}$.

Proposition 5.2. *Let R be an associative algebra over a ring of scalars Φ with $1/2 \in \Phi$, and let P be a semiprime (resp. prime) ideal of $R^{(-)}$. Let I be the maximum of the ideals of R contained in P. Then, I is a semiprime (resp. prime) ideal of R, $P/I = Z((R/I)^{(-)})$, and P is a nondegenerate (resp. strongly prime) ideal of $R^{(-)}$.*

Proof. Let us assume that P is a semiprime ideal of $R^{(-)}$. If $a \in R$ satisfies that $(\widehat{R}a\widehat{R})^2 \subset I$, then $[\widehat{R}a\widehat{R}, \widehat{R}a\widehat{R}] \subset I \subset P$, so, $\widehat{R}a\widehat{R} \subset P$ by semiprimeness of P as an ideal of $R^{(-)}$, which implies that $a \in I$. We have shown that I is a semiprime ideal of R.

If $a+I\in Z(R/I)$, then $[a,R]\subset I\subset P$, which implies that $a+P\in Z(R^{(-)}/P)$ but, $Z(R^{(-)}/P)=0$ by semiprimeness of P again, and $a\in P$. Conversely, if $a\in P$, then $[a,[a,R]]\subset [P,P]\subset I$ by (5.1), which implies that $a+I$ is an absolute zero divisor of $(R/I)^{(-)}$, hence $a+I$ lies in the Kostrikin radical $\mathrm{Kos}((R/I)^{(-)})$ of $(R/I)^{(-)}$. But $\mathrm{Kos}((R/I)^{(-)})=Z((R/I)^{(-)})$ [13, 4.3(2)].

We have that $P/I=Z((R/I)^{(-)})=\mathrm{Kos}((R/I)^{(-)})$, hence

$$R^{(-)}/P\cong (R/I)^{(-)}/P/I=(R/I)^{(-)}/\,\mathrm{Kos}\left((R/I)^{(-)}\right)$$

is a nondegenerate Lie algebra, i.e. P is a nondegenerate ideal.

If P is a prime ideal of $R^{(-)}$, P is, in particular, semiprime, and I is semiprime. Now, if $a,b\in R$ satisfy $a\widehat{R}b\subseteq I$, then $b\widehat{R}a\widehat{R}b\widehat{R}a\subseteq I$, and $b\widehat{R}a\subseteq I$ by semiprimeness of I. Thus, $[\widehat{R}a\widehat{R},\widehat{R}b\widehat{R}]\subseteq I\subseteq P$, and either $\widehat{R}a\widehat{R}\subseteq P$ or $\widehat{R}b\widehat{R}\subseteq P$ by primeness of P. Therefore, either $\widehat{R}a\widehat{R}\subseteq I$ or $\widehat{R}b\widehat{R}\subseteq I$, hence either $a\in I$ or $b\in I$, which shows that I is a prime ideal of R. □

The rest of the section is devoted to taking care of Lie algebras of the form $\mathrm{Skew}(R,*)$, for an associative algebra with involution $(R,*)$.

5.4 Given an associative algebra with involution $(R,*)$, quotients of R by a $*$-ideal I of R are called $*$-quotients. They inherit the involution which will be denoted also $*$, so that the canonical projection $\pi\colon R\longrightarrow R/I$ becomes a $*$-epimorphism: $\pi(x^*)=(\pi(x))^*$, for any $x\in R$.

5.5 Let $(R,*)$ be an associative algebra with involution over a ring of scalars Φ with $\frac{1}{2}\in\Phi$, and P be an ideal of $K=\mathrm{Skew}(R,*)$. The set $\mathcal{I}$ of the $*$-ideals I of R such that $K\cap I=\mathrm{Skew}(I,*)\subseteq P$ is closed under the sum, so that $I:=\sum_{M\in\mathcal{I}}M$ is the maximum of the elements in $\mathcal{I}$. Indeed, every element x in a sum of ideals of $\mathcal{I}$ has the form $x=x_1+\dots x_n$, where $x_1\in I_1,\dots,x_n\in I_n$, for some $I_1,\dots,I_n\in\mathcal{I}$. If, in addition $x\in K$, then $x=\frac{1}{2}(x-x^*)=\frac{1}{2}[(x_1+\cdots+x_n)-(x_1+\cdots+x_n)^*]=\frac{1}{2}(x_1-x_1^*)+\cdots+\frac{1}{2}(x_n-x_n^*)\in(K\cap I_1)+\cdots+(K\cap I_1)\subseteq P$.

The next result is aimed at reducing the study of a prime ideal P of $\mathrm{Skew}(R,*)$ to the particular case in which R is $*$-prime and, at the same time, no nonzero $*$-ideal I of R satisfies $\mathrm{Skew}(I,*)\subseteq P$.

Lemma 5.3. *Let $(R,*)$ be an associative algebra with involution over a ring of scalars Φ with $\frac{1}{2}\in\Phi$. Let I be a $*$-ideal of R and $\pi:R\longrightarrow R/I$ the canonical projection. Then*

(i) $\mathrm{Skew}(I,*)$ *is an ideal of* $\mathrm{Skew}(R,*)$, *and we have* $\mathrm{Skew}(R,*)/\mathrm{Skew}(I,*)\cong\pi(\mathrm{Skew}(R,*))=\mathrm{Skew}(R/I,*)$.

Let P be an ideal of $K=\mathrm{Skew}(R,*)$, and I be the maximum of the elements of $\mathcal{I}$, as in (5.5).

(ii) *If P is a prime (respectively, semiprime) ideal of K, then I is a $*$-prime (respectively, semiprime) ideal of R, and $K/P\cong\mathrm{Skew}(R/I,*)/\pi(P)$, which implies that $\pi(P)$ is a prime (respectively, semiprime) ideal of $\mathrm{Skew}(R/I,*)$. Moreover, there is no nonzero $*$-ideal of R/I whose skew-symmetric elements are contained in $\pi(P)$.*

Proof. (i) Idealness of $\mathrm{Skew}(I,*)$ in $\mathrm{Skew}(R,*)$, and the isomorphism follow from the fact that

$$\mathrm{Ker}(\pi|_{\mathrm{Skew}(R,*)}) = \mathrm{Ker}\,\pi \cap \mathrm{Skew}(R,*) = I \cap \mathrm{Skew}(R,*) = \mathrm{Skew}(I,*).$$

Given any $x \in \mathrm{Skew}(R,*)$, $\pi(x) = x + I \in \mathrm{Skew}(R/I,*)$ since $(x+I)^* = x^*+I = (-x)+I = -(x+I)$. Conversely, if $x+I \in \mathrm{Skew}(R/I,*)$, then we have $x+I = \frac{1}{2}[(x+I)-(x+I)^*] = \frac{1}{2}[(x+I)-(x^*+I)] = \frac{1}{2}(x-x^*)+I = \pi(\frac{1}{2}(x-x^*))$, where $\frac{1}{2}(x-x^*) \in \mathrm{Skew}(R,*)$. This shows the second equality.
(ii) Let us assume that P is prime, and consider M_1, M_2, $*$-ideals of R such that $M_1M_2 \subseteq I$. Then also $M_2M_1 = M_2^*M_1^* = (M_1M_2)^* \subseteq I^* = I$ since M_1, M_2, I are $*$-invariant, and, then

$$\begin{aligned}[\mathrm{Skew}(M_1,*), \mathrm{Skew}(M_2,*)] &\subseteq (M_1M_2 + M_2M_1) \cap \mathrm{Skew}(R,*)\\ &\subseteq I \cap \mathrm{Skew}(R,*) \subseteq P,\end{aligned}$$

which implies, since P is a prime ideal of K, that either $\mathrm{Skew}(M_1,*) \subseteq P$ or $\mathrm{Skew}(M_2,*) \subseteq P$ and, therefore, either $M_1 \subseteq I$ or $M_2 \subseteq I$. This shows that I is a $*$-prime $*$-ideal of R.

If P is semiprime, the argument above can be adapted, taking $M_1 = M_2$, to show that I is a $*$-semiprime ideal of R and, therefore, a semiprime ideal of R.

On the other hand, using (i), $\pi(P)$ is an ideal of $\pi(\mathrm{Skew}(R,*)) = \mathrm{Skew}(R/I,*)$, and, since $\mathrm{Ker}(\pi|_{\mathrm{Skew}(R,*)}) = \mathrm{Skew}(I,*) \subseteq P$,

$$K/P \cong \big(K/\mathrm{Skew}(I,*)\big)/\big(P/\mathrm{Skew}(I,*)\big) \cong \mathrm{Skew}(R/I,*)/\pi(P).$$

Finally, a nonzero $*$-ideal of R/I has the form M/I, where M is a $*$-ideal of R strictly containing I, hence, there exists $x \in \mathrm{Skew}(M,*) \setminus P$, therefore $x+I \in \mathrm{Skew}(M/I,*)$, but $x+I \notin \pi(P) = (P+I)/I$ [otherwise $x+I = p+I$ for some $p \in P$, hence $x-p \in I \cap K = \mathrm{Skew}(I,*) \subseteq P$, and $x \in P$, which is a contradiction]. □

5.6 Let $(R,*)$ be an associative algebra with involution over an algebraically closed field $\mathbb{F}$, and let $K = \mathrm{Skew}(R,*)$. We say that K or $(R,*)$ (or R, for short) is *of class*

- A_n if $R = T \oplus T^{op}$ with the exchange involution, where $T = M_n(\mathbb{F})$, so that $K \cong M_n(\mathbb{F})^{(-)}$.
- BD_n if $R = M_n(\mathbb{F})$ under the transpose involution.
- C_n if $R = M_n(\mathbb{F})$ under the symplectic involution (necessarily $n = 2m$ is even).

5.7 Let $(R,*)$ be a $*$-prime associative algebra with involution with extended centroid C, $*$-extended centroid C^* (notice that $C^* \subset C$), centroid Γ and $*$-centroid Γ^* ($\Gamma^* \subset \Gamma$). We define $\widetilde{R} := RC^* \otimes_{C^*} \mathbb{F}$ where $\mathbb{F}$ is the algebraic closure of the field C^*. By [10, Theorem 8] and [24, Theorem 2.11(b)], $\widetilde{R}$ is a $*$-closed $*$-prime algebra over $\mathbb{F}$ with skew elements $\widetilde{K} = KC^* \otimes_{C^*} \mathbb{F}$ (cf. [24, Section 5]).

In [24], it is shown that the classes listed in (5.6) correspond to PI algebras R, producing PI algebras $\widetilde{R}$. Some of those classes for small n's are exceptions to the general results proved in [24]. Therefore, those cases will need to be treated separately.

Lemma 5.4. *Let $(R, *)$ be an associative algebra with involution over a ring of scalars Φ with $\frac{1}{2} \in \Phi$. Let $K = \mathrm{Skew}(R, *)$ and P be a prime ideal of K such that no nonzero $*$-ideal M of R satisfies $\mathrm{Skew}(M, *) = M \cap K \subseteq P$. Then, R is $*$-prime. If, in addition, $\widetilde{R}$ is not of class A_2 or BD_4, then $P = Z(K)$, and P is a nondegenerate ideal of K.*

Proof. The greatest $*$-ideal of R whose skew symmetric elements are contained in P is zero, hence R is $*$-prime by Lemma 5.3(ii).

Since $[Z(K), Z(K)] = 0 \subseteq P$, and P is a semiprime ideal of K, $Z(K) \subseteq P$.

Since R is $*$-prime, R itself provides a way to express R as a subdirect product of $*$-prime rings, hence [24, Theorem 6.4] applies, yielding that either $0 \equiv P$ in the notation of [24, page 26], which means $P \subseteq Z(K)$, as we wanted to prove, or there exists a nonzero standard Lie ideal of K contained in P, i.e. there exists a $*$-ideal M of R such that $0 \neq [M \cap K, K] \subseteq P$. In this latter situation, $M \cap K = \mathrm{Skew}(M, *) \subseteq P$ by semiprimeness of P as an ideal of K, which is a contradiction.

We have shown $P = Z(K)$, but $Z(K) = \mathrm{Kos}(K)$ by [13, 4.8], hence $K/P = K/\mathrm{Kos}(K)$ is a nondegenerate Lie algebra. □

We will next study the cases not covered by Lemma 5.4.

The following technical result is mostly a part of the Lie folklore.

Lemma 5.5. (i) *If a is an absolute zero divisor in a Lie algebra L over a ring of scalars Φ with $\frac{1}{2} \in \Phi$, then $\mathrm{ad}_{[a,x_1]}\,\mathrm{ad}_{[a,x_2]}\,\mathrm{ad}_{[a,x_3]} = 0$, for any $x_1, x_2, x_3 \in L$. In particular, $\mathrm{ad}^3_{[a,x]} = 0$, for any $x \in L$.*

(ii) *If, in addition, L is a subalgebra of $R^{(-)}$, where R is an associative Φ-algebra, and $y = ra + \sum_{i=1}^n s_i[a, x_i]$, for some $r, s_i \in \widehat{Z(R)}$, $x_i \in L$, then $\mathrm{ad}^3_y(L) = 0$.*

(iii) *If a is an element in a Lie algebra L over a ring of scalars Φ with $\frac{1}{2} \in \Phi$ such that $[a, [a, L]] \subseteq P$, for an ideal P of L, then $\mathrm{ad}_{[a,x_1]}\,\mathrm{ad}_{[a,x_2]}\,\mathrm{ad}_{[a,x_3]}(L) \subseteq P$, for any $x_1, x_2, x_3 \in L$. In particular, $\mathrm{ad}^3_{[a,x]}(L) \subseteq P$, for any $x \in L$.*

(iv) *If, in addition, L is a subalgebra of $R^{(-)}$, where R is an associative Φ-algebra, and $y = ra + \sum_{i=1}^n s_i[a, x_i]$, for some $r, s_i \in \widehat{Z(R)}$, $x_i \in L$, then $\mathrm{ad}^3_y(L) \subseteq SP$, where $S \subseteq \widehat{Z(R)}$ is the linear span over Φ of the monomials of degree less than or equal to three in the elements r and s_i, $i = 1, \ldots, n$.*

Proof. In this proof, for any $x \in L$, we will write $X := ad_x : L \longrightarrow L$. Thus, a being an absolute zero divisor of L just means

$$A^2 = 0. \tag{1}$$

Since $\mathrm{ad}\colon L \longrightarrow \mathrm{End}_\Phi(L)^{(-)}$ is a Lie algebra homomorphism

$$0 = \mathrm{ad}_{[a,[a,x]]} = [A, [A, X]] = AAX + XAA - 2AXA,$$

hence, using (1), for any $x \in L$,

$$AXA = 0, \tag{2}$$

and

$$\begin{aligned} \mathrm{ad}_{[a,x_1]}\,\mathrm{ad}_{[a,x_2]}\,\mathrm{ad}_{[a,x_3]} &= [A, X_1][A, X_2][A, X_3] \\ &= (AX_1 - X_1A)(AX_2 - X_2A)(AX_3 - X_3A) \\ &=_{(1)(2)} -AX_1X_2A(AX_3 - X_3A) =_{(1)(2)} 0, \end{aligned}$$

which shows (i).

To prove (ii), first notice that y is not necessarily an element of L, hence ad_y is a map defined on $R^{(-)}$, but we just want to show that the restriction of ad_y^3 to L vanishes. This fact, taking into account that $r, s_i \in \widehat{Z(R)}$, can be obtained as a consequence of (1), (i), and

$$\begin{aligned} A[A, X_i] &= AAX_i - AX_iA =_{(1)(2)} 0, \\ [A, X_i]A &= AX_iA - X_iAA =_{(1)(2)} 0. \end{aligned}$$

Now, (iii) follows from (i) applied to L/P, while (iv) can be obtained by slightly modifying the proof of (ii). □

The following lemma is aimed at dealing with those cases not covered by Lemma 5.4.

Proposition 5.6. *Let R be an associative algebra over a ring of scalars Φ with $\frac{1}{2} \in \Phi$. If L is a Lie subalgebra of $R^{(-)}$ such that for every $a \in L$, $a^2 \in Z(R)$, then every prime ideal of L is nondegenerate.*

Proof. Before going into studying prime ideals of L, we will establish some consequences of the fact that $a^2 \in Z(R)$ for any $a \in L$. To begin with, it can be readily checked that, for any $a, b \in L$,

$$\mathrm{ad}_a^3(b) = 4a^2[a, b] \in I_{a^2}, \tag{1}$$

where $I_{a^2} = a^2L \cap L$ is an ideal of L. This is the first step to show, by induction on n, that

$$\mathrm{ad}_a^{2n+1}(b) = (4a^2)^n\,[a, b] \in I_{a^{2n}}, \tag{2}$$

where $I_{a^{2n}} = a^{2n}L \cap L$ is also an ideal of L.

For any $x \in L$, $\mathrm{Id}_L(x)$ will denote the ideal of L generated by x. Extending the notation introduced in (1), for any $x \in R$, I_x will denote the set $xL \cap L$. Notice that I_z is an ideal of L if $z \in Z(R)$. Moreover, for any $z \in Z(R)$ and $x, y \in L$ and any operator f in the multiplication algebra of L, $[zy, f(x)] = z[y, f(x)]$. This shows

$$[zL, \mathrm{Id}_L(x)] \subseteq z\,\mathrm{Id}_L(x) = \mathrm{Id}_L(zx), \tag{3}$$

where the last equality makes sense when $zx \in L$. Since an element of I_z has the form zy for some $y \in L$, (3) also shows that, when $zx \in L$, $[I_z, \mathrm{Id}_L(x)] \subseteq z\,\mathrm{Id}_L(x) = \mathrm{Id}_L(zx)$ and, by induction, for any positive integer n, when $z^nx \in L$,

$$\mathrm{ad}_{I_z}^n(\mathrm{Id}_L(x)) \subseteq z^n\,\mathrm{Id}_L(x) = \mathrm{Id}_L(z^nx). \tag{4}$$

Let P be a prime ideal of L.

(5) If $a \in L\backslash P$ is ad-nilpotent modulo P, then $I_{a^2} \subset P$ and, in particular, $\frac{1}{4}\,\mathrm{ad}_a^3(b) =_{(1)} a^2[a,b] \in P$ for every $b \in L$:

Since a is ad-nilpotent module P, there exists a positive integer n such that, for every $b \in L$, $\mathrm{ad}_a^{2n+1}(b) =_{(2)} (4a^2)^n[a,b] \in P$. Since $a \notin P$, $\mathrm{Id}_L(a) \not\subseteq P$, hence $[\mathrm{Id}_L(a), \mathrm{Id}_L(a)] \not\subseteq P$ because P is a semiprime ideal of L. Semiprimeness of P also implies that there must be $b \in L$ such that $[a,b] \notin P$, hence $\mathrm{Id}_L([a,b]) \not\subseteq P$. Therefore,

$$\mathrm{ad}^n_{I_{a^2}}(\mathrm{Id}_L([a,b])) \subseteq_{(4)} \mathrm{Id}_L(a^{2n}[a,b]) \subseteq P$$

implies $I_{a^2} \subset P$ by primeness of P as an ideal of L.

Let $x \in L$.

(6) If a and $[a,x]$ are ad-nilpotent module P with $[a,x] \in L \setminus P$ (this implies $a \in L\backslash P$), then $I_{ax+xa} \subset P$:

Notice that $ax + xa = (a+x)^2 - a^2 - x^2 \in Z(R)$, so that I_{ax+xa} is an ideal of L. By semiprimeness of P, the fact that $[a,x] \notin P$ implies the existence of $y \in L$ such that $[[a,x],y] \notin P$, as above. Now, two elements in I_{ax+xa} have the form $(ax+xa)t_1$, $(ax+xa)t_2$, for some $t_1, t_2 \in L$. For any operator f in the multiplication algebra of L

$$\begin{aligned}
&\Big[(ax+xa)t_1, \big[(ax+xa)t_2, f([[a,x],y])\big]\Big] \\
&= (ax+xa)^2\Big[t_1, \big[t_2, f([[a,x],y])\big]\Big] \text{ (because } ax+xa \in Z(R)) \\
&= \Big[t_1, \big[t_2, f((ax+xa)^2[[a,x],y])\big]\Big] \text{ (because } ax+xa \in Z(R)) \\
&= \Big[t_1, \big[t_2, f\big(([a,x]^2 + 4a^2x^2)[[a,x],y]\big)\big]\Big] \\
&\quad \text{(because } (ax+xa)^2 = [a,x]^2 + 4a^2x^2 \text{ since } x^2, a^2 \in Z(R)) \\
&= \Big[t_1, \big[t_2, f([a,x]^2[[a,x],y])\big]\Big] + \Big[t_1, \big[t_2, f(4a^2x^2[[a,x],y])\big]\Big] \\
&= \Big[t_1, \big[t_2, f([a,x]^2[[a,x],y])\big]\Big] + \Big[t_1, \big[t_2, f(4a^2[x^2[a,x],y])\big]\Big] \\
&\quad \text{(because } x^2 \in Z(R)) \\
&= \Big[t_1, \big[t_2, f([a,x]^2[[a,x],y])\big]\Big] + \Big[t_1, \big[t_2, f(a^2[\mathrm{ad}_x^2([a,x]),y])\big]\Big] \text{ (using (1))} \\
&= \Big[t_1, \big[t_2, f([a,x]^2[[a,x],y])\big]\Big] + \Big[t_1, \big[t_2, f([\mathrm{ad}_x^2(a^2[a,x]),y])\big]\Big] \text{ (using } a^2 \in Z(R)) \\
&\in \Big[t_1, \big[t_2, f(P)\big]\Big] + \Big[t_1, \big[t_2, f([\mathrm{ad}_x^2(P),y])\big]\Big] \text{ (using (5))} \subseteq P.
\end{aligned}$$

We have shown $\big[I_{ax+xa}, \big[I_{ax+xa}, \mathrm{Id}_L([a,x],y])\big]\big] \subseteq P$, which implies $I_{ax+xa} \subseteq P$ by primeness of P since $\mathrm{Id}_L([a,x],y]) \not\subseteq P$.

(7) Given $x_i, y_i \in L$, $i = 1, 2, \ldots, n$ there exists an ideal I of L such that

$$[(x_{i_1}y_{i_1} + y_{i_1}x_{i_1}) \cdots (x_{i_r}y_{i_r} + y_{i_r}x_{i_r})L, I] \subseteq \bigcap_{j=1}^{r} I_{x_{i_j}y_{i_j}+y_{i_j}x_{i_j}},$$

for any subset $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$. Moreover, if $x_i, y_i \in L\backslash P$, for all $i = 1, \ldots, n$, the ideal I can be chosen not contained in P:

For any $x, y \in L$, the equalities $xy + yx = (x+y)^2 - x^2 - y^2$ and $xy + yx = -(x-y)^2 + x^2 + y^2$ allow us to use (1) and (3) to obtain

$$\Big[(xy+yx)L, \big[[\mathrm{Id}_L([x,L]), \mathrm{Id}_L([y,L])], \mathrm{Id}_L([x+y,L])\big]\Big]$$
$$\subseteq \big[(xy+yx)L, \mathrm{Id}_L([x,L]) \cap \mathrm{Id}_L([y,L]) \cap \mathrm{Id}_L([x+y,L])\big] \subseteq L,$$

and

$$\Big[(xy+yx)L, \big[[\mathrm{Id}_L([x,L]), \mathrm{Id}_L([y,L])], \mathrm{Id}_L([x-y,L])\big]\Big]$$
$$\subseteq \big[(xy+yx)L, \mathrm{Id}_L([x,L]) \cap \mathrm{Id}_L([y,L]) \cap \mathrm{Id}_L([x-y,L])\big] \subseteq L.$$

If $x, y \in L \backslash P$, then either $x + y \in L\backslash P$ or $x - y \in L\backslash P$, and then we can define either

$$M = \big[[\mathrm{Id}_L([x,L]), \mathrm{Id}_L([y,L])], \mathrm{Id}_L([x+y,L])\big]$$

or

$$M = \big[[\mathrm{Id}_L([x,L]), \mathrm{Id}_L([y,L])], \mathrm{Id}_L([x-y,L])\big],$$

respectively, and M will be an ideal of L not contained in P, by primeness of P [the elements x, y and $x+y$ or $x-y$ are not contained in P, so their Lie products with L cannot be contained in P by semiprimeness of P, as in (5); thus M is the product of three ideals, each one not contained in P].

Applying the above to every pair of elements $x_i, y_i \in L$, we obtain, for any $i = 1, \ldots, n$, an ideal M_i of L such that

$$[(x_i y_i + y_i x_i)L, M_i] \subseteq L, \tag{8}$$

and if, in addition, $x_i, y_i \in L\backslash P$, then $M_i \not\subseteq P$.

We define

$$N = \Big[M_1, \big[M_2, \ldots, [M_{n-1}, M_n] \cdots\big]\Big].$$

and

$$I = N^{(n)}.$$

Notice that I is an ideal of L, and, by primeness of P, $I \not\subseteq P$ if $x_i, y_i \in L\backslash P$, for all $i = 1, \ldots, n$.

We will now show by induction on r that, for any $1 \le r \le n$, and any subset $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$,

$$[(x_{i_1} y_{i_1} + y_{i_1} x_{i_1}) \cdots (x_{i_r} y_{i_r} + y_{i_r} x_{i_r})L, N^{(r)}] \subseteq \bigcap_{j=1}^{r} I_{x_{i_j} y_{i_j} + y_{i_j} x_{i_j}}, \tag{9}$$

which will prove (3) since $I \subseteq N^{(r)}$ for any $1 \le r \le n$.

For $r = 1$,

$$[(x_{i_1} y_{i_1} + y_{i_1} x_{i_1})L, N^{(1)}] = (x_{i_1} y_{i_1} + y_{i_1} x_{i_1})[L, N^{(1)}] \subseteq (x_{i_1} y_{i_1} + y_{i_1} x_{i_1})L$$

since $x_{i_1}y_{i_1}+y_{i_1}x_{i_1}\in Z(R)$, and also

$$\begin{aligned}[(x_{i_1}y_{i_1}+y_{i_1}x_{i_1})L, N^{(1)}] &\subseteq [(x_{i_1}y_{i_1}+y_{i_1}x_{i_1})L, N]\\ &\subseteq [(x_{i_1}y_{i_1}+y_{i_1}x_{i_1})L, M_{i_1}]\subseteq L\end{aligned}$$

by (8). Thus, we have shown

$$[(x_{i_1}y_{i_1}+y_{i_1}x_{i_1})L, N^{(1)}]\subseteq I_{x_{i_1}y_{i_1}+y_{i_1}x_{i_1}}.$$

If we assume that (9) is true for some $1\le r<n$, then using that $x_{i_{r+1}}y_{i_{r+1}}+y_{i_{r+1}}x_{i_{r+1}}\in Z(R)$,

$$\begin{aligned}&\Big[(x_{i_1}y_{i_1}+y_{i_1}x_{i_1})\dots(x_{i_r}y_{i_r}+y_{i_r}x_{i_r})(x_{i_{r+1}}y_{i_{r+1}}+y_{i_{r+1}}x_{i_{r+1}})L, N^{(r+1)}\Big]\\ &\subseteq (x_{i_{r+1}}y_{i_{r+1}}+y_{i_{r+1}}x_{i_{r+1}})\Big[(x_{i_1}y_{i_1}+y_{i_1}x_{i_1})\dots(x_{i_r}y_{i_r}+y_{i_r}x_{i_r})L, N^{(r+1)}\Big]\\ &= (x_{i_{r+1}}y_{i_{r+1}}+y_{i_{r+1}}x_{i_{r+1}})\Big[(x_{i_1}y_{i_1}+y_{i_1}x_{i_1})\dots(x_{i_r}y_{i_r}+y_{i_r}x_{i_r})L, [N^{(r)}, N^{(r)}]\Big]\\ &\subseteq (x_{i_{r+1}}y_{i_{r+1}}+y_{i_{r+1}}x_{i_{r+1}})\Big[[(x_{i_1}y_{i_1}+y_{i_1}x_{i_1})\dots(x_{i_r}y_{i_r}+y_{i_r}x_{i_r})L, N^{(r)}], N^{(r)}\Big]\\ &= \Big[\Big[(x_{i_1}y_{i_1}+y_{i_1}x_{i_1})\dots(x_{i_r}y_{i_r}+y_{i_r}x_{i_r})L, N^{(r)}\Big], (x_{i_{r+1}}y_{i_{r+1}}+y_{i_{r+1}}x_{i_{r+1}})N^{(r)}\Big]\\ &\subseteq \left(\bigcap_{j=1}^{r} I_{x_{i_j}y_{i_j}+y_{i_j}x_{i_j}}\right)\cap I_{x_{i_{r+1}}y_{i_{r+1}}+y_{i_{r+1}}x_{i_{r+1}}} = \bigcap_{j=1}^{r+1} I_{x_{i_j}y_{i_j}+y_{i_j}x_{i_j}},\end{aligned}$$

by the induction assumption and the fact that

$$\begin{aligned}&(x_{i_{r+1}}y_{i_{r+1}}+y_{i_{r+1}}x_{i_{r+1}})N^{(r)} = (x_{i_{r+1}}y_{i_{r+1}}+y_{i_{r+1}}x_{i_{r+1}})[N^{(r-1)}, N^{(r-1)}]\\ &\quad\subseteq (x_{i_{r+1}}y_{i_{r+1}}+y_{i_{r+1}}x_{i_{r+1}})[N^{(0)}, N^{(0)}] = [(x_{i_{r+1}}y_{i_{r+1}}+y_{i_{r+1}}x_{i_{r+1}})N, N]\\ &\quad\subseteq [(x_{i_{r+1}}y_{i_{r+1}}+y_{i_{r+1}}x_{i_{r+1}})L, M_{r+1}]\\ &\quad\subseteq L\cap(x_{i_{r+1}}y_{i_{r+1}}+y_{i_{r+1}}x_{i_{r+1}})L = I_{x_{i_{r+1}}y_{i_{r+1}}+y_{i_{r+1}}x_{i_{r+1}}}\end{aligned}$$

by (8).

The final part of the proof consists of showing that having nonzero absolute zero divisors of L/P yields a contradiction.

Let $a\in L\backslash P$ such that $[a,[a,L]]\subseteq P$, i.e. $0\ne a+P\in L/P$ is an absolute zero divisor of L/P.

(10) If $x\in L$ satisfies $[a,x]\notin P$, then $I_{ax+xa}\subset P$:
By Lemma 5.5(iii) $\mathrm{ad}^3_{[a,x]}(L)\subset P$, and we can apply (6).

(11) Every element of $\mathrm{Id}_L(a)$ is ad-nilpotent of index ≤ 3 module P:
Given any $y_1,y_2\in L$, $[y_2,a]=y_2a-ay_2=2y_2a-(ay_2+y_2a)$, and using the fact that $ay_2+y_2a\in Z(R)$,

$$\begin{aligned}[y_1,[y_2,a]] &= [y_1, 2y_2a-(ay_2+y_2a)] = 2[y_1,y_2a] = 2y_1y_2a-2y_2ay_1\\ &= 2(y_1y_2+y_2y_1)a-2y_2(ay_1+y_1a)\\ &= 2(y_1y_2+y_2y_1)a-2(ay_1+y_1a)y_2\end{aligned}$$

since $ay_1+y_1a\in Z(R)$. Now, by induction on n, any product

$$\Big[y_1,\Big[y_2,\dots,[y_n,a]\dots\Big]\Big] = \mathrm{ad}_{y_1}\mathrm{ad}_{y_2}\dots\mathrm{ad}_{y_n}(a),$$

for $y_1, \ldots, y_n \in L$ has the form

$$\mathrm{ad}_{y_1}\,\mathrm{ad}_{y_2}\ldots\mathrm{ad}_{y_n}(a) = \lambda a + \sum_{k=1}^{h} \mu_k [a, b_k] + \sum_{l=1}^{m} \gamma_l c_l$$

where $b_k \in \{y_1, \ldots, y_n\}$, $c_l \in L$, $\lambda, \mu_k, \gamma_l \in \widehat{Z(R)}$ are products of elements of the form $ay_i + y_i a$ and $y_j y_i + y_i y_j$, and al least one of the factors of each γ_l is of the form $ay_i + y_i a$. If some $y_i \in P$, then $\mathrm{ad}_{y_1}\,\mathrm{ad}_{y_2}\ldots\mathrm{ad}_{y_n}(a) \in P$, and, if $[y_i, a] \in P$ for some i, then we can use the fact that ad_{y_i} is a derivation to write

$$\begin{aligned}
\mathrm{ad}_{y_1}\,\mathrm{ad}_{y_2}\ldots\mathrm{ad}_{y_n}(a) &= \mathrm{ad}_{y_1}\ldots\mathrm{ad}_{y_{i-1}}\,\mathrm{ad}_{y_i}\big(\mathrm{ad}_{y_{i+1}}\ldots\mathrm{ad}_{y_n}(a)\big) \\
&= \mathrm{ad}_{y_1}\ldots\mathrm{ad}_{y_{i-1}}\,\mathrm{ad}_{[y_i,y_{i+1}]}\,\mathrm{ad}_{y_{i+2}}\ldots\mathrm{ad}_{y_n}(a) \\
&\quad + \mathrm{ad}_{y_1}\ldots\mathrm{ad}_{y_{i-1}}\,\mathrm{ad}_{y_{i+1}}\,\mathrm{ad}_{[y_i,y_{i+2}]}\ldots\mathrm{ad}_{y_n}(a) \\
&\qquad\vdots \\
&\quad + \mathrm{ad}_{y_1}\ldots\mathrm{ad}_{y_{i-1}}\,\mathrm{ad}_{y_{i+1}}\ldots\mathrm{ad}_{y_{n-1}}\,\mathrm{ad}_{[y_i,y_n]}(a) \\
&\quad + \mathrm{ad}_{y_1}\ldots\mathrm{ad}_{y_{i-1}}\,\mathrm{ad}_{y_{i+1}}\ldots\mathrm{ad}_{y_n}([y_i, a]),
\end{aligned}$$

where the last term lies in P, i.e. we can write $\mathrm{ad}_{y_1}\,\mathrm{ad}_{y_2}\ldots\mathrm{ad}_{y_n}(a)$ as a sum of similar elements of smaller length modulo P. Therefore, any element $t \in \mathrm{Id}_L(a)$ has the form

$$t = \lambda a + \sum_{k=1}^{h} \mu_k [a, b_k] + \sum_{l=1}^{m} \gamma_l c_l + p$$

where $b_k \in \{y_1, \ldots, y_n\}$, $c_l \in L$, $p \in P$, $\lambda, \mu_k, \gamma_l \in \widehat{Z(R)}$ are products of elements of the form $ay_i + y_i a$ and $y_j y_i + y_i y_j$, and at least one of the factors of each γ_l is of the form $ay_i + y_i a$, for some set $\{y_1, \ldots, y_n\}$ of elements in L such that all $y_i, [y_i, a] \in L \backslash P$. By (7), there exists an ideal I of L such that $I \not\subseteq P$ and

$$[\lambda L, I] \subseteq L, \quad [\mu_k L, I] \subseteq L, \quad [\gamma_l L, I] \subseteq L, \quad [\pi L, I] \subseteq L \tag{12}$$

for any monomial π of degree less than or equal to three in λ, μ_k, and γ_l, and any $k = 1, \ldots, h$, $l = 1, \ldots, m$. Also, using (3) and (9),

$$[\gamma_l L, I] \subseteq P, \tag{13}$$

for all $l = 1, \ldots, m$. Hence, for any $y \in I$

$$\begin{aligned}
[t, y] &= \left[\lambda a + \sum_{k=1}^{h} \mu_k [a, b_k] + \sum_{l=1}^{m} \gamma_l c_l + p, y\right] \\
&\subseteq_{(13)} \left[\lambda a + \sum_{k=1}^{h} \mu_k [a, b_k], y\right] + P = \mathrm{ad}_s(y) + P,
\end{aligned} \tag{14}$$

where $p \in P$, $s = \lambda a + \sum_{k=1}^{h} \mu_k [a, b_k] \in R$, and $\mathrm{ad}_s(I) \subseteq L$ by (12). This latter fact implies that, for $r \geq 1$, we also have

$$\begin{aligned}\mathrm{ad}_s(I^{(r)}) &= \mathrm{ad}_s([I^{(r-1)}, I^{(r-1)}]) \subseteq [\mathrm{ad}_s(I^{(r-1)}), I^{(r-1)}] \subseteq [L, I^{(r-1)}] \\ &\subseteq I^{(r-1)},\end{aligned} \tag{15}$$

using that ad_s is a derivation.

For any $x \in L$, $y \in I$,

$$\big[[t, x], y\big] = \big[[t, y], x\big] + \big[t, [x, y]\big] \subseteq \big[[s, y], x\big] + \big[s, [x, y]\big] + P = \big[[s, x], y\big] + P \tag{16}$$

using (14) on y and $[x, y]$. We will prove by induction on n that for any $x \in L$, $y \in I^{(n)}$,

$$[\mathrm{ad}_t^n(x), y] \subseteq [\mathrm{ad}_s^n(x), y] + P. \tag{17}$$

The case $n = 1$ is (16), so let us assume that (17) is true for some $n \geq 0$ and prove it for $n + 1$: If we assume that $y \in I^{(n+1)}$, then

$$\begin{aligned}\big[\mathrm{ad}_t^{n+1}(x), y\big] &= \big[[t, \mathrm{ad}_t^n(x)], y\big] \\ &\subseteq_{(16)} \big[[s, \mathrm{ad}_t^n(x)], y\big] + P = \Big[\mathrm{ad}_s\big(\mathrm{ad}_t^n(x)\big), y\Big] + P \\ &= \mathrm{ad}_s\big([\mathrm{ad}_t^n(x), y]\big) - \big[\mathrm{ad}_t^n(x), \mathrm{ad}_s(y)\big] + P \\ &\quad \text{(using } \mathrm{ad}_s \text{ is a derivation)} \\ &\subseteq \mathrm{ad}_s\big([\mathrm{ad}_s^n(x), y]\big) - \big[\mathrm{ad}_s^n(x), \mathrm{ad}_s(y)\big] + P \\ &\quad \text{(by the induction assumption since } \mathrm{ad}_s(y) \in I^{(n)} \text{ by (15))} \\ &= [\mathrm{ad}_s^{n+1}(x), y] + P,\end{aligned}$$

using again that ad_s is a derivation.

In particular, we have

$$\Big[\mathrm{ad}_t^3(L), I^{(3)}\Big] \subseteq_{(17)} \Big[\mathrm{ad}_s^3(L), I^{(3)}\Big] + P \subseteq_{\text{Lemma 5.5(iv)}} [SP, I^{(3)}] + P,$$

where $S \subseteq \widehat{Z(R)}$ is the span over Φ of the monomials of degree less than or equal to three in the elements $\lambda, \mu_k, k = 1, \ldots, n$, appearing in the description of t, but

$$\begin{aligned}[SP, I^{(3)}] + P &=_{(S \subseteq \widehat{Z(R)})} [P, SI^{(3)}] + P = \big[P, S[I^{(2)}, I^{(2)}]\big] + P \\ &=_{(S \subseteq \widehat{Z(R)})} \big[P, [SI^{(2)}, I^{(2)}]\big] + P \subseteq \big[P, [SL, I]\big] \\ &\quad + P \subseteq_{(12)} [P, L] + P \subseteq P,\end{aligned}$$

and we havee shown that $[\mathrm{ad}_t^3(L), I^{(3)}] \subseteq P$, which readily implies $[\mathrm{Id}_L(\mathrm{ad}_t^3(L)), I^{(3)}] \subseteq P$. Hence $\mathrm{Id}_L(\mathrm{ad}_t^3(L)) \subseteq P$ by primeness of P, using that $I \not\subseteq P$ yields $I^{(3)} \not\subseteq P$ by semiprimeness of P. This shows that $\mathrm{ad}_t^3(L) \subseteq P$, proving (11).

(18) $\mathrm{Id}_L(a) \subseteq P$, which is a contradiction since we were assuming $a \in L \backslash P$:

Given $b, c, d \in \mathrm{Id}_L(a)$, $[c, d] = cd - dc = 2cd - (cd + dc)$, and, using the fact that $cd + dc \in Z(R)$,

$$\begin{aligned}[b,[c,d]] &= [b, 2cd - (cd+dc)] = 2[b, cd] = 2bcd - 2cdb \\ &= 2(bc+cb)d - 2c(db+bd) \\ &= 2(bc+cb)d - 2(db+bd)c \end{aligned} \tag{19}$$

since $db+bd \in Z(R)$. On the other hand, if $\mathrm{Id}_L(a) \not\subseteq P$, then $[\mathrm{Id}_L(a), \mathrm{Id}_L(a)] \not\subseteq P$ by semiprimeness of P, and we can find $c, d \in \mathrm{Id}_L(a)$ such that $[c, d] \notin P$, which implies $c \notin P$, and $d \notin P$. Since $[x, \mathrm{Id}_L(a)] \subseteq P$ readily implies $[\mathrm{Id}_L(x), \mathrm{Id}_L(a)] \subseteq P$, which yields $\mathrm{Id}_L(x) \subseteq P$ by primeness of P, the submodules

$$\begin{aligned} A_1 &:= \{y \in \mathrm{Id}_L(a) \mid [c, y] \in P\}, \\ A_2 &:= \{y \in \mathrm{Id}_L(a) \mid [d, y] \in P\}, \\ A_3 &:= \{y \in \mathrm{Id}_L(a) \mid \big[[c, d], y\big] \in P\} \end{aligned}$$

of $\mathrm{Id}_L(a)$ are proper. Hence $A_1 \cup A_2 \cup A_3 \neq \mathrm{Id}_L(a)$, because $\frac{1}{2} \in \Phi$, and we can find $b \in \mathrm{Id}_L(a)$ such that

$$[b, c], [b, d], \big[[b, [c, d]\big] \in L \backslash P. \tag{20}$$

In particular $b, c, d \in L\backslash P$, and we can apply (3) to find an ideal I of L not contained in P, such that

$$\Big[\big[b, [c, d]\big], I\Big] \subseteq_{(19)} I_{bc+cb} + I_{bd+db}. \tag{21}$$

Because of (11) and (20), we can apply (6) to obtain that $I_{bc+cb} \subseteq P$ and $I_{bd+db} \subseteq P$. Therefore, (21) yields $\Big[\big[b, [c, d]\big], I\Big] \subseteq P$, which readily implies $\Big[\mathrm{Id}_L\big(\big[b, [c, d]\big]\big), I\Big] \subseteq P$, and this is impossible by primeness of P.

We have shown that L/P is a nondegenerate Lie algebra. □

Lemma 5.7. *If L is a subalgebra of* $\mathrm{Skew}(R, *)$*, for a $*$-prime associative algebra R over a ring of scalars Φ with $\frac{1}{2} \in \Phi$, such that $\widetilde{R}$ is of class A_2 or BD_4, and P is a prime ideal of L, then L/P is nondegenerate.*

Proof. If $\widetilde{R}$ is of class A_2, L is a Φ-subalgebra of $M := M_2(\mathbb{F})^{(-)}$, for an algebraically closed field $\mathbb{F}$ and $L/(L \cap Z(M))$ imbeds in $M/Z(M) \cong \mathrm{sl}_2(\mathbb{F})$ [the epimorphism $M \longrightarrow \mathrm{sl}_2(\mathbb{F})$ given by $A \mapsto A - \frac{1}{2}t(A)I_2$ has kernel $Z(M)$]. Thus, we can see $L/(L \cap Z(M))$ as a subalgebra of $\mathrm{sl}_2(\mathbb{F}) \leq M_2(\mathbb{F})^{(-)} =: R_1^{(-)}$ and, for any $a \in L/(L \cap Z(M))$, $a^2 \in Z(R_1)$, so that Proposition 5.6 applies to $L/(L \cap Z(M))$.

Now, given a prime ideal P of L, since $[L \cap Z(M), L \cap Z(M)] = 0 \subseteq P$, $L \cap Z(M) \subseteq P$, and

$$L/P \cong (L/(L \cap Z(M)))/(P/(L \cap Z(M)).$$

Hence, $P/(L \cap Z(M))$ is a prime ideal of $L/(L \cap Z(M))$, and L/P is nondegenerate by Proposition 5.6.

If $\widetilde{R}$ is of class BD_4, L is a Φ-subalgebra of $\mathrm{Skew}(Q, *)$, where $Q = M_4(\mathbb{F})$, and $*$ is the transpose involution, for an algebraically closed field $\mathbb{F}$. For any $1 \leq i, j, \leq 4$, $i \neq j$, let $E_{ij} = e_{ij} - e_{ji}$, where e_{ij} is the usual matrix unit. Then, it can be readily seen that $\mathrm{Skew}(Q, *)$ is the direct sum of the ideals I_1 and I_2 where (there is a misprint in Herstein's counterexample [14, page 40]) I_1 is the vector subspace spanned by $\{a := E_{12} - E_{34}, b := E_{13} + E_{24}, c := E_{14} - E_{23}\}$ and I_2 is the vector subspace spanned by $\{\widehat{a} := E_{12} + E_{34}, \widehat{b} := E_{13} - E_{24}, \widehat{c} := E_{14} + E_{23}\}$. Moreover, I_i is isomorphic to $\mathrm{sl}_2(\mathbb{F})$, and hence it is simple:

> By direct computation, the multiplication table of I_1 is given by $[a, b] = 2c$, $[a, c] = -2b$, $[b, c] = 2a$, so that the basis $h = ia$, $e = \frac{1}{2}(ib - c)$, $f = \frac{1}{2}(ib + c)$, where i is an element in $\mathbb{F}$ such that $i^2 = -1$, behaves like the natural basis of $\mathrm{sl}_2(\mathbb{F})$ ($[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$).
> Analogously, the multiplication table of I_2 is given by $[\widehat{a}, \widehat{b}] = -2\widehat{c}$, $[\widehat{a}, \widehat{c}] = 2\widehat{b}$, $[\widehat{b}, \widehat{c}] = -2\widehat{a}$, and we can take $h = i\widehat{a}$, $e = \frac{1}{2}(i\widehat{b} + \widehat{c})$, $f = \frac{1}{2}(i\widehat{b} - \widehat{c})$ to show $I_2 \cong \mathrm{sl}_2(\mathbb{F})$.

Thus, L can be seen as a subalgebra of $\mathrm{sl}_2(\mathbb{F}) \oplus \mathrm{sl}_2(\mathbb{F}) \leq \big(M_2(\mathbb{F}) \oplus M_2(\mathbb{F})\big)^{(-)} =: R_1^{(-)}$, and, for any $a \in L$, $a^2 \in Z(R_1)$. Therefore, Proposition 5.6 applies to L, and L/P is nondegenerate, for any prime ideal P of L. □

The next result is a consequence of Lemmas 5.4 and 5.7.

Theorem 5.8. *Let $(R, *)$ be an associative algebra with involution over a ring of scalars Φ with $\frac{1}{2} \in \Phi$. If $L = \mathrm{Skew}(R, *)$ and P a semiprime (resp. prime) ideal of L, then P is a nondegenerate (resp. strongly prime) ideal of L.*

Proof. By [1, Th. 1.1], every semiprime Lie algebra is a subdirect product of prime Lie algebras, hence, given a semiprime ideal P of L, there exist prime ideals P_α of L such that $P = \cap_{\alpha \in A} P_\alpha$ and L/P is a subdirect product of the algebras L/P_α. Since a subdirect product of nondegenerate Lie algebras is nondegenerate, we just need to prove the theorem in the case when P is a prime ideal of L.

Thus, since P is prime, we can use Lemma 5.3 and assume that R is $*$-prime with no non nonzero $*$-ideal of R whose skew-symmetric elements are contained in P. If $\tilde{R}$ is not of class A_2 or BC_4, then L/P is nondegenerate by Lemma 5.4. Otherwise, Lemma 5.7 applies, and L/P is also nondegenerate. □

As in the previous section, we can extend Theorem 5.8 to quotients of algebras of the form $\mathrm{Skew}(R, *)$. By (5.1), it also applies to Lie algebras L which are quotients of $R^{(-)}$, for an associative algebra R.

Corollary 5.9. *Let L be a Lie algebra over a ring of scalars Φ with $\frac{1}{2} \in \Phi$ which is a quotient of* $\mathrm{Skew}(R, *)$*, where* $(R, *)$ *is an associative algebra with involution. If P is a semiprime* (*resp. prime*) *ideal of L, then P is a nondegenerate* (*resp. strongly prime*) *ideal of L.*

References

[1] Amitsur, S.A.: A general theory of radicals. Am. J. Math. **76**, 126–136 (1954)

[2] D'Amour, A.: Quadratic Jordan systems of hermitian type. J. Algebra **149**, 197–233 (1992)

[3] Anquela, J.A., Cortés, T.: Primitive Jordan pairs and triple systems. J. Algebra **184**, 632–678 (1996)

[4] Anquela, J.A., Cortés, T.: Local and subquotient inheritance of simplicity in Jordan systems. J. Algebra **240**, 680–704 (2001)

[5] Anquela, J.A., Cortés, T.: Growing hearts in associative systems. J. Algebra Appl. **11**(5), 1250083 (11 p.) (2012)

[6] Anquela, J.A., Cortés, T.: Imbedding Lie algebras in strongly prime algebras. Commun. Algebra **41**, 2467–2475 (2013)

[7] Anquela, J.A., Cortés, T., García, E.: Herstein's theorems and simplicity of hermitian Jordan systems. J. Algebra **246**, 193–214 (2001)

[8] Anquela, J.A., Cortés, T., Loos, O., McCrimmon, K.: An elemental characterization of strong primeness in Jordan systems. J. Pure Appl. Algebra **109**, 23–36 (1996)

[9] Anquela, J.A., Cortés, T.: McCrimmon, K.: Imbedding Jordan Systems in Primitive Systems (to appear)

[10] Baxter, W.E., Martindale, W.S. III.: The extended centroid in ∗-prime rings. Commun. Algebra **10**(8), 847–874 (1982)

[11] Beidar, K.I., Mikhalev, A.V.: The structure of nondegenerate alternative algebras. J. Sov. Math. **47**, 2525–2536 (1989)

[12] García, E.: Tits–Kantor–Koecher algebras of strongly prime hermitian Jordan pairs. J. Algebra **277**, 559–571 (2004)

[13] García, E., Gomez Lozano, M.: A characterization of the Kostrikin radical of a Lie algebra. J. Algebra **346**, 266–283 (2011)

[14] Herstein, I.N.: Topics in Ring Theory. The University of Chicago Press, Chicago (1969)

[15] Humphreys, J.E.: Introduction to Lie Algebras and Representation Theory. Springer, New York (1980)

[16] Jacobson, N.: Structure and Representations of Jordan Algebras, vol. 39. Am. Math. Soc. Colloq. Publ., Providence (1968)

[17] Jacobson, N.: Lie Algebras. Dover Publications Inc., New York (1979)

[18] Kostrikin, A.I., Zelmanov, E.I.: A theorem on sandwich algebras. Trudy Mat. Inst. Steklov **183**, 106–111 (1990)

[19] Loos, O.: Jordan pairs. Lecture Notes in Mathematics, vol. 460. Springer, New York (1975)

[20] McCrimmon, K.: On Herstein's theorems relating Jordan and associative algebras. J. Algebra **13**, 382–392 (1969)

[21] McCrimmon, K.: Absolute zero divisors and local nilpotence in alternative algebras. Proc. Amer. Math. Soc **47**(2), 293–299 (1975)

[22] McCrimmon, K.: A Taste of Jordan Algebras. Springer, New York (2004)

[23] McCrimmon, K., Zelmanov, E.: The structure of strongly prime quadratic Jordan algebras. Adv. Math. **69**(2), 133–222 (1988)

[24] Martindale, W.S. III., Miers, C.R.: Herstein's Lie theory revisited. J. Algebra **98**, 14–37 (1986)

[25] Medvedev, Yu.A.: Absolute zero divisors in special finitely generated Jordan algebras. Sib. Math. J. **29**(2), 325–327 (1988)

[26] Medvedev, Yu.A., Zelmanov, E.I.: Some counter examples in the theory of Jordan algebras. In: González, S., Myung, H.C. (eds.) Nonassociative Algebraic Models (Universidad de Zaragoza, 1989), pp. 1–16. Nova Science Publishers, New York (1992)

[27] Pchelintsev, S.V.: Prime algebras and absolute zero divisors. Math. USSR Izv. **28**(1), 79–98 (1987)

[28] Shestakov, I.P.: Absolute zero divisors and radicals of finitely generated alternative algebras. Algebra Log. **15**(5), 582–602 (1976)

[29] Skosyrskii, V.G.: Prime Jordan algebras and the Kantor construction. Algebra Log. **33**(3), 169–179 (1995)

[30] Strade, H.: Simple Lie Algebras over Fields of Positive Characteristic. I Structure Theory. De Gruyter, Berlin (2004)

[31] Zelmanov, E.I.: Absolute zero-divisors in Jordan pairs and Lie algebras. Math. USSR Sb. **40**(4), 549–565 (1981)

[32] Zelmanov, E.I.: Absolute zero-divisors and algebraic Jordan algebras. Sib. Math. J. **23**(6), 841–854 (1982)

[33] Zhevlakov, K.A., Slinko, A.M., Shestakov, I.P., Shirshov, A.I.: Rings that are Nearly Associative. Academic Press, New York (1982)

José A. Anquela and Teresa Cortés
Departamento de Matemáticas
Universidad de Oviedo
C/Calvo Sotelo s/n
33007 Oviedo, Spain
e-mail: anque@orion.ciencias.uniovi.es

Teresa Cortés
e-mail: cortes@orion.ciencias.uniovi.es

Esther García
Departamento de Matemática Aplicada
Ciencia e Ingeniería de los Materiales y Tecnología Electrónica
Universidad Rey Juan Carlos
28933 Móstoles
Madrid, Spain
e-mail: esther.garcia@urjc.es

Miguel Gómez Lozano
Departamento de Álgebra, Geometría y Topología
Universidad de Málaga
29071 Málaga, Spain
e-mail: magomez@agt.cie.uma.es

Received: May 8, 2014.
Revised: August 22, 2014.
Accepted: November 3, 2014.