Communications in Algebra

ISSN: 0092-7872 (Print) 1532-4125 (Online) Journal homepage: https://www.tandfonline.com/loi/lagb20

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To cite this article: Esther García, Miguel Gómez Lozano & Guillermo Vera De Salas (2020) Jordan supersystems related to Lie superalgebras, Communications in Algebra, 48:3, 992-1000, DOI: 10.1080/00927872.2019.1670196

To link to this article: https://doi.org/10.1080/00927872.2019.1670196

Published online: 07 Oct 2019.

COMMUNICATIONS IN ALGEBRA®
2020, VOL. 48, NO. 3, 992–1000
https://doi.org/10.1080/00927872.2019.1670196

Jordan supersystems related to Lie superalgebras

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ABSTRACT

Given a Lie superalgebra and an even ad-nilpotent element of index ≤ 3, one can obtain a Jordan superalgebra attached to that element; inspired by that construction we build a Jordan superpair attached to an odd ad-nilpotent element of index ≤ 4. We introduce inner ideals for Lie superalgebras, and we prove that the associated subquotients are Jordan superpairs. All three constructions agree when considering abelian inner ideals generated by one element.

ARTICLE HISTORY

Received 3 April 2019
Revised 11 September 2019
Communicated by J. L. Gomez Pardo

KEYWORDS

Ad-nilpotent element; Jordan superstructure; Lie superalgebra; subquotient

2000 MATHEMATICS SUBJECT CLASSIFICATION

17B60; 17C50

1. Introduction

Local algebras of Jordan systems were introduced by Meyberg [11], used by Zelmanov and revisited by D'Amour and McCrimmon in their classification of linear and quadratic Jordan systems [3, 4, 14]. Ever since their introduction, they have played a prominent role in the structure theory of Jordan systems, mainly due to the fact that nice properties flow between the system and their local algebras (see for example [1, 2] or [12]).

In [6], the first two authors together with A. Fernández López attached a Jordan algebra to any Lie algebra L with an ad-nilpotent element x of index less than or equal to three. Their construction extended the fact that every Lie algebra with an $\mathfrak{sl}_2$-triple $(e, [e, f], f)$ is automatically 5-graded relative to the eigenspaces of $\mathrm{ad}_{[e,f]}$ and $L_2 = \mathrm{ad}_e^2(L)$ is a unital Jordan algebra. Although their object imitates the construction of a "local" algebra of a Lie algebra, they did not get a Lie algebra again but a Jordan algebra, so this object was called the Jordan algebra of L at x. Furthermore, any $\mathbb{Z}$-graded Lie algebra $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ comes together with a Jordan pair $V = (L_{-n}, L_n)$ and any element x of L_n is ad-nilpotent of index less than or equal to three, so one can construct the local algebra of V at x (in the sense of Meyberg [11]) and this Jordan algebra coincides with the Jordan algebra of L at x.

The Jordan algebras of Lie algebras, together with their extension to subquotients (Jordan pairs) associated to abelian inner ideals of Lie algebras, have provided a new way of connecting the Lie and the Jordan settings. For example, they were used by E. Zelmanov in his proof of the Lie version of the Kurosh problem: every finitely generated PI Lie algebra over a field of characteristic zero in which every commutator in generators is ad-nilpotent is nilpotent [15, §2]. They

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were also used by J. Hennig in her classification of ad-integrable simple, locally finite Lie algebras over algebraically closed fields of characteristic >3 [8, Theorem 2]. This construction was also mimicked in [13] to construct a quasi-Jordan algebra from a Leibniz algebra and an ad-nilpotent element of index less than or equal to three.

In this article, we extend these ideas to the supersetting, and Jordan superstuctures are attached to Lie superalgebras at ad-nilpotent homogeneous elements. When the ad-nilpotent element is even, one directly obtains a Jordan superalgebra by using the Grassmann envelope. Nevertheless, when the ad-nilpotent element is odd, our construction cannot be so directly imitated. In that case, we consider a triple product, and after doubling and slightly modifying this product we obtain a Jordan superpair.

We also generalize the notion of subquotient to the Lie supersetting. It comes attached to an abelian Lie inner ideal of a Lie superalgebra, and it is indeed a Jordan superpair. Moreover, in the particular case of an abelian inner ideal of the form $[x,[x,L]]$, the subquotient agrees with the Jordan superobject obtained in the previous sections.

We expect that Jordan superstructures and subquotients attached to Lie superalgebras prove to be a useful tool to get a notion of socle and a Wedderburn–Artin theory for Lie superalgebras, as in the non-supersetting [5, 7].

The article is organized as follows. When x is even, we easily obtain a Jordan superalgebra by using the Grassmann envelope. But when we deal with an odd ad-nilpotent element x of index less than or equal to 4 we first define a triple product in $[x,[x,L]]$, and then we double this triple and change a sign in one of the associated triple products to get a Jordan superpair. We introduce subquotients associated to abelian inner ideals of Lie superalgebras and show that they are Jordan superpairs. Finally, we show that the Jordan superalgebras/superpairs obtained previously agree with the subquotients associated to abelian inner ideals of the form $[x,[x,L]]$.

2. Preliminaries

In this article, we will deal with Lie superalgebras and Jordan superstructures defined over a ring of scalars Φ with $\frac{1}{2}, \frac{1}{3}$.

A Φ-superalgebra $(L = L_{\bar 0} + L_{\bar 1}, [\ ,\ ])$ is a Lie superalgebra if the product is superanticommutative and satisfies the Jacobi superidentity:

- $[a,b] = -(-1)^{|a||b|}[b,a]$
- $[[a,b],c] + (-1)^{|a|(|b|+|c|)}[[b,c],a] + (-1)^{|c|(|a|+|b|)}[[c,a],b] = 0$

for every homogeneous $a,b,c \in L$. Here $|a|$ denotes the parity of a homogeneous element $a \in L$, i.e., $|a| = 0$ if $a \in L_{\bar 0}$ and $|a| = 1$ if $a \in L_{\bar 1}$.

Let A be an associative superalgebra, i.e., a $\mathbb{Z}_2$-graded associative algebra. Then A with the product $[a,b] = ab - (-1)^{|a||b|}ba$ for $a,b \in A_{\bar 0} \cup A_{\bar 1}$ is a Lie superalgebra denoted by $A^{(-)}$. In particular, if L is a Lie superalgebra then End L becomes an associative superalgebra and $(\text{End } L)^{(-)}$ with product $[f,g] = fg - (-1)^{|f||g|}gf$ for homogeneous elements $f,g \in \text{End } L$ becomes a Lie superalgebra. The set ad L of adjoint maps is a Lie superideal of $(\text{End } L)^{(-)}$, so if we denote by capital letters the adjoint maps associated to elements, i.e., $A = \text{ad}_a, B = \text{ad}_b$, etc., we have $[A,B] = AB - (-1)^{|a||b|}BA$ for homogeneous elements $a,b \in L_{\bar 0} \cup L_{\bar 1}$.

A Φ-superalgebra $(J = J_{\bar 0} + J_{\bar 1}, \cdot)$ is a Jordan superalgebra if the product is supercommutative and satisfies the Jordan superidentity:

- $a \cdot b = (-1)^{|a||b|} b \cdot a$

- $(a\cdot b)\cdot(c\cdot d)+(-1)^{|b||c|}(a\cdot c)\cdot(b\cdot d)+(-1)^{|b||d|+|c||d|}(a\cdot d)\cdot(b\cdot c)$
  $=((a\cdot b)\cdot c)\cdot d+(-1)^{|b||c|+|b||d|+|c||d|}((a\cdot d)\cdot c)\cdot b$
  $+(-1)^{|a||b|+|a||c|+|a||d|+|c||d|}((b\cdot d)\cdot c)\cdot a$

for homogeneous $a,b,c,d\in J$, see [10].

A pair of $\mathbb{Z}_2$-graded Φ-modules $V=(V^+,V^-)$ is a (linear) Jordan superpair if there exist two trilinear maps $\{,,\}^\sigma: V^\sigma\times V^{-\sigma}\times V^\sigma\to V^\sigma, \sigma=\pm$, both supersymmetric in the outer variables, and that satisfy (JSP15):

$$\{a,b,\{c,d,e\}^\sigma\}^\sigma=\{\{a,b,c\}^\sigma,d,e\}^\sigma-(-1)^{|a||b|+|a||c|+|b||c|}\{c,\{b,a,d\}^{-\sigma},e\}^\sigma$$
$$+(-1)^{|a||c|+|a||d|+|b||c|+|b||d|}\{c,d,\{a,b,e\}^\sigma\}^\sigma,\ \sigma=\pm$$

for homogeneous $a,c,e\in V^\sigma$ and homogeneous $b,d\in V^{-\sigma},\sigma=\pm$.

We will also deal with (1, 1)-Jordan supertriples, which are a particular case (ϵ,δ)-Freudenthal–Kantor supertriple systems, $\epsilon=\pm1,\delta=\pm1$ [9, §3]. We say that a $\mathbb{Z}_2$-graded Φ-module $M=M_{\bar 0}+M_{\bar 1}$ with a graded triple product $\{,,\}: M\times M\times M\to M$ is a (1, 1)-Jordan supertriple if

- $\{a,b,c\}=(-1)^{|a||b|+|a||c|+|b||c|}\{c,b,a\}$ and
- $\{a,b,\{c,d,e\}\}=\{\{a,b,c\},d,e\}+(-1)^{|a||b|+|a||c|+|b||c|}\{c,\{b,a,d\},e\}$
  $+(-1)^{|a||c|+|a||d|+|b||c|+|b||d|}\{c,d,\{a,b,e\}\}$

for homogeneous elements $a,b,c,d,e\in M$. The second identity resembles (JSP15) but there is a change of sign in the second summand of its right side. Notice that every (1, 1)-Jordan supertriple M with triple product $\{,,\}$ gives rise to a Jordan superpair $V=(V^+,V^-)=(M,M)$ with products $\{a,b,c\}^+:=\{a,b,c\}$ and $\{b,c,d\}^-:=-\{b,c,d\}$ for every $a,c\in V^+$ and $b,d\in V^-$.

3. Jordan superalgebras at even homogeneous ad-nilpotent elements

3.1. Let $L=L_{\bar 0}+L_{\bar 1}$ be a Lie superalgebra, and let $x\in L_0$ such that $\mathrm{ad}_x^3L=0$. Such an element will be called Jordan element of L. In the Φ-module $[x,[x,L]]$ we can define a new product

$$[x,[x,a]]\cdot[x,[x,b]]=\frac{1}{2}[x,[x,[a,[x,b]]]].$$

If we denote by $\bar a:=[x,[x,a]]$ and $\bar b:=[x,[x,b]]$ the product just means that $\bar a\cdot\bar b=\frac{1}{2}\overline{[a,[x,b]]}$.

The (nonassociative) algebra $([x,[x,L]],\cdot)$ is $\mathbb{Z}_2$-graded with homogeneous parts $[x,[x,L]]_{\bar 0}=[x,[x,L_{\bar 0}]]$ and $[x,[x,L]]_{\bar 1}=[x,[x,L_{\bar 1}]]$. The parity of an homogeneous element $\bar a$ coincides with the parity of a as an element in the Lie superalgebra L, i.e., $|\bar a|=|a|$ for every homogeneous element $a\in L_{\bar 0}\cup L_{\bar 1}$. This nonassociative superalgebra is a Jordan superalgebra by using the Grassmann envelope. We recall this fact in the following proposition.

Proposition 3.2. *Let $L=L_{\bar 0}+L_{\bar 1}$ be a Lie superalgebra and $x\in L_{\bar 0}$ be a Jordan element. Then $([x,[x,L]],\cdot)$ is a Jordan superalgebra.*

Proof. Let us check that the Grassmann envelope of $[x,[x,L]]$ is a Jordan algebra with the induced product. Let us consider $\tilde x=x\otimes 1\in G(L)$, which is a Jordan element of the Lie algebra $G(L)$. By [6, Theorem 2.4(ii)] and [6, Remark 2.14] we can consider the Jordan algebra $[\tilde x,[\tilde x,G(L)]]$ of $G(L)$ at $\tilde x$ with product

$$[\tilde x,[\tilde x,\tilde a]]\cdot[\tilde x,[\tilde x,\tilde b]]=\frac{1}{2}[\tilde x,[\tilde x,[\tilde a,[\tilde x,\tilde b]]]]$$

for any $\tilde a,\tilde b\in G(L)$.

The map $\varphi : [\tilde{x}, [\tilde{x}, G(L)]] \to G([x, [x, L]])$ given by $\varphi([\tilde{x}, [\tilde{x}, a \otimes \xi_{i_1}\xi_{i_2}...\xi_{i_k}]]) = [x, [x, a]] \otimes \xi_{i_1}\xi_{i_2}...\xi_{i_k}$ is an isomorphism, so $G([x, [x, L]])$ is a Jordan algebra, giving that $[x, [x, L]]$ is a Jordan superalgebra. □

Remark 3.3. The induced triple product on $[x, [x, L]]$ is given by

$$\{\bar{a}, \bar{b}, \bar{c}\} = (-1)^{|b||c|}\frac{1}{4}X^2ACX^2(b)$$

for homogeneous $\bar{a}, \bar{b}, \bar{c} \in [x, [x, L]]$. Indeed,

$$\begin{aligned}
4\{\bar{a}, \bar{b}, \bar{c}\} &= 4\Big(\bar{a} \cdot (\bar{b} \cdot \bar{c}) + (-1)^{|b||c|+|a||c|}\bar{c} \cdot (\bar{a} \cdot \bar{b}) - (-1)^{|a||b|}\bar{b} \cdot (\bar{a} \cdot \bar{c})\Big) \\
&= X^2\big[a, \big[x, \big[b, [x, c]\big]\big]\big] + (-1)^{|b||c|+|a||c|}X^2[c, [x, [a, [x, b]]]] \\
&\quad - (-1)^{|a||b|}X^2\big[b, [x, [a, [x, c]]]\big] \\
&= \overline{[[a, x], [[b, x], c]]} + (-1)^{|b||c|+|a||c|}\overline{[[c, x], [[a, x], b]]} - (-1)^{|a||b|}\overline{[[b, x], [[a, x], c]]} \\
&= X^2\big[\big[[a, x], [b, x]\big], c\big] + (-1)^{|a||b|+|b||c|+|a||c|}X^2[c, [x, [[b, x], a]]] \\
&= X^2[[a, [x, [b, x]]], c] = (-1)^{|c|(|a|+|b|)}X^2[c, [a, [x, [x, b]]]] = (-1)^{|b||c|}X^2ACX^2(b).
\end{aligned}$$

Remark 3.4. An equivalent construction of the Jordan superalgebra $([x, [x, L]], \cdot)$ is the following: in L define a new product by $a \bullet b = \frac{1}{2}[a, [x, b]]$ for any $a, b \in L$, and denote $L^{(x)}$ the (nonassociative) $\mathbb{Z}_2$-graded algebra $(L, \bullet)$, with $L^{(x)}_{\bar{0}} = L_{\bar{0}}$ and $L^{(x)}_{\bar{1}} = L_{\bar{1}}$. If we define $\mathrm{Ker}_L(x) := \{a \in L | [x, [x, a]] = 0\}$, then $\mathrm{Ker}_L(x)$ is the kernel of the $\mathbb{Z}_2$-graded algebra homomorphism $\varphi : L^{(x)} \to [x, [x, L]]$ given by $\varphi(a) = [x, [x, a]]$, so $L^{(x)}/\mathrm{Ker}_L(x)$ and $[x, [x, L]]$ are isomorphic as Jordan superalgebras.

4. Jordan superalgebras at odd homogeneous ad-nilpotent elements

Now we turn to odd ad-nilpotent elements. Notice that for every homogeneous element $x \in L_{\bar{1}}$ we have $\mathrm{ad}_{[x,x]} = 2\mathrm{ad}_x^2$. When dealing with ad-nilpotent elements of $L_{\bar{1}}$ we will require $\mathrm{ad}_x^4 L = 0$. In this case the element $y = [x, x] \in L_0$ verifies $\mathrm{ad}_y^2 = \mathrm{ad}_{[x,x]}^2 = 4\mathrm{ad}_x^4 = 0$.

Remark 4.1. Given such an element $x \in L_{\bar{1}}$ with $\mathrm{ad}_x^4 = 0$, if we consider the Φ-module $[x, [x, L]]$ and we define the bilinear product as in 3.1 ($[x, [x, a]] \cdot [x, [x, b]] = \frac{1}{2}[x, [x, [a, [x, b]]]]$ for every $a, b \in L$) then $[x, [x, L]]$ is $\mathbb{Z}_2$-graded with $[x, [x, L]]_{\bar{0}} = [x, [x, L_{\bar{1}}]]$ and $[x, [x, L]]_{\bar{1}} = [x, [x, L_{\bar{0}}]]$. The parity of the homogeneous elements of $[x, [x, L]]$ changes and $|[x, [x, a]]| = |a| + \bar{1}$ for any homogeneous element $a \in L_{\bar{0}} \cup L_{\bar{1}}$. Moreover,

$$\bar{a} \cdot \bar{b} = -(-1)^{|\bar{a}||\bar{b}|}\bar{b} \cdot \bar{a}$$

for homogeneous $\bar{a} = [x, [x, a]], \bar{b} = [x, [x, b]] \in [x, [x, L]]$, i.e., $([x, [x, L]], \cdot)$ is super-anticommutative. To avoid this situation and get a Jordan superstructure, we define a trilinear product on $[x, [x, L]$.

4.2. For an element $x \in L_1$ with $\mathrm{ad}_x^4 = 0$, we consider the trilinear map $\{,,\}$ on $[x, [x, L]]$ defined by

$$\{\bar{a}, \bar{b}, \bar{c}\} := \frac{1}{4}\big[[x, [x, a]], \big[b, [x, [x, c]]\big]\big] = \frac{1}{4}X^2ABX^2(c) \tag{4.1}$$

for every homogeneous $\bar{a} = [x, [x, a]], \bar{b} = [x, [x, b]]$ and $\bar{c} = [x, [x, c]] \in [x, [x, L]]$ (notice that $[[x, [x, a]], [b, [x, [x, c]]]] = \frac{1}{4}[[[x, x], a], [b, [[x, x], c]]] = X^2ABX^2(c)$ because $\mathrm{ad}_{[x,x]}\mathrm{ad}_b\mathrm{ad}_{[x,x]} = 0$). The

Φ-module $[x,[x,L]]$ is $\mathbb{Z}_2$-graded with respect to this trilinear product and $[x,[x,L]]_{\bar{i}} = [x,[x,L_{\bar{i}}]], \bar{i} \in \{\bar{0},\bar{1}\}$.

We have that

$$\big[[x,[x,a]],\big[b,[x,[x,c]]\big]\big] = \big[\big[[x,[x,a]],b\big],[x,[x,c]]\big]$$

for every $a,b,c \in L_{\bar{0}} \cup L_{\bar{1}}$ because $[[x,[x,a]],[x,[x,c]]] = \frac{1}{4}[[[x,x],a],[[x,x],c]] = 0$ since $[x,x]$ has $\mathrm{ad}^2_{[x,x]} = 0 = \mathrm{ad}_{[x,x]}\mathrm{ad}_a\mathrm{ad}_{[x,x]} = 0$. This implies that the triple product is supersymmetric in the outer variables:

$$\{\bar{a},\bar{b},\bar{c}\} = (-1)^{|a||b|+|a||c|+|b||c|}\{\bar{c},\bar{b},\bar{a}\}. \tag{4.2}$$

Moreover,

$$\{\bar{a},\bar{b},\bar{c}\} = (-1)^{|b||c|}\{\bar{a},\bar{c},\bar{b}\} \tag{4.3}$$

because $4\ \{\bar{a},\bar{b},\bar{c}\} = [[x,[x,a]],[b,[x,[x,c]]]] = (-1)^{|b||c|+1}[[x,[x,a]],[[x,[x,c]],b]] = \frac{1}{4}(-1)^{|b||c|+1}[[[x,x],a],\ [[[x,x],c],b]] = \frac{1}{4}(-1)^{|b||c|}[[x,x],[a,[c,[[x,x],b]]]] \ = (-1)^{|b||c|}X^2ACX^2(b) = 4(-1)^{|b||c|}\{\bar{a},\bar{c},\bar{b}\}$. From equations (4.2) and (4.3) we get that the triple product defined in (4.1) is supercommutative on its three variables.

Lemma 4.3. *For a homogeneous element $x \in L_{\bar{1}}$ with $\mathrm{ad}_x^4 = 0$, the trilinear map given in (4.1) satisfies*

$$\{\bar{a},\bar{b},\{\bar{c},\bar{d},\bar{e}\}\} = \{\{\bar{a},\bar{b},\bar{c}\},\bar{d},\bar{e}\} + (-1)^{|a||b|+|a||c|+|b||c|}\{\bar{c},\{\bar{b},\bar{a},\bar{d}\},\bar{e}\}$$
$$+ (-1)^{|a||c|+|a||d|+|b||c|+|b||d|}\{\bar{c},\bar{d},\{\bar{a},\bar{b},\bar{e}\}\} \quad (*)$$

for every $a,b,c,d,e \in L_{\bar{0}} \cup L_{\bar{1}}$.

Proof. For every $a,b,c,d,e \in L_{\bar{0}} \cup L_{\bar{1}}$,

$$\begin{aligned}
8\ \{\bar{a},\bar{b},\{\bar{c},\bar{d},\bar{e}\}\} &= \big[\big[[x,[x,a]],b\big],\big[\big[[x,[x,c]],d\big],[x,[x,e]]\big]\big] \\
&= \big[\big[\big[[x,[x,a]],b\big],\big[[x,[x,c]],d\big]\big],[x,[x,e]]\big] \\
&\quad + (-1)^{(|a|+|b|)(|c|+|d|)}\big[\big[[x,[x,c]],d\big],\big[\big[[x,[x,a]],b\big],[x,[x,e]]\big]\big] \\
&= \big[\big[\big[\big[[x,[x,a]],b\big],[x,[x,c]]\big],d\big],[x,[x,e]]\big] \\
&\quad + (-1)^{(|a|+|b|)|c|}\big[\big[[x,[x,c]],\big[\big[[x,[x,a]],b\big],d\big]\big],[x,[x,e]]\big] \\
&\quad + (-1)^{(|a|+|b|)(|c|+|d|)}\big[\big[[x,[x,c]],d\big],\big[\big[[x,[x,a]],b\big],[x,[x,e]]\big]\big] \\
&= 8\ \{\{\bar{a},\bar{b},\bar{c}\},\bar{d},\bar{e}\} + (-1)^{(|a|+|b|)|c|}\big[\big[[x,[x,c]],\big[\big[[x,[x,a]],b\big],d\big]\big],[x,[x,e]]\big] \\
&\quad + (-1)^{|a||c|+|a||d|+|b||c|+|b||d|}8\ \{\bar{c},\bar{d},\{\bar{a},\bar{b},\bar{e}\}\}
\end{aligned}$$

Let us see that $(-1)^{(|a|+|b|)|c|}[[[x,[x,c]],[[[x,[x,a]],b],d]],[x,[x,e]]]$ coincides with the second summand on the right side of equality (*): from the definition of the triple product,

$$\big[\big[[x,[x,c]],\big[\big[[x,[x,a]],b\big],d\big]\big],[x,[x,e]]\big] = 4\ \{\bar{c},\overline{\big[\big[[x,[x,a]],b\big],d\big]},\bar{e}\}$$

and

$$\begin{aligned}
\overline{[[[x,[x,a]],b],d]} &= [x,[x,[[[x,[x,a]],b],d]]] = (-1)^{1+|b|}[[[x,[x,a]],[x,b]],[x,d]] \\
&\quad + (-1)^{|b|}[[x,[x,a],[x,b]],[x,d]] + [[[x,[x,a]],b],[x,[x,d]]] \\
&= 4\ \{\bar{a},\bar{b},\bar{d}\} = (-1)^{|a||b|}4\ \{\bar{b},\bar{a},\bar{d}\}
\end{aligned}$$

hence

$$(-1)^{(|a|+|b|)|c|}\left[[[x,[x,c]],\left[[[x,[x,a]],b],d\right]],[x,[x,e]]\right] \\ = (-1)^{|a||b|+|a||c|+|b||c|}8\ \{\bar{c},\{\bar{b},\bar{a},\bar{d}\},\bar{e}\}$$

and we have shown (*). □

Remark 4.4. We have just shown that when $x \in L_{\bar{1}}$ has $\mathrm{ad}_x^4 = 0, [x,[x,L]]$ with the trilinear map $\{,,\}$ given in (4.1) is a (1,1)-Jordan supertriple in the sense of [9, §3]. As mentioned in Section 2, if we double $[x,[x,L]]$ and twist one of the triple products we have that $([x,[x,L]],[x,[x,L]])$ is a Jordan superpair.

4.5. Another Jordan structure can be defined from an ad-nilpotent element $x \in L_{\bar{1}}$: suppose that $x \in L_{\bar{1}}$ has $\mathrm{ad}_x^6 = 0$. Then $y = [x,x] \in L_{\bar{0}}$ is a Jordan element ($\mathrm{ad}_y^3 = (2\mathrm{ad}_x^2)^3 = 0$), and we can define a Jordan superalgebra on the Φ-module $[y,[y,L]] = \mathrm{ad}_x^4 L$ as in 3.2. The product is now given by

$$[y,[y,a]]\cdot[y,[y,b]] = \frac{1}{2}[y,[y,[a,[y,b]]]]$$

or, equivalently,

$$\mathrm{ad}_x^4 a\cdot \mathrm{ad}_x^4 b = \frac{1}{4}\mathrm{ad}_x^4\left[a,\mathrm{ad}_x^2 b\right].$$

5. Subquotients associated to Abelian inner ideals

5.1. Let $L = L_{\bar{0}} + L_{\bar{1}}$ be a Lie superalgebra. We say that $B = B_{\bar{0}} + B_{\bar{1}} \subset L$ is an inner ideal of L if $[B,[B,L]] \subset B$, and B is abelian if $[B,B] = 0$. Inner ideals can be easily produced from homogeneous ad-nilpotent elements.

Example 5.2. Let $L = L_{\bar{0}} + L_{\bar{1}}$ a Lie superalgebra and let $x \in L_{\bar{0}}$ with $\mathrm{ad}_x^3 = 0$ or $x \in L_{\bar{1}}$ with $\mathrm{ad}_x^4 = 0$. Then

$$[x] := [x,[x,L]] \quad (x) := \Phi x + [x,[x,L]]$$

are inner ideals of L. Moreover, $[x]$ is an abelian inner ideal.

Conversely, given an abelian inner ideal $B = B_{\bar{0}} + B_{\bar{1}}$, any homogeneous $b \in B_{\bar{0}}$ is a Jordan element and gives rise to the inner ideals $[b]$ and (b) contained in B. If $b \in B_{\bar{1}}$ then $0 = [b,b]$ implies $0 = \mathrm{ad}_{[b,b]} = 2\mathrm{ad}_b^2$ so $[b] = 0$ and $(b) = \Phi b$.

Proposition 5.3. *Let L be a Lie superalgebra and B an abelian inner ideal of L. Let us consider* $\mathrm{Ker}\ B := \{x \in L | [B,[B,x]] = 0\}$. *Then $(B, L/\mathrm{Ker}\ B)$ is a Jordan superpair with products:*

$$\{a,\bar{x},b\} := [a,[x,b]] = [[a,x],b] \\ \{\bar{x},a,\bar{y}\} := \overline{[x,[a,y]]} = \overline{[[x,a],y]}$$

for $a,b \in B$ and $x,y \in L$ (here $\bar{x},\bar{y},\overline{[x,[a,y]]}$ and $\overline{[[x,a],y]}$ denote equivalence classes in the quotient $L/\mathrm{Ker}\ B$). This Jordan superpair is called the subquotient of L associated to B.

Proof. First notice that $[a,[x,b]] = [[a,x],b]$ and $\overline{[x,[a,y]]} = \overline{[[x,a],y]}$ for every $a,b \in B$ and every $x,y \in L$ because B is abelian and the definition of Ker B.

The products are well defined: clearly $\{a,\bar{0},b\} = 0$, and if we take homogeneous $\bar{x},\bar{y} \in L/\mathrm{Ker}\ B$ with $\bar{x} = \bar{0}$ or $\bar{y} = \bar{0}$ then for homogeneous $a,b,c \in L$ we have that

$$\left[b,\left[c,[x,[a,y]]\right]\right] = \left[b,\left[[c,x],[a,y]\right]\right] + (-1)^{|c||x|}\left[b,\left[x,\left[c,[a,y]\right]\right]\right] \\ = (-1)^{|c||x|}\left[[b,x],[c,[a,y]]\right] + (-1)^{|c||x|+|b||x|}\left[x,\left[b,\left[c,[a,y]\right]\right]\right] = 0$$

Let us see that the triple products are supersymmetric in the outer variables:

$$\begin{aligned}\{a,\bar{x},b\} &= [a,[x,b]] = (-1)^{1+|x||b|}[a,[b,x]] = (-1)^{|b||x|+|a|(|b|+|x|)}[[b,x],a]] \\ &= (-1)^{|b||x|+|a||b|+|a||x|}[b,[x,a]] = (-1)^{|b||x|+|a||b|+|a||x|}\{b,\bar{x},a\} \\ \{\bar{x},a,\bar{y}\} &= \overline{[x,[a,y]]} = \overline{[[x,a],y]} + (-1)^{|x||a|}\overline{[a,[x,y]]} \\ &= (-1)^{|x||y|+|x||a|+|y||a|}\overline{[y,[a,x]]} = (-1)^{|x||y|+|x||a|+|y||a|}\{y,a,\bar{x}\}\end{aligned}$$

Let us prove (JSP15). For homogeneous $a,b,c \in B$ and homogeneous $x,y,z \in L$,

- $\{a,\bar{x},\{b,\bar{y},c\}\} = \big[[a,x],[[b,y],c]\big]$
$= \big[\big[\big[[a,x],b\big],y\big],c\big] + (-1)^{(|a|+|x|)b}\big[\big[b,[[a,x],y]\big],c\big] + (-1)^{(|b|+|y|)(|a|+|x|)}\big[[b,y],[[a,x],c]\big]$
$= \{\{a,\bar{x},b\},\bar{y},c\} - (-1)^{|a||b|+|x||b|+|a||x|}\{b,\{\bar{x},a,\bar{y}\},c\}$
$+ (-1)^{(|b|+|y|)(|a|+|x|)}\{b,\bar{y},\{a,\bar{x},c\}\}.$
- $\{\bar{x},a,\{\bar{y},b,\bar{z}\}\} = \overline{[[x,a],[[y,b],z]]}$
$= \overline{[[[[x,a],y],b],z]} + (-1)^{|y|(|x|+|a|)}\overline{[[y,[[x,a],b]],z]} + (-1)^{(|y|+|b|)(x+a)}\overline{[[y,b],[[x,a],z]]}$
$= \{\{\bar{x},a,\bar{y}\},b,\bar{z}\} - (-1)^{|y||x|+|y||a|+|a||x|}\{\bar{y},\{a,\bar{x},b\},\bar{z}\}$
$+(-1)^{(|y|+|b|)(|x|+|a|)}\{\bar{y},b,\{\bar{x},a,\bar{z}\}\}$

Therefore, $(B, L/\mathrm{Ker}\ B)$ is a Jordan superpair. □

Remark 5.4. Let $x \in L_{\bar{0}}$ be a Jordan element or $x \in L_{\bar{1}}$ with $\mathrm{ad}_x^4 = 0$. Then $B = [x] = [x,[x,L]]$ is an abelian inner ideal and we can build the subquotient $([x], L/\mathrm{Ker}[x])$. In this particular case, for homogeneous $a,b,c \in L$ the triple product

$$\begin{aligned}\{[x,[x,a]],\ b+\mathrm{Ker}[x],\ [x,[x,c]]\} &= \big[[x,[x,a]],\big[b,[x,[x,c]]\big]\big] \\ &= (-1)^{|b||c|+1+|x|}X^2ACX^2(b)\end{aligned}$$

coincides, up to a scalar, with the triple product we have already defined in $[x]$, see Remark 3.3 when $[x]$ is even and 4.2 when x is odd. In the following result we are going to prove that the Jordan superpair structures defined in this section and in the previous ones coincide.

Corollary 5.5. *Let L be a Lie superalgebra, take $x \in L_{\bar{0}}$ with $\mathrm{ad}_x^3 = 0$ or $x \in L_{\bar{1}}$ with $\mathrm{ad}_x^4 = 0$, and let us consider the subquotient associated to the abelian inner ideal [x].*

(a) *When $x \in L_{\bar{0}}$, if we consider the Jordan superpair structure induced on $([x,[x,L]],[x,[x,L]])$ by Remark 3.3, then the pair of maps*

$$(\Psi_1,\Psi_2) : ([x,[x,L]],[x,[x,L]]) \to ([x], L/\mathrm{Ker}[x])$$

given by

$$\Psi_1 = -\frac{1}{2}\mathrm{id} \quad \textit{and} \quad \Psi_2([x,[x,a]]) = \frac{1}{2}a + \mathrm{Ker}[x]$$

is an isomorphism of Jordan superpairs.

(b) *When $x \in L_{\bar{1}}$, if we consider the Jordan superpair structure defined on $([x,[x,L]],[x,[x,L]])$ by Remark 4.4, then the pair of maps*

$$(\Psi_1,\Psi_2) : ([x,[x,L]],[x,[x,L]]) \to ([x], L/\mathrm{Ker}[x])$$

given by

$$\Psi_1 = \frac{1}{2}\mathrm{id} \quad \textit{and} \quad \Psi_2([x,[x,a]]) = \frac{1}{2}a + \mathrm{Ker}[x]$$

is an isomorphism of Jordan superpairs.

Proof. In both cases, the pair of maps given by

$$\Psi_1([x,[x,a]]) = (-1)^{|x|+1}\frac{1}{2}[x,[x,a]] \in [x], \text{ and}$$
$$\Psi_2([x,[x,a]]) = \frac{1}{2}a + \mathrm{Ker}[x] \in L/\mathrm{Ker}[x],$$

for every $a \in L$, are well defined (if $[x,[x,a]] = [x,[x,b]]$, then $[x,[x,a-b]] = 0$ implies $a-b \in \mathrm{Ker}[x]$). They are clearly bijective. Let us see that they are Jordan superpair homomorphisms.

(a) Suppose that $x \in L_{\bar{0}}$ and take homogeneous $a,b,c \in L$.
- $\Psi_1(\{[x,[x,a]],[x,[x,b]],[x,[x,c]]\}) = \Psi_1((-1)^{|b||c|}\frac{1}{4}X^2ACX^2(b))$
$= (-1)^{|b||c|+1}\frac{1}{8}X^2ACX^2(b) = \{-\frac{1}{2}[x,[x,a]], \frac{1}{2}b + \mathrm{Ker}[x], -\frac{1}{2}[x,[x,c]]\}$
$= \{\Psi_1([x,[x,a]]), \Psi_2([x,[x,b]]), \Psi_1([x,[x,c]])\}$.
- $\Psi_2(\{[x,[x,a]],[x,[x,b]],[x,[x,c]]\}) = \Psi_2((-1)^{|b||c|}\frac{1}{4}X^2ACX^2(b))$
$= (-1)^{|b||c|}\frac{1}{8}ACX^2(b) + \mathrm{Ker}[x] = -\frac{1}{8}[a,[[x,[x,b]],c]] + \mathrm{Ker}[x]$
$= \{\frac{1}{2}a + \mathrm{Ker}[x], -\frac{1}{2}[x,[x,b]], \frac{1}{2}c + \mathrm{Ker}[x]\}$
$= \{\Psi_2([x,[x,a]]), \Psi_1([x,[x,b]]), \Psi_2([x,[x,c]])\}$.

(b) Suppose that $x \in L_{\bar{1}}$ and take homogeneous $a,b,c \in L$.
- $\Psi_1(\{[x,[x,a]],[x,[x,b]],[x,[x,c]]\}) = \Psi_1((-1)^{|b||c|}\frac{1}{4}X^2ACX^2(b))$
$= (-1)^{|b||c|}\frac{1}{8}X^2ACX^2(b) = \{\frac{1}{2}[x,[x,a]], \frac{1}{2}b + \mathrm{Ker}[x], \frac{1}{2}[x,[x,c]]\}$
$= \{\Psi_1([x,[x,a]]), \Psi_2([x,[x,b]]), \Psi_1([x,[x,c]])\}$.
- $\Psi_2(\{[x,[x,a]],[x,[x,b]],[x,[x,c]]\}) = \Psi_2\big(-\frac{1}{4}[[[x,[x,a]],b],[x,[x,c]]]\big)$
$= -\frac{1}{4}\Psi_2((-1)^{|b||c|}X^2ACX^2(b)) = \frac{1}{8}(-1)^{1+|b||c|}ACX^2(b) + \mathrm{Ker}[x]$
$= \frac{1}{8}[a,[[x,[x,b]],c]] + \mathrm{Ker}[x] = \{\frac{1}{2}a + \mathrm{Ker}[x], \frac{1}{2}[x,[x,b]], \frac{1}{2}c + \mathrm{Ker}[x]\}$
$= \{\Psi_2([x,[x,a]]), \Psi_1([x,[x,b]]), \Psi_2([x,[x,c]])\}$.

□

Acknowledgements

The authors would like to thank the referee for his/her comments on a preliminary version of this paper.

Funding

All authors were partially supported by Ministerio de Ciencia, Innovación y Universidades, MTM2017-84194-P (AEI/FEDER, UE), and by the Junta de Andalucía, FQM264.

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