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Ad-nilpotent elements of skew-index in semiprime associative algebras with involution

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Abstract:	In this paper we study ad-nilpotent elements of a semiprime associative algebra R with involution $*$ whose indices of ad-nilpotence differ on $\Skew(R,*)$ and on R. The existence of such an ad-nilpotent element a implies the existence of a GPI of R, and determines a big part of its structure. When moving to the symmetric Martindale algebra of quotients $Q_m^s(R)$ of R, a remains ad-nilpotent of the original indices in $\Skew(Q_m^s(R),*)$ and $Q_m^s(R)$. There exists an idempotent e that orthogonally decomposes $a=ea+(1-e)a$ and either both ea and $(1-e)a$ are ad-nilpotent of the same index (in this case the index of ad-nilpotence of a in $\Skew(Q_m^s(R),*)$ is congruent with 0 modulo 4), or ea and $(1-e)a$ have different indices of ad-nilpotence (in this case the index of ad-nilpotence of a in $\Skew(Q_m^s(R),*)$ is congruent with 3 modulo 4). Furthermore we show that $Q_m^s(R)$ has a finite $\mathbb{Z}$-grading induced by a $*$-complete family of orthogonal idempotents and that $eQ_m^s(R)e$, which contains ea, is isomorphic to an algebra of matrices over its extended centroid. All this information is used to produce examples of these types of ad-nilpotent elements for any possible index of ad-nilpotence n.
Keywords:	Ad-nilpotent element; semiprime algebra; GPI; involution; matrix algebra; grading
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AD-NILPOTENT ELEMENTS OF SKEW-INDEX IN SEMIPRIME ASSOCIATIVE ALGEBRAS WITH INVOLUTION

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Abstract. In this paper we study ad-nilpotent elements of a semiprime associative algebra R with involution $*$ whose indices of ad-nilpotence differ on $\mathrm{Skew}(R,*)$ and on R. The existence of such an ad-nilpotent element a implies the existence of a GPI of R, and determines a big part of its structure. When moving to the symmetric Martindale algebra of quotients $Q_m^s(R)$ of R, a remains ad-nilpotent of the original indices in $\mathrm{Skew}(Q_m^s(R),*)$ and $Q_m^s(R)$. There exists an idempotent e that orthogonally decomposes $a = ea + (1-e)a$ and either both ea and $(1-e)a$ are ad-nilpotent of the same index (in this case the index of ad-nilpotence of a in $\mathrm{Skew}(Q_m^s(R),*)$ is congruent with 0 modulo 4), or ea and $(1-e)a$ have different indices of ad-nilpotence (in this case the index of ad-nilpotence of a in $\mathrm{Skew}(Q_m^s(R),*)$ is congruent with 3 modulo 4). Furthermore we show that $Q_m^s(R)$ has a finite $\mathbb{Z}$-grading induced by a $*$-complete family of orthogonal idempotents and that $eQ_m^s(R)e$, which contains ea, is isomorphic to an algebra of matrices over its extended centroid. All this information is used to produce examples of these types of ad-nilpotent elements for any possible index of ad-nilpotence n.

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1. Introduction

Let R be an associative algebra, and let $a \in R$. The map $\mathrm{ad}_a : R \to R$ defined by $\mathrm{ad}_a(x) := ax - xa$ is called an inner derivation of R. It is a derivation of the Lie algebra $R^{(-)}$ with bracket product given by $[x,y] := xy - yx$ for every $x, y \in R$. An element $a \in R$ is ad-nilpotent if the map ad_a is nilpotent. Suppose that R is an associative algebra with involution $*$ and let K and $H(R,*)$ respectively denote the sets of skew-symmetric and of symmetric elements of R. We say that an element $a \in K$ is ad-nilpotent (of K) of index n if $\mathrm{ad}_a^n K = 0$ but $\mathrm{ad}_a^{n-1} K \neq 0$. Since the seminal work [15] by Posner, derivations (or some of their generalizations) forcing a prime or semiprime ring to be PI have been broadly studied (see e.g. [13] or [6]). In this paper we focus on the ad-nilpotent elements of a semiprime associative algebra with involution that produce GPIs.

The study of ad-nilpotent elements of the skew-symmetric elements of a prime ring with involution began in 1991 with the work of Martindale and Miers [14].

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Later on, their result was extended to prime associative superalgebras (see [11]) and to semiprime rings with involution (see [2] and [12]).

Martindale and Miers result in the prime setting separates ad-nilpotent elements of K between those which are ad-nilpotent of R of the same index (this may occur when $n \equiv_4 1, 3$) and those that are nilpotent elements and produce a GPI in the central closure of R (this may happen if $n \equiv_4 0, 3$). A similar phenomenon occurs when R is semiprime under the right torsion constraints (see [2, Proposition 3.4 and Theorem 5.6]): for any ad-nilpotent element $a \in R$ there exists a family of orthogonal central idempotents ϵ_i such that $R = \bigoplus \epsilon_i R$, $a = \sum \epsilon_i a$, each $\epsilon_i a$ is ad-nilpotent of index n_i in $K_i = \text{Skew}(\epsilon_i R, *)$, and either

(a) $\epsilon_i a$ is ad-nilpotent in the whole $\epsilon_i R$ of the same index n_i, or
(b) $\epsilon_i a$ is nilpotent of index $[\frac{n+1}{2}] + 1$, the ideal generated by $a^{[\frac{n+1}{2}]}$ is essential in $\epsilon_i R$ and the elements of $\epsilon_i R$ satisfy certain GPI involving $a^{[\frac{n+1}{2}]}$.

Elements of type (a) occur when $n_i \equiv_4 1, 3$ and will be called ad-nilpotent elements of *full-index*. Elements of type (b) occur when $n_i \equiv_4 0, 3$ and will be called elements of *skew-index*. Notice that ad-nilpotent elements of skew-index are also ad-nilpotent elements of $\epsilon_i R$, but the indices of ad-nilpotence in K_i and in $\epsilon_i R$ differ. The goal of this paper is to describe ad-nilpotent elements of skew-index in semiprime associative algebras and to show how they determine a big part of their structure.

The smallest possible index for an element of skew-index is $n = 3$. Ad-nilpotent elements of skew-index 3 are called *Clifford elements* because associated to them there is a Jordan algebra of Clifford type (see [9] and [5]). Our paper is a natural generalization of the careful study of Clifford elements carried out in [4] (alternatively, see [8, Section 8.4]): If R is a prime ring with involution and $a \in R$ is a Clifford element then it satisfies $a^3 = 0$, $a^2 \neq 0$ and $a^2 K a^2 = 0$, a^2 and a are von Neumann regular elements and there is an element $b \in H(R, *)$ such that $a^2 b a^2 = a^2$, $ba^2 b = b$ and $b^2 = 0$ (which also has a square root $\sqrt{b} \in K$, $\sqrt{b}^2 = b$, such that $a\sqrt{b}a = a$, $\sqrt{b}a\sqrt{b} = \sqrt{b}$, which is also a Clifford element). The existence of a Clifford element determines much of the structure of the prime ring: it forces $\text{Skew}(C(R), *) = 0$ for the extended centroid $C(R)$, makes R a GPI ring (so R has socle), and the related $*$-orthogonal idempotents $a^2 b, ba^2$ induce a 5-grading on R and a compatible 3-grading on K with $a \in K_1$ (and $\sqrt{b} \in K_{-1}$) with R_{-2}, R_2 being isomorphic to $C(R)$ as vector spaces and K_{-1}, K_1 being Clifford inner ideals of the Lie algebra K (see [3] for details). We generalize these results to ad-nilpotent elements of skew-index.

Since they produce GPIs and we are working with semiprime associative algebras with involution, the best setting to study these elements is the symmetric Martindale ring of quotients $Q_m^s(R)$. Accordingly, we show that these elements remain ad-nilpotent of the same index in $\mathcal{K} := \text{Skew}(Q_m^s(R), *)$ and produce a $*$-complete family of orthogonal idempotents in $Q_m^s(R)$ which induces a grading on $Q_m^s(R)$ compatible with the involution. When restricting the grading to $\mathcal{K}$ we obtain a grading with shorter support. We can consider this result an extension of Smirnov's description of simple graded algebras with involution with $\text{Supp}(K) \neq \text{Supp}(R)$, see [16, Theorem 5.4], which deepens Zelmanov's classification of simple Lie algebras with a $\mathbb{Z}$-grading carried out in [17]. Furthermore we show that, given an ad-nilpotent element of skew-index, there is an associated set of matrix units making a related subalgebra isomorphic to a ring of matrices, which produces a clear-cut extension of the relevant properties of Clifford elements.

Our last section is devoted to constructing matrix examples of ad-nilpotent elements both of full-index and of skew-index of all possible ad-nilpotence indices n. We highlight that this section completes the work of Martindale and Miers in [14]. In [14, §4.Examples] Martindale and Miers constructed examples of ad-nilpotent elements of skew-index in complex matrices with the transpose involution, and they claimed that they were giving examples for both $n \equiv_4 3$ and $n \equiv_4 0$, covering the possibilities of [14, Main Theorem(2b)], but, as it turns out, they actually addressed the case $n \equiv_4 3$ twice: for each $n \equiv_4 0$ they constructed a skew-symmetric matrix W which, as they showed, satisfies $\mathrm{ad}_W^n(K) = 0$; but it is easily checked that it also satisfies $\mathrm{ad}_W^{n-1}(K) = 0$, so that its index of ad-nilpotence is actually $n-1$, which is congruent to 3 modulo 4.

2. Preliminaries

2.1. In this paper we will deal with semiprime associative algebras R with involution $*$ over a ring of scalars Φ with $\frac{1}{2} \in \Phi$ ($\lambda R \neq 0$ for every nonzero $\lambda \in \Phi$). If we define the bracket product as $[x, y] := xy - yx$ for every $x, y \in R$, R turns into a Lie algebra denoted by $R^{(-)}$. The set of skew-symmetric elements $\{x \in R \mid x^* = -x\}$, which will be denoted by K, is a Lie subalgebra of $R^{(-)}$.

Given a Lie algebra L, we say that $a \in L$ is ad-nilpotent of L of index n if $\mathrm{ad}_a^n L = 0$ and $\mathrm{ad}_a^{n-1} L \neq 0$, where ad_a denotes the usual adjoint map $\mathrm{ad}_a x := [a, x]$ for every $x \in L$. In [2], a deep study of ad-nilpotent elements in semiprime associative algebras with involution was carried out. Following the classification of ad-nilpotent elements obtained in [2, Proposition 3.4 and Theorem 5.6], we introduce the following definitions:

2.2. Let R be a semiprime associative algebra with involution $*$. Let $a \in K$.

We say that a is ad-nilpotent of full-index if a is ad-nilpotent of R and of K of the same index n. By [2, Theorem 5.6], under the adequate torsion requirements, this occurs when $n \equiv_4 1$ or $n \equiv_4 3$.

We say that a is ad-nilpotent of skew-index n if it satisfies all the following conditions:

- a is ad-nilpotent of K of index n with $n \equiv_4 0$ or $n \equiv_4 3$,
- a is a nilpotent element of index $t+1$ for $t := \left[\frac{n+1}{2}\right]$ (in particular t is even and a is an ad-nilpotent of R of index $n+1$ or $n+2$),
- a^t generates an essential ideal in R,
- and
  - if $n \equiv_4 0$, $a^t x a^t = 0$ for every $x \in K$.
  - if $n \equiv_4 3$, $a^t x a^{t-1} - a^{t-1} x a^t = 0$ for every $x \in K$.

Notice that under the adequate torsion requirements, this last condition follows from [2, Theorem 5.6].

2.3. Given an associative algebra R over Φ, we define a permissible map of R as a pair (I, f) where I is an essential ideal of R and f is a homomorphism of right R-modules. For permissible maps (I, f) and (J, g) of R, define an equivalence relation $\equiv$ by $(I, f) \equiv (J, g)$ if there exists an essential ideal M of R, contained in $I \cap J$, such that $f(x) = g(x)$ for all $x \in M$. The quotient set $Q_m^r(R)$ will be called the right Martindale algebra of quotients of R. Suppose from now on that R is semiprime.

Then we can define an addition and a multiplication in $Q_m^r(R)$ coming respectively from the addition and the composition of homomorphisms (see [1, Chapter 2]):

- $[I,f]+[J,g]:=[I\cap J, f+g]$,
- $[I,f]\cdot[J,g]:=[(I\cap J)^2, f\circ g]$.

The map $i: R\hookrightarrow Q_m^r(R)$ defined by $i(r):=[R,L_r]$, where $L_r: R\to R$ is the left multiplication map $L_r(x):=rx$, is a monomorphism of associative algebras (called the usual embedding of R into $Q_m^r(R)$), i.e., R can be considered as a subalgebra of its right Martindale algebra of quotients. Moreover, given any $0\neq q:=[I,f]\in Q_m^r(R)$ we have that $0\neq qI\subseteq R$. Therefore every subalgebra S of $Q_m^r(R)$ which contains R is semiprime because every nonzero ideal of S has nonzero intersection with R. We also recall the following useful property: for every $q\in Q_m^r(R)$ and every essential ideal J of R, $qJ=0$ or $Jq=0$ imply $q=0$.

The symmetric Martindale ring of quotients of R is defined as

$$Q_m^s(R):=\{q\in Q_m^r(R)|\ \exists \text{ an essential ideal } I \text{ of } R \text{ such that } qI+Iq\subseteq R\}.$$

Since $R\subseteq Q_m^s(R)\subseteq Q_m^r(R)$, $Q_m^s(R)$ is again semiprime. When R has an involution, the involution is uniquely extended to $Q_m^s(R)$ ([1, Proposition 2.5.4]).

2.4. The extended centroid $C(R)$ of a semiprime algebra R is defined as the center of $Q_m^s(R)$. It is commutative and unital von Neumann regular. The ring of scalars Φ is contained in $C(R)$ under the usual embedding of R into $Q_m^s(R)$.

The central closure of R, denoted by $\hat{R}$, is defined as the subalgebra of $Q_m^s(R)$ generated by R and $C(R)$, i.e., $\hat{R}:=C(R)+C(R)R$; so the elements of R can be identified with elements in its central closure. The algebra $\hat{R}$ is semiprime since $R\subseteq\hat{R}\subseteq Q_m^r(R)$, and it is centrally closed, meaning that $\hat{R}$ coincides with its central closure.

Since the extended centroid $C(R)$ of a semiprime R is von Neumann regular, given an element $\lambda\in C(R)$ there exists $\lambda'\in C(R)$ such that $\lambda\lambda'\lambda=\lambda$ and $\lambda'=\lambda'\lambda\lambda'$. Let us define $\epsilon_\lambda:=\lambda\lambda'$. Then ϵ_λ is an idempotent of $C(R)$ such that $\epsilon_\lambda\lambda=\lambda$. Moreover, if R is semiprime with involution $*$ and $\lambda\in\mathrm{Skew}(C(R),*)$, then $-\lambda=\lambda^*=(\lambda\lambda'\lambda)^*=\lambda\lambda'^*\lambda$, which implies that λ' can be taken in $\mathrm{Skew}(C(R),*)$ (replace λ' by $\frac{1}{2}(\lambda'-\lambda'^*)$). In this case $\epsilon_\lambda=\lambda\lambda'\in H(C(R),*)$ is a symmetric idempotent of $C(R)$.

The following result relates the extended centroid and the center of the local algebra at an idempotent element, and can be easily deduced from [1, Corollary 2.3.12].

Lemma 2.5. *Let R be a semiprime centrally closed associative algebra and let e be an idempotent of R such that the ideal generated by e in R is essential. Then $C(R)\cong Z(eRe)$.*

Proof. The homomorphism $\varphi: C(R)\to Z(eRe)$ defined by $\varphi(\lambda)=\lambda e=e\lambda e$ is an isomorphism: by [1, Corollary 2.3.12], φ is surjective; moreover, if $\varphi(\lambda)=0$, then the ideals λR and ReR are orthogonal, which implies that $\lambda=0$ because ReR is an essential ideal. □

The following technical lemma, which collects two results about $*$-identities, was proved in [2, Lemma 5.1].

Lemma 2.6. *Let R be a semiprime associative algebra with involution $*$ over a ring of scalars Φ with $\frac{1}{2}\in\Phi$. Let $k\in K$ and $h\in H(R,*)$. Then:*

(1) *$hKh = 0$ implies $hRh \subseteq H(C(R),*)h$. Moreover, R satisfies $hxhyh = hyhxh$ for every $x, y \in R$, and if $\mathrm{Id}_R(h)$ is essential this identity is a strict GPI in R and $\mathrm{Skew}(C(R),*) = 0$.*
(2) *$hKh = 0$ and $hKk = 0$ imply $hRk = 0$.*
(3) *$kKk = 0$ implies $k = 0$.*

In particular, if there is an element $a \in R$ which is ad-nilpotent of skew-index n, then since $t = \left[\frac{n+1}{2}\right]$ is even we have $a^tKa^t = 0$ with $a^t \in H(R,*)$ and $\mathrm{Id}_R(a^t)$ essential, so item (1) applies and shows that $\mathrm{Skew}(C(R),*) = 0$ and that R satisfies a strict GPI (in particular $Q_m^r(R)$ is von Neumann regular; see [1, Section 6.3] for more structural consequences).

3. Main

3.1. Let R be an associative algebra over a ring of scalars Φ with $\frac{1}{2} \in \Phi$. Let $a \in K$ be a nilpotent element of index $t+1$ such that $a^t \in H(R,*)$ is von Neumann regular – as occurs when a is an ad-nilpotent element of skew-index, see Theorem 3.5 below. In this situation we can associate a $*$-Rus inverse to a, i.e., an element $b \in H(R,*)$ satisfying $a^tba^t = a^t$, $ba^tb = b$ and $ba^sb = 0$ for every $s < t$, see [10, Lemma 2.4] and [7, Lemma 3.2] (which works also when $a \in K$). Define $e_{ij} := a^{i-1}ba^{t+1-j}$, $e_i := e_{ii}$ for every $i, j = 1, \ldots, t+1$, and $e := \sum_{i=1}^{t+1} e_i$. The element e is a symmetric idempotent which we call a $*$-Rus idempotent associated to a. It satisfies $ea = ae = \sum_{i=2}^{t+1} e_{i,i-1}$, $ea^t = a^t$ and $eb = b = be$. The set $\{e_{ij}\}_{i,j=1}^{t+1}$ is a set of matrix units for eRe. Notice that $e_{\frac{t+2}{2}} \in H(R,*)$ and let $S := e_{\frac{t+2}{2}} R e_{\frac{t+2}{2}}$. Then the subalgebra eRe and $\mathcal{M}_{t+1}(S)$ are $*$-isomorphic under the isomorphism

$$\Psi : \mathcal{M}_{t+1}(S) \to eRe \text{ defined by } \Psi((x_{ij})_{i,j=1}^{t+1}) := \sum_{i,j=1}^{t+1} e_{i,\frac{t+2}{2}} x_{ij} e_{\frac{t+2}{2},j}$$

where each $x_{ij} = e_{\frac{t+2}{2}} x_{ij} e_{\frac{t+2}{2}} \in e_{\frac{t+2}{2}} R e_{\frac{t+2}{2}}$, and the involution in $\mathcal{M}_{t+1}(S)$ is given by

$$A^* := D^{-1} \bar{A}^{\mathrm{tr}} D$$

for every $A = \sum_{ij} a_{ij} e_{ij} \in \mathcal{M}_{t+1}(S)$, where $\bar{A}^{\mathrm{tr}} := \sum_{ij} a_{ij}^* e_{ji}$ and

$$D := \sum_{i=1}^{t+1} (-1)^i e_{i,t+2-i} = D^{-1} \in \mathcal{M}_{t+1}(S).$$

When considering the following $*$-complete family of orthogonal idempotents

$$\{f_i := e_{i+1},\ i = 0, \ldots, t, i \neq \frac{t}{2}\} \cup \{f_{\frac{t}{2}} := 1 - e + e_{\frac{t+2}{2}}\},$$

which satisfy $f_i^* = f_{t-i}$ for every i, we obtain a grading in R which is compatible with the involution:

$$R = R_{-t} \oplus \cdots \oplus R_0 \oplus \cdots \oplus R_t$$

where $R_j := \sum_{k-l=j} f_k R f_l$ (notice that $R_j^* = R_j$ for each j). With respect to this grading we have

$$ea \in R_1,\ (1-e)a \in R_0,\ a^t \in R_t \text{ and } b \in R_{-t}.$$

This grading is called the grading of R induced by a and its $*$-Rus inverse b.

In the above argument, the element a can be replaced by ea without changing the grading in R: the element $b = eb$ is also a $*$-Rus inverse for ea and gives rise to the same set of matrix units

$$e_{ij} = a^{i-1}ba^{t+1-j} = a^{i-1}ebea^{t+1-j} = (ea)^{i-1}b(ea)^{t+1-j},$$

so the grading in R induced by ea and its $*$-Rus inverse b coincides with the grading of R induced by a and b.

When a is an ad-nilpotent element of K of skew-index, the GPIs satisfied in R allow a more precise description of this grading, as we will show in the following theorem.

Theorem 3.2. *Let R be a semiprime associative algebra with involution $*$ over a ring of scalars Φ with $\frac{1}{2} \in \Phi$, let $K := \mathrm{Skew}(R, *)$ and let $a \in K$ be an ad-nilpotent element of skew-index n. Let $t := [\frac{n+1}{2}]$ and suppose that a^t is von Neumann regular. Let us consider the grading in R*

$$R = R_{-t} \oplus \cdots \oplus R_0 \oplus \cdots \oplus R_t \qquad (\star)$$

induced by a and its $*$-Rus inverse b. Let e be a $*$-Rus idempotent associated to a. Then:

1. *The grading $(\star)$ restricted to K has $K_{-t} = 0 = K_t$.*
2. *S is a semiprime commutative algebra with identity involution. In particular, the involution in $eRe \cong \mathcal{M}_{t+1}(S)$ under this isomorphism is given by*
$$A^* = D^{-1}A^{\mathrm{tr}}D \text{ for any } A \in \mathcal{M}_{t+1}(S).$$
3. *As Φ-modules, both R_t and R_{-t} are isomorphic to S.*
4. *If $t > 2$, both $K_{-(t-1)}$ and K_{t-1} are isomorphic to S.*

Moreover, if R is centrally closed, $S \cong C(R)$.

Proof. Since the grading $(\star)$ is compatible with the involution, we can restrict it to K,

$$K = K_{-t} \oplus K_{-t+1} \oplus \cdots \oplus K_0 \oplus \cdots \oplus K_{t-1} \oplus K_t.$$

(1) Let us show that $K_{-t} = 0 = K_t$: if $x \in K_{-t} = R_{-t} \cap K$ then $x = f_0 k f_t = e_1 k e_{t+1}$ for some $k \in K$, so $x = ba^t k a^t b \in ba^t K a^t b = 0$. Similarly, if $x \in K_t = R_t \cap K$ then $x = f_t k f_0 = e_{t+1} k e_1$ for some $q \in K$, so $x = a^t bkba^t \in a^t K a^t = 0$.
(2) We claim that $S = e_{\frac{t+2}{2}} R e_{\frac{t+2}{2}}$ does not contain skew-symmetric elements: let $k := \frac{t+2}{2}$; if $x = -x^* \in e_k R e_k$ then $x = e_k x e_k = e_{k,t+1}(e_{t+1,k} x e_{k,1}) e_{1,k}$, but $e_{t+1,k} x e_{k,1} = e_{t+1} e_{t+1,k} x e_{k,1} e_1$ is a skew-symmetric element of R_t, so it is zero by (1). Therefore $x = 0$, the involution in S is the identity and hence S is commutative.
(3) $R_t = f_t R f_0 = e_{t+1} R e_1 \cong S$ as a Φ-module, and analogously for R_{-t}.
(4) Since $t > 2$, $R_{-(t-1)} = \sum_{k-l=-(t-1)} f_k R f_l = e_1 R e_t + e_2 R e_{t+1} \subseteq eRe \cong \mathcal{M}_{t+1}(S)$, and under this isomorphism the elements of $R_{-(t-1)}$ are of the form

$$x = \lambda e_{1,t} + \mu e_{2,t+1}, \qquad \lambda, \mu \in S,$$

whence $x = \frac{\lambda+\mu}{2} u + \frac{\lambda-\mu}{2} v$ for $u := e_{1,t} + e_{2,t+1} \in H(R, *)$ and $v := e_{1,t} - e_{2,t+1} \in K$. Therefore $K_{-(t-1)} \subseteq Sv$. A similar argument applies to K_{t-1}.

Moreover, if R is centrally closed, by Lemma 2.5, since the ideal of R generated by $e_{\frac{t+2}{2}}$ is essential because it contains $a^t = a^{\frac{t}{2}} e_{\frac{t+2}{2}} a^{\frac{t}{2}}$, we get $S = e_{\frac{t+2}{2}} R e_{\frac{t+2}{2}} = Z(e_{\frac{t+2}{2}} R e_{\frac{t+2}{2}}) \cong C(R)$ as associative algebras. ∎

The last theorem allows to describe $ea \in eRe \cong \mathcal{M}_{t+1}(S)$ in detail. Now we show how is a related to ea.

Theorem 3.3. *Let R be a semiprime associative algebra with involution $*$ over a ring of scalars Φ with $\frac{1}{2} \in \Phi$, let $K := \mathrm{Skew}(R, *)$ and let $a \in K$ be an ad-nilpotent element of skew-index n. Set $t := [\frac{n+1}{2}]$ and suppose that R is free of $\binom{2t-2}{t-1}$-torsion and a^t is von Neumann regular. Then for any $*$-Rus-idempotent $e \in R$ associated to a, $a = ea + (1-e)a$, and*

(1) if $n \equiv_4 0$:
- *ea is nilpotent of index $t+1$ and ad-nilpotent of skew-index $n-1$ in K.*
- *$(1-e)a$ is nilpotent of index t and ad-nilpotent of full-index $n-1$ in K.*

(2) if $n \equiv_4 3$:
- *ea is nilpotent of index $t+1$ and ad-nilpotent of skew-index n in K.*
- *$(1-e)a$ is nilpotent of index $\leq t-1$ and ad-nilpotent in K of index $\leq n-2$.*
- *$ea^{t-1} = a^{t-1}$.*

Proof. Let $b \in H(R, *)$ be a $*$-Rus-inverse of a and let e be the associated $*$-Rus idempotent.

Suppose that $n \equiv_4 0$. Let us see that ea is ad-nilpotent of index $n-1$ in K: for every $k \in K$,

$$\begin{aligned}
\mathrm{ad}_{ea}^{n-1} k = \mathrm{ad}_{ea}^{2t-1} k &= \binom{n-1}{t-1}(ea^{t-1}kea^t - ea^tkea^{t-1}) = \\
&= \binom{n-1}{t-1}(ea^{t-1}ka^t - a^tkea^{t-1}) = \\
&= \binom{n-1}{t-1}((a^tba^{t-1} + a^{t-1}ba^t)ka^t - a^tk(a^tba^{t-1} + a^{t-1}ba^t)) = \\
&= \binom{n-1}{t-1}(a^t(ba^{t-1}k)a^t - a^t(ka^{t-1}b)a^t) = \\
&= \binom{n-1}{t-1}a^t((ba^{t-1}k) - (ba^{t-1}k)^*)a^t = 0
\end{aligned}$$

because $(ba^{t-1}k)^* = ka^{t-1}b$ and $a^tKa^t = 0$. Thus ea is nilpotent of index $t+1$ (since $(ea)^t = a^t \neq 0$) and ad-nilpotent of index $\leq n-1$. Let us see that its index of ad-nilpotence is $n-1$. Suppose on the contrary that $\mathrm{ad}_{ea}^{n-2} K = 0$. Then for every $k \in K$,

$$0 = a \cdot \mathrm{ad}_{ea}^{n-2} k = \binom{2t-2}{t}ea^{t-1}ka^t - \binom{2t-2}{t-1}a^tkea^{t-1}.$$

Since $ea^{t-1} = a^tba^{t-1} + a^{t-1}ba^t$ and $a^tka^t = 0$ we obtain

$$\binom{2t-2}{t}a^tba^{t-1}ka^t - \binom{2t-2}{t-1}a^tka^{t-1}ba^t = 0,$$

and since $a^tx^*a^t = a^txa^t$ for all $x \in R$ and $(ba^{t-1}k)^* = ka^{t-1}b$ we get

$$\left(\binom{2t-2}{t-1} - \binom{2t-2}{t}\right)a^tka^{t-1}ba^t = 0.$$

Now, again from $a^tka^t=0$ and $ea^{t-1}=a^tba^{t-1}+a^{t-1}ba^t$, we find

$$\left(\binom{2t-2}{t-1}-\binom{2t-2}{t}\right)(a^tka^{t-1}ba^t+a^tka^tba^{t-1})=$$
$$=\left(\binom{2t-2}{t-1}-\binom{2t-2}{t}\right)a^tkea^{t-1}=0.$$

Since $\binom{2t-2}{t-1}-\binom{2t-2}{t}$ divides $\binom{2t-2}{t-1}$ and R is $\binom{2t-2}{t-1}$-torsion free we have $a^tKea^{t-1}=0$, so by Lemma 2.6(2) we get $a^tRea^{t-1}=0$ with a^t generating an essential ideal of R, and thus $ea^{t-1}=0$, a contradiction. Thus ea is ad-nilpotent of index $n-1$ in K.

Since $ea^t=a^t$, $(1-e)a$ is nilpotent of index less than or equal to t. Let us see that its index of nilpotence is t. Suppose on the contrary that $ea^{t-1}=a^{t-1}$. Then, for every $k\in K$,

$$\mathrm{ad}_a^{n-1}k=\mathrm{ad}_a^{2t-1}k=\binom{2t-1}{t-1}(-1)^t(a^{t-1}ka^t-a^tka^{t-1})=$$
$$=\binom{2t-1}{t-1}(-1)^t(ea^{t-1}ka^t-a^tkea^{t-1})=\mathrm{ad}_{ea}^{2t-1}k=\mathrm{ad}_{ea}^{n-1}k=0$$

would mean that a has index of ad-nilpotence $\le n-1$ in K, a contradiction. Hence $(1-e)a^{t-1}\neq 0$.

Let us see that $(1-e)a$ is ad-nilpotent of index $n-1$: since $(1-e)a^t=0$ we get that $\mathrm{ad}_{(1-e)a}^{n-1}K=\mathrm{ad}_{(1-e)a}^{2t-1}K=0$. In addition, $\mathrm{ad}_{(1-e)a}^{n-2}K=\binom{2t-2}{t-1}(1-e)a^{t-1}K(1-e)a^{t-1}\neq 0$ by Lemma 2.6(3). Thus $(1-e)a$ is nilpotent of index t and ad-nilpotent of index $2t-1=n-1$.

Suppose that $n\equiv_4 3$. Let us see that in this case $ea^{t-1}=a^{t-1}$: for every $k\in K$, using that $a^tka^t=0$, $a^{t-1}ka^t=a^tka^{t-1}$ and $a^tba^t=a^t$,

$$(ea^{t-1}-a^{t-1})ka^t=(a^{t-1}ba^t+a^tba^{t-1}-a^{t-1})ka^t=a^tba^{t-1}ka^t-a^{t-1}ka^t=$$
$$=a^tba^tka^{t-1}-a^{t-1}ka^t=a^tka^{t-1}-a^{t-1}ka^t=0.$$

Hence $(ea^{t-1}-a^{t-1})Ka^t=0$. Since $ea^{t-1}-a^{t-1}\in K$, $a^t\in H(R,*)$, $a^tKa^t=0$ and the ideal generated by a^t is essential in R, we have by Lemma 2.6(2) that $ea^{t-1}-a^{t-1}=0$. In particular we get that $(1-e)a$ is nilpotent of index $\le t-1$. Moreover, since in this case $n-2=2t-2$, $\mathrm{ad}_{(1-e)a}^{2t-3}K=0$, implying that the index of ad-nilpotence of $(1-e)a$ in K must be $\le n-2$.

Let us see that ea is ad-nilpotent of index n: since $n=2t-1$, $\mathrm{ad}_{ea}^nK=0$ follows as above. In addition, $\mathrm{ad}_{ea}^{n-1}K=\binom{2t-2}{t-1}ea^{t-1}Kea^{t-1}\neq 0$ by Lemma 2.6(3). So ea is nilpotent of index $t+1$ and ad-nilpotent of index $\le n$. □

Remarks 3.4. Let e be a $*$-Rus idempotent associated to the ad-nilpotent element a of skew-index n with a^t von Neumann regular ($t=[\frac{n+1}{2}]$), and consider the grading of K associated to them by Theorem 3.2.

(1) When a is a Clifford element (i.e., $n=3$) we have $a=ea=a^2ba+aba^2$ by Theorem 3.3(2) (since $t-1=1$), and $a\in K_1$ in the grading.

(2) When $n\equiv_4 3$ and R is free of $\binom{2t-2}{t-1}$-torsion we obtain that a^{t-1} is also von Neumann regular: by Theorem 3.3(2) we have $a^{t-1}=ea^{t-1}$, so $a^{t-1}=e_{t,1}+e_{t+1,2}\in eRe\cong\mathcal{M}_{t+1}(S)$ by Theorem 3.2(2) and we get $a^{t-1}=a^{t-1}ca^{t-1}$, $c=ca^{t-1}c$ for $c:=e_{1,t}+e_{2,t+1}\in K$. If $t>2$ then $c^2=0$, while when a is Clifford we have $n=3$, $t=2$ and $c=e_{1,2}+e_{2,3}$ satisfies

$c^2 = e_{1,3} = e_{1,t+1} = b$, so c is a square root of b. In addition c is also a Clifford element and $c \in K_{-1}$ in the grading.

(3) Suppose R centrally closed. While when $t > 2$ we have $K_{-(t-1)}, K_{t-1}$ isomorphic to $C(R)$ as Φ-modules by Theorem 3.2(4), when $t = 2$ they may be larger: since $t = 2$ we have $n \in \{3, 4\}$; in either case, $a' := ea$ is a Clifford element generating the same grading by Theorem 3.3. We can show that $a'Ka' = C(R)a'$ by using $a' = a^2ba + aba^2$, $a^2Ka^2 = 0$ and $a^2xa^2 = \lambda_x a^2$ with $\lambda_x \in C(R)$ for $x \in R$ to show $a'Ka' \subseteq C(R)a'$, and $a'ca' = a'$ with $c \in K$ to show the equality. Then as a Φ-module $K_1 = C(R)a' \oplus X$ with $X := \{a^2k + ka^2 \mid k \in K, a'ka' = 0\}$ and analogously for K_{-1} with c in place of a' (see [4, Proposition 4.4 and related results] for the details, which can be easily adapted to our context). The Φ-module X can be 0, for example in the ring of 3×3 matrices over a field (see [4, Remark 4.6(2)]).

The extra hypothesis of a^t being von Neumann regular required in Theorems 3.2 and 3.3 is not too restrictive. When R is a $*$-prime associative algebra, $a^tKa^t = 0$ implies von Neumann regularity by Lemma 2.6(1). In general, if R is semiprime we can move to the symmetric Martindale algebra of quotients $Q_m^s(R)$ because, as we will show in the following theorem, any ad-nilpotent element a of skew-index n is still ad-nilpotent in $\mathcal{K} = \mathrm{Skew}(Q_m^s(R), *)$ of skew-index n with a^t von Neumann regular in $Q_m^s(R)$. Although the liftings of GPIs and $*$-GPIs respectively to the maximal right ring of quotients and the Martindale symmetric ring of quotients are well known (see for example [1, Theorems 6.4.1 and 6.4.7]), we will include all the calculations for the sake of completeness.

Theorem 3.5. *Let R be a semiprime associative algebra with involution $*$ over a ring of scalars Φ with $\frac{1}{2} \in \Phi$. Let $a \in K$ be an ad-nilpotent element of skew-index n. Let $t := [\frac{n+1}{2}]$, let $Q_m^s(R)$ be the symmetric Martindale ring of quotients of R and denote $\mathcal{K} := \mathrm{Skew}(Q_m^s(R), *)$. Then a is an ad-nilpotent element of skew-index n of $\mathcal{K}$, and a^t is von Neumann regular in $Q_m^s(R)$.*

Proof. Let us see that $a^t\mathcal{K}a^t = 0$: let $q \in \mathcal{K}$ and let I be an essential ideal of R such that $Iq + qI \subseteq R$. By Lemma 2.6(1) we know that for any $y \in I$ there exists $\lambda_y \in H(C(R), *)$ with $a^tya^t = \lambda_y a^t$. From $a^tKa^t = 0$ we have $a^txa^t = a^tx^*a^t$ for every $x \in R$. Thus

$$\begin{aligned} a^tya^tqa^t = a^t(ya^tq)^*a^t = -a^tqa^ty^*a^t = \\ = -a^tqa^tya^t = -\lambda_y a^tqa^t = -a^tya^tqa^t \end{aligned}$$

so $2a^tya^tqa^t = 0$ for every y in the essential ideal I of R, so $a^tqa^t = 0$.

Suppose now that $n \equiv_4 3$. In this case we will show that not only $a^t\mathcal{K}a^t = 0$ but also $a^tqa^{t-1} = a^{t-1}qa^t$ for every $q \in \mathcal{K}$. Let $q \in \mathcal{K}$ and let I be an essential ideal of R such that $Iq + qI \subseteq R$. From $a^tka^{t-1} = a^{t-1}ka^t$ for every $k \in K$ and $a^t\mathcal{K}a^t = 0$ we get $a^tqa^t = a^tq^*a^t$ for every $q \in Q_m^s(R)$, whence

$$\begin{aligned} a^tya^tqa^{t-1} &= a^t(ya^tq - (ya^tq)^*)a^{t-1} + a^t(ya^tq)^*a^{t-1} = \\ &= a^{t-1}(ya^tq - (ya^tq)^*)a^t - a^tqa^ty^*a^{t-1} = \\ &= a^{t-1}(ya^tq - (ya^tq)^*)a^t = -a^{t-1}(ya^tq)^*a^t = \\ &= a^{t-1}qa^ty^*a^t = a^{t-1}qa^tya^t. \end{aligned}$$

As we know, for any $y \in I$ there is $\lambda_y \in H(C(R),*)$ such that $a^t y a^t = \lambda_y a^t$. Therefore, for every $x \in R$, if we multiply $a^t y a^t q a^{t-1} - a^{t-1} q a^t y a^t = 0$ by $a^t x$ on the left we obtain

$$0 = a^t x a^t y a^t q a^{t-1} - a^t x a^{t-1} q a^t y a^t = \lambda_y a^t x a^t q a^{t-1} - \lambda_y a^t x a^{t-1} q a^t =$$
$$= a^t y a^t x a^t q a^{t-1} - a^t y a^t x a^{t-1} q a^t = a^t y a^t x (a^t q a^{t-1} - a^{t-1} q a^t),$$

so $a^t q a^{t-1} - a^{t-1} q a^t = 0$ because $a^t I a^t$ generates an essential ideal of R.

• If $n \equiv_4 0$, for any $q \in \mathcal{K}$,

$$\mathrm{ad}_a^n(q) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} a^i q a^{n-i} = (-1)^t \binom{n}{t} a^t q a^t = 0.$$

• If $n \equiv_4 3$, for any $q \in \mathcal{K}$,

$$\mathrm{ad}_a^n(q) = (-1)^{t-1} \binom{n}{t-1} a^t q a^{t-1} + (-1)^t \binom{n}{t} a^{t-1} q a^t =$$
$$= (-1)^{t-1} \binom{n}{t-1} (a^t q a^{t-1} - a^{t-1} q a^t) = 0.$$

Moreover, since a^t generates an essential ideal of R, it also generates an essential ideal of $Q_m^s(R)$.

Let us see that a^t is von Neumann regular in $Q_m^s(R)$. Since $Q_m^s(R) = Q_m^s(\hat{R})$ we will suppose in the rest of this proof that R is centrally closed. As we know, for every $x \in R$ there exists $\lambda_x \in H(C(R),*)$ such that $a^t x a^t = \lambda_x a^t$. Since $C(R)$ is von Neumann regular there exists $\lambda'_x \in C(R)$ such that $\lambda_x \lambda'_x \lambda_x = \lambda_x$ and $\epsilon_x := \lambda_x \lambda'_x$ is an idempotent of $C(R)$, i.e., for every $x \in R$ we have $a^t \lambda'_x x a^t = \epsilon_x a^t$. Let us consider the family $\{\epsilon_x\}_{x \in R}$ of these idempotents and take a maximal subfamily $\{\epsilon_{x_\gamma}\}_{\gamma \in \Delta}$ of nonzero orthogonal idempotents. Note that for every $\gamma \in \Delta$ there exists $c_{x_\gamma} := \lambda'_{x_\gamma} x_\gamma \in R$ such that $a^t c_{x_\gamma} a^t = \epsilon_{x_\gamma} a^t$.

Let us prove that $I := \sum_{\gamma \in \Delta} \epsilon_{x_\gamma} R$ is an essential ideal of R: by [2, Proposition 2.10] there exists an idempotent $\epsilon \in C(R)$ such that $\epsilon \, \epsilon_{x_\gamma} = \epsilon_{x_\gamma}$ for every $\gamma \in \Delta$ and $\mathrm{Ann}_R(I) = (1-\epsilon)R$. We claim that $\epsilon = 1$; otherwise, if $\epsilon \neq 1$, we can produce a new orthogonal idempotent that does not belong to Δ, which contradicts the maximality of Δ: since R is semiprime and the ideal generated by a^t is essential, $a^t R a^t R(1-\epsilon) \neq 0$ and for every $x \in R$ such that $0 \neq a^t x a^t R(1-\epsilon)$ we have $0 \neq (1-\epsilon) a^t x a^t = (1-\epsilon)\epsilon_x \lambda_x a^t$, i.e., $(1-\epsilon)\epsilon_x$ is a new orthogonal idempotent, a contradiction. Therefore $\epsilon = 1$ and I is an essential ideal of R.

Define $c : I \to R$ by $c(\sum_\gamma \epsilon_{x_\gamma} y_\gamma) := \sum_\gamma c_{x_\gamma} y_\gamma$. It is clear that c is a homomorphism of right R-modules; moreover, for every $\delta \in \Delta$,

$$L_{\epsilon_{x_\delta}} c(\sum_\gamma \epsilon_{x_\gamma} y_\gamma) = \epsilon_{x_\delta} c_{x_\delta} y_\delta = L_{c_{x_\delta}}(\sum_\gamma \epsilon_{x_\gamma} y_\gamma) \in R,$$

where $L_{\epsilon_{x_\delta}} : R \to R$ and $L_{c_{x_\delta}} : R \to R$ are the corresponding left multiplication maps, so $[R, L_{\epsilon_{x_\delta}}] \cdot [I, c] = [R, L_{c_{x_\delta}}]$, and by the usual embedding of R into $Q_m^s(R)$ we obtain $Iq \subseteq R$ for $q := [I, c]$. Furthermore, since each ϵ_{x_δ} lies in $C(R)$, with the same argument we can prove that $qI \subseteq R$. Thus $q \in Q_m^s(R)$.

Finally, for every $\gamma \in \Delta$ we have $\epsilon_{x_\gamma}(a^t q a^t - a^t) = a^t c_{x_\gamma} a^t - \epsilon_{x_\gamma} a^t = 0$ which implies that $a^t q a^t - a^t \in \mathrm{Ann}_R(I) = 0$, i.e., $a^t q a^t = a^t$. □

4. Examples

In this section we construct examples of ad-nilpotent elements of full-index and of skew-index for any possible index of ad-nilpotence.

4.1. Let m be a natural number, let F a field of characteristic zero (or big enough) with involution denoted by $\overline{\alpha}$ for any $\alpha \in F$, and denote the simple associative algebra $\mathcal{M}_m(F)$ by R and its standard matrix units by e_{ij}, $1 \leq i, j \leq m$. We endow R with the involution $* : R \to R$ given by

$$X^* := D^{-1}\overline{X}^{\mathrm{tr}} D$$

where $D := \sum_{i=1}^{m} (-1)^i e_{i,m+1-i} \in R$ and $\overline{X}^{\mathrm{tr}} := (\overline{x}_{ji})_{i,j=1}^{m}$ for $X = (x_{ij})_{i,j=1}^{m} \in R$. As before, we denote the set of skew-symmetric elements of R with respect to the involution $*$ by K. When m is odd (the only case we actually need) we have $D^{-1} = D$ and

$$e_{ij}^* = (-1)^{i+j} e_{m-j+1,m-i+1},$$

and thus $A = (a_{ij})_{i,j=1}^{m} \in K$ if and only if

$$\overline{a_{ij}} = (-1)^{i+j+1} a_{m-j+1,m-i+1}$$

for all $1 \leq i, j \leq m$; in particular $\overline{a_{i,m-i+1}} = -a_{i,m-i+1}$, so $a_{i,m-i+1} \in \mathrm{Skew}(F, -)$ for all $1 \leq i \leq m$.

4.2. Ad-nilpotent elements of full-index: Let $R := \mathcal{M}_m(F)$ with the involution $*$ given in 4.1, and let m be odd. As in 4.1, consider

$$A_1 := \sum_{i=1}^{m-1} e_{i,i+1} \in K,$$

which is a nilpotent element of index m and ad-nilpotent of R of index $2m - 1$ (see [2, Lemma 4.2]). If the involution $-$ in the field F is not the identity, for any $0 \neq \alpha \in \mathrm{Skew}(F, -)$, the element $0 \neq \alpha e_{m,1}$ is skew-symmetric in R, and

$$\mathrm{ad}_{A_1}^{2m-2}(\alpha e_{m,1}) = \binom{2m-2}{m-1} A_1^{m-1} \alpha e_{m,1} A_1^{m-1} = \binom{2m-2}{m-1} \alpha e_{1,m} \neq 0.$$

Thus A_1 is an ad-nilpotent element of K (and of R) whose index of ad-nilpotence is $n = 2m - 1 \equiv_4 1$.

In the same associative algebra R, take any $1 < t \leq \frac{m-1}{2}$ and consider the matrix

$$A_2 := \sum_{i=1}^{t-1} (e_{i,i+1} + e_{m-i,m-i+1}) \in K,$$

which is nilpotent of index t. The element A_2 is ad-nilpotent of R of index $2t-1$ (see [2, Lemma 4.2]). Moreover, $0 \neq B := e_{t,1} + (-1)^t e_{m,m-t+1} \in K$ and $\mathrm{ad}_{A_2}^{2t-2} B \neq 0$. Therefore A_2 is ad-nilpotent of K (and of R) of index $n = 2t - 1$. If t is even then $n \equiv_4 3$, while if t is odd then $n \equiv_4 1$.

4.3. Ad-nilpotent elements of skew-index: Inspired by Theorem 3.2 we will construct the examples of ad-nilpotent elements of skew-index in matrix algebras over fields with identity involution.

• $n \equiv_4 3$: Let $m > 1$ be some odd number. Let us consider $R = \mathcal{M}_m(F)$ where F is a field with identity involution and R is an algebra with the involution $*$ given in 4.1. Take any k such that $2k \leq m$. Let us consider the element

$$A_1 := \sum_{i=k}^{m-k} e_{i,i+1} \in K$$

which is nilpotent of index $l = m - 2k + 2$ and ad-nilpotent of R of index $2l - 1$ (see [2, Lemma 4.2]). Nevertheless, its index of ad-nilpotence in K is lower: Indeed, any $B = \sum_{i,j=1}^{m} b_{ij} e_{ij} \in K$ has $b_{k+l-1,k} = 0$ and $b_{k+l-2,k} = b_{k+l-1,k+1}$ by 4.1 since $\mathrm{Skew}(F, -) = 0$, so

$$\begin{aligned}
\mathrm{ad}_{A_1}^{2l-3} B &= \binom{2l-3}{l-2}(A_1^{l-2} B A_1^{l-1} - A_1^{l-1} B A_1^{l-2}) = \\
&= \binom{2l-3}{l-2}(e_{k,k+l-2} + e_{k+1,k+l-1}) B e_{k,k+l-1} - \\
&- \binom{2l-3}{l-2} e_{k,k+l-1} B (e_{k,k+l-2} + e_{k+1,k+l-1}) = \\
&= \binom{2l-3}{l-2}(b_{k+l-2,k} - b_{k+l-1,k+1}) e_{k,k+l-1} = 0.
\end{aligned}$$

Furthermore, for $C := e_{k+l-2,k} - e_{k+l-2,k}^* = e_{k+l-2,k} + e_{k+l-1,k+1} \in K$ we have $\mathrm{ad}_{A_1}^{2l-4} C \neq 0$, so the index of ad-nilpotence of A_1 in K is $2l - 3 \equiv_4 3$. For any odd l we have built an ad-nilpotent matrix A_1 of index $n := 2l - 3 \equiv_4 3$.

• $n \equiv_4 0$: Take any $n \equiv_4 0$. Then $n = 2t$ for some even number t. Let $m := 3t + 3$. In the associative algebra $R = \mathcal{M}_m(F)$ where F is a field with identity involution and R has the involution $*$ given in 4.1, let us define $A := A_1 + A_2$ where

$$A_1 := \sum_{i=t+2}^{2t+1} e_{i,i+1} \quad \text{and} \quad A_2 := \sum_{i=1}^{t-1} (e_{i,i+1} + e_{m-i,m-i+1}).$$

By construction, $A_1 \in K$ is nilpotent of index $t + 1$ and ad-nilpotent of R of index $2t + 1$. Moreover, by taking $k = t + 2$ this matrix corresponds to the matrix A_1 defined in case $n \equiv_4 3$, so it is ad-nilpotent of K of index $2t - 1$. Similarly, $A_2 \in K$ is nilpotent of index t, and it is ad-nilpotent of K (and of R) of index $2t - 1$.

The matrix A, which is an orthogonal sum of A_1 and A_2, is nilpotent of index $t + 1$ and ad-nilpotent of R of index $2t + 1$. Let us see that $\mathrm{ad}_A^{2t} K = 0$: for any $B = \sum_{ij} b_{ij} e_{ij} \in K$ we have

$$\begin{aligned}
\mathrm{ad}_A^{2t} B &= \binom{2t}{t} A^t B A^t = \binom{2t}{t} e_{t+2,2t+2} B e_{t+2,2t+2} = \\
&= \binom{2t}{t} b_{2t+2,t+2} e_{t+2,2t+2} = 0
\end{aligned}$$

because $b_{2t+2,t+2} \in \mathrm{Skew}(F, -) = 0$. Furthermore, for $C := e_{t,t+2} - e_{t,t+2}^* = e_{t,t+2} - e_{2t+2,2t+4} \in K$ we have $\mathrm{ad}_A^{2t-1} C \neq 0$, so A is ad-nilpotent of K of index $n = 2t \equiv_4 0$.

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