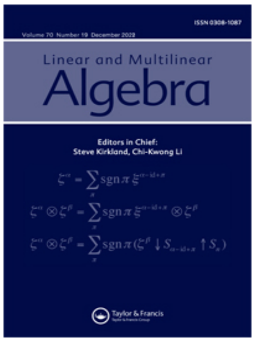

Linear and Multilinear Algebra

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/glma20

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LINEAR AND MULTILINEAR ALGEBRA
2022, VOL. 70, NO. 19, 4056–4070
https://doi.org/10.1080/03081087.2020.1862742

Decompositions of matrices into diagonalizable and square-zero matrices

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ABSTRACT

In order to find a suitable expression of an arbitrary square matrix over an arbitrary field, we prove that every square matrix over an infinite field is always representable as a sum of a diagonalizable matrix and a nilpotent matrix of order less than or equal to two. In addition, each 2×2 matrix over any field admits such a representation. We, moreover, show that, for all natural numbers $n \geq 3$, every $n \times n$ matrix over a finite field having no less than $n+1$ elements also admits such a decomposition. The latter completes a recent example due to Breaz [Matrices over finite fields as sums of periodic and nilpotent elements. Linear Algebra Appl. 2018;555:92–97]. As a consequence of these decompositions, we show that every nilpotent matrix over a field can be expressed as the sum of a potent matrix and a square-zero matrix. This somewhat improves on recent results due to Abyzov et al. [On some matrix analogues of the little Fermat theorem. Mat Zametki. 2017;101(2):163–168] and Shitov [The ring $\mathbb{M}_{8k+4}(\mathbb{Z}_2)$ is nil-clean of index four. Indag Math (N.S.). 2019;30:1077–1078].

ARTICLE HISTORY
Received 9 March 2020
Accepted 4 December 2020

COMMUNICATED BY
S. Gaubert

KEYWORDS
Companion matrix; diagonalizable matrix; nilpotent matrix; Jordan normal form; rational form; irreducible polynomial

2010 MATHEMATICS SUBJECT CLASSIFICATIONS
15A24; 15B33; 16U99

1. Introduction and preliminaries

As both nilpotent and potent elements will play a key role in our further explorations, let us start our article with recalling that an element x of an arbitrary ring R is said to be *nilpotent* if there is a positive integer i such that $x^i = 0$, and an element y from R is said to be *potent*, or more exactly *m-potent*, if there is a natural number $m \geq 2$ with $y^m = y$. In particular, the idempotents are always 2-potent elements.

After defining the notion of nil-clean rings by Diesl in [1], this type of rings became of great interest. In fact, in [2] was proven that each matrix from the ring $\mathbb{M}_n(\mathbb{F}_2)$ of $n \times n$ matrices over the field $\mathbb{F}_2$ consisting of two elements is nil-clean, that is, a sum of an idempotent matrix and a nilpotent matrix. This result was strengthened by Šter in [3] proving that $\mathbb{M}_n(\mathbb{F}_2)$ is actually a nil-clean ring of index at most 4. Lately, this result was significantly improved by Shitov [4]. Likewise, an important work was done by de Seguins Pazzis

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in [5], where a valuable discussion on the decomposition of a matrix as a sum of an idempotent and a square-zero matrix is provided. We also refer to [17] for some related nilpotent decompositions.

On the other vein, Abyzov and Mukhametgaliev showed in [6] that, for all naturals $n \geq 1$, any element of the ring $\mathbb{M}_n(F)$ is presented as a sum of a nilpotent and a q-potent element, provided that F is a field of cardinality q. Also, a recent paper by Breaz [7] deals with the more exact presentation of matrices over fields of odd cardinality q as a sum of a q-potent matrix and a nilpotent matrix of order 3. Besides, it was constructed in [7, Example 6] an ingenious example of a 3×3 matrix over the field $\mathbb{F}_3$ of three elements that cannot be presented as the sum of a 3-potent and a nilpotent matrix of order 2 (in other terms, the latter matrix is also called *a square-zero matrix* or just *2-nilpotent matrix*). However, the given construction illustrates something more general, namely that such a matrix cannot be presented as a sum of a diagonalizable matrix and a 2-nilpotent matrix. As here a crucial role is played by the finiteness of the field, we ask what can be said for such type of presentations, provided the field is infinite or even if the field is finite containing enough elements of number greater than the size of the matrix.

So, we quite naturally come to the study of the following intriguing and non-trivial question, which motivates the writing up of the present paper.

Question: When can every square matrix over a field K be expressed as

$$D + Q$$

where D is a diagonalizable matrix and Q is a nilpotent matrix with $Q^2 = 0$?

Let us notice that diagonalizable matrices over finite fields are q-potent, with q the size of the field. In this sense, our question is related to the work in [7], but our requirements about the order of nilpotence are stronger. In fact, in [6, Theorem 2] it was shown that some square matrices over finite fields are expressible as a sum of a potent and a nilpotent but the order of the existing nilpotent is, in general, greater than 2. So we will provide the matrix expressions with a more precise treating of possible nilpotent elements.

In what follows we shall give an almost complete solution to that query by using some different decompositions which entirely rely on the rational normal form of matrices. Our work is organized as follows: In the current first section, we shall address the previously mentioned example from [7] to the most elementary 2×2 case by giving a more detailed thought to it. In the second section, we explore in detail the general case of possible decompositions of the required form as we will demonstrate that the aforementioned Breaz's example is just a simple consequence of deeper facts (see Remark 2.2, Lemma 2.4 and Theorem 2.6). The next third section pertains to the special decomposition of matrices into semi-simple and 2-nilpotent ones (see Proposition 3.1). Our final fourth section is devoted to showing that our question also has a positive answer for a special class of matrices that do no fit the hypothesis of Theorem 2.6: 4×4 matrices over the field with 3 elements (see Proposition 4.4). We also put there two challenging problems which being answered will contribute substantially to the object of our investigation.

And so, we foremost start with the following useful observations for such decompositions of 2×2 matrices, addressing also the aforementioned [7, Example 6].

Given any field K, every matrix in A in $\mathbb{M}_2(K)$ admits such a decomposition. Indeed, A is either diagonalizable itself, or it is similar to the companion matrix of a degree two polynomial $p(x) = x^2 + ax + b$ in $K[x]$:

$$C(p(x)) = \begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix}.$$

As soon as $a \neq 0$, the matrix $C(p(x))$ can be written as

$$C(p(x)) = \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & -a \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}}_{Q}$$

where D is diagonalizable because it has two different eigenvalues.

If $a = 0$ then $p(x) = x^2 + b$:

(i) If $K \neq \mathbb{F}_2$ then

$$C(p(x)) = \begin{pmatrix} 0 & -b \\ 1 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & -b-1 \\ 0 & 0 \end{pmatrix}}_{Q}$$

where D is diagonalizable because it has two different eigenvalues ± 1.

(ii) If $K = \mathbb{F}_2$ and $p(x) = x^2$ then $C(p(x))$ is nilpotent of order 2.

(iii) If $K = \mathbb{F}_2$ and $p(x) = x^2 + 1$ then

$$C(p(x)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}_{Q}.$$

Nevertheless, there are matrices in $\mathbb{M}_3(\mathbb{F}_3)$ that do not admit such a decomposition, as was shown in [7, Example 6]: Let $K = \mathbb{F}_3$ and consider the companion matrix of the irreducible polynomial $p(x) = x^3 + 2x^2 + 2x + 2 \in K[x]$

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

By construction the characteristic (minimal) polynomial of A is irreducible and its trace is non-zero. Suppose that $A = D + Q$ where D is diagonalizable and Q is 2-nilpotent. Since D has the same trace as A, in the list of eigenvalues of D there must be at least a repetition, so there exists $a \in K$ such that $\dim S_a \geq 2$, where S_a denotes the eigenspace associated to such multiple eigenvalue. On the other hand, if $Q^2 = 0$ then the rank of Q is at most 1, so the kernel of Q has dimension at least two. Therefore, $\mathrm{Ker} Q \cap S_a \neq 0$ and there exists a non-zero vector $v \in \mathrm{Ker} Q \cap S_a$. This vector is an eigenvector of A associated to a ($Av = (D + Q)v = av$) so A itself admits an eigenvalue, which is not possible because its minimal polynomial is irreducible.

2. A general decomposition

Since every matrix over a field K is similar to a direct sum of companion matrices, we are going to focus on such matrices.

Lemma 2.1: *Let K be a field, let $n \geq 3$ and let $A \in \mathbb{M}_n(K)$ be the companion matrix of a polynomial $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$. Then*

- *If $c_{n-1} = 0$ and $|K| \geq n$ then A admits a decomposition into $D + Q$ where D is diagonalizable with no multiple eigenvalues and $Q^2 = 0$ with $\operatorname{rank}(Q) \leq 1$.*
- *If $c_{n-1} \neq 0$ and $|K| \geq n + 1$ then A admits a decomposition into $D + Q$, where D is diagonalizable with no multiple eigenvalues and $Q^2 = 0$ with $\operatorname{rank}(Q) \leq 1$.*

Proof: Let $A = C(p(x))$ where

$$C(p(x)) = \begin{pmatrix} 0 & 0 & 0 & 0 & -c_0 \\ 1 & 0 & 0 & 0 & -c_1 \\ 0 & 1 & 0 & 0 & \vdots \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 1 & -c_{n-1} \end{pmatrix}$$

for $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$.

Take n different elements $a_1, \ldots, a_n$ in the field such $\sum_{i=1}^n a_i = -c_{n-1}$ (notice that the cardinality of K was chosen to assure the existence of these pairwise different elements) and consider the polynomial $q(x) = (x - a_1)(x - a_2) \cdots (x - a_n) = x^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0$. Then

$$C(p(x)) = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & -b_0 \\ 1 & 0 & 0 & 0 & -b_1 \\ 0 & 1 & 0 & 0 & \vdots \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 1 & -b_{n-1} \end{pmatrix}}_{C(q(x))} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & -c_0 + b_0 \\ 0 & 0 & 0 & 0 & -c_1 + b_1 \\ 0 & 0 & 0 & 0 & \vdots \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 0 & -c_{n-1} + b_{n-1} \end{pmatrix}}_{Q} \tag{1}$$

where $C(q(x))$ is diagonalizable because it corresponds to a polynomial with n different roots, while $Q^2 = 0$ because $-c_{n-1} + b_{n-1} = 0$.

Notice that the n different elements $a_1, \ldots, a_n$ in K can be taken

- as soon as $|K| \geq n$ if $c_{n-1} = 0$ (recall that the sum of all the different elements of a finite field of at least 3 elements is zero).
- as soon as $|K| \geq n + 1$ if $c_{n-1} \neq 0$: first choose n different elements in K whose sum is different from zero. Let $0 \neq \gamma$ be the sum of those n different elements of K. Then multiply each of those different elements by $\frac{-c_{n-1}}{\gamma}$ to get n different elements whose sum is $-c_{n-1}$.

■

Remark 2.2: In the proof of Lemma 2.1, formula labelled by (1) gives an explicit decomposition ($C(p(x)) = C(q(x)) + Q$ where $C(q(x))$ is diagonalizable and $Q^2 = 0$) for the companion matrix A of any polynomial of the form $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$. Here we present another decomposition. The requirements for the size of the field are the same: when $c_{n-1} = 0$ we need that $|K| \geq n$ and when $c_{n-1} \neq 0$ we need that $|K| \geq n + 1$.

Given any polynomial $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0 \in K[x]$ and its companion matrix A, take n different elements $a_1, \ldots, a_n \in K$ such that $a_1 + \cdots + a_n = -c_{n-1}$ (those elements exist because we are assuming that $|K| \geq n$ if $c_{n-1} = 0$ or that $|K| \geq n + 1$ if $c_{n-1} \neq 0$). Let us consider the following matrix

$$B = \underbrace{\begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 1 & a_2 & 0 & 0 & 0 \\ 0 & 1 & a_3 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & a_n \end{pmatrix}}_{\hat{D}} + \underbrace{\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & 0 & x_{n-1} \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right)}_{\hat{Q}}.$$

We claim that the elements $x_1, \ldots, x_{n-1}$ can be chosen in K so that the characteristic polynomial of B coincides with $p(x)$. Indeed, if $q(x) = x^n + d_{n-1}x^{n-1} + \cdots + d_0$ denotes the characteristic polynomial of B, $d_{n-1} = c_{n-1}$ because the traces of A and B coincide. Moreover, by the Faddeev–LeVerrier algorithm [8, 6.7], d_{n-2} depends on the $a_1, \ldots, a_n$ and on x_{n-1}, and x_{n-1} can be taken such that $d_{n-2} = c_{n-2}$. Again, by the Faddeev–LeVerrier algorithm, d_{n-3} depends on the $a_1, \ldots, a_n$ and on x_{n-1} and x_{n-2}, and x_{n-2} can be taken such that $d_{n-3} = c_{n-3}$. We can repeat this process until we get the precise $x_1, \ldots, x_{n-1} \in K$ that make $q(x) = p(x)$.

Finally, A and B are two non-derogatory matrices with the same characteristic polynomial, so there exists an invertible P such that

$$A = P^{-1}\hat{D}P + P^{-1}\hat{Q}P = D + Q.$$

Remark 2.3: The decompositions of companion matrices obtained in Lemma 2.1 and in Remark 2.2 have the following properties:

- Each matrix D is diagonalizable with no multiple eigenvalues.
- $Q^2 = 0$ and $\operatorname{rank}(Q) \leq 1$.
- If K is a field with q elements, then $D^q = D$.

In the following result we get a converse validity for the decomposition of Lemma 2.1.

Lemma 2.4: (i) *If every companion matrix in $\mathbb{M}_n(\mathbb{F}_2)$ can be decomposed as $D+Q$, where D is diagonalizable, $Q^2 = 0$ and* $\operatorname{rank}(Q) \leq 1$, *then $n = 2$.*

(ii) *Let K be a field with at least 3 elements. Let $n \geq 3$. If every companion matrix in $\mathbb{M}_n(K)$ can be decomposed as $D+Q$, where D is diagonalizable, $Q^2 = 0$ and* $\operatorname{rank}(Q) \leq 1$, *then $|K| \geq n + 1$.*

(iii) *Let K be a field with at least 3 elements. Let $n \geq 3$. If every zero trace companion matrix in $\mathbb{M}_n(K)$ can be decomposed as $D+Q$ where D is diagonalizable, $Q^2 = 0$ and* $\operatorname{rank}(Q) \leq 1$, *then $|K| \geq n$.*

Proof: (i) Let $K = \mathbb{F}_2$ and suppose that $n \geq 3$. Let A be the (nilpotent) companion matrix of the polynomial $p(x) = x^n \in \mathbb{F}_2[x]$, and let us show that it cannot be written as $D + Q$. Otherwise, since the trace of D must be zero and A is not nilpotent of order two, the eigenvalue 1 occurs in D with multiplicity at least two (it must occur an even number of times due to the condition on the trace). Then the eigenspace S_1 has dimension ≥ 2 and $S_1 \cap \mathrm{Ker}Q \neq 0$, leading to a non-zero eigenvector v of A associated to the eigenvalue 1, a contradiction.

(ii) Let K be a field with at least three elements, $n \geq 3$, and suppose that $|K| = q \leq n$. Let A be the companion matrix of a non-zero trace polynomial of degree n in $K[x]$ with no roots in K, and suppose that A is expressed as $D + Q$. The trace of D coincides with the trace of A and it is non-zero. In order to get a non-zero trace diagonalizable matrix D of size $n \times n$ one needs to repeat at least one eigenvalue $a \in K$ (recall that K consists on q different elements and that their sum is zero), leading to an eigenspace S_a of dimension at least 2 that hits the kernel of Q (notice that $\dim \mathrm{Ker}Q \geq n - 1$ because $\mathrm{rank}(Q) \leq 1$). This will lead to a non-zero vector v in such intersection that verifies $A(v) = D(v) + Q(v) = av$, i.e. v is an eigenvector of A associated to the eigenvalue a, which is a contradiction.

(iii) To get a counterexample when $|K| < n$, suppose that $|K| = q < n$ and define A as the companion matrix of a zero trace polynomial of degree n in $K[x]$ with no roots in K. Suppose that A can be decomposed as $D + Q$ where D is diagonalizable and Q satisfies $Q^2 = 0$ and $\mathrm{rank}(Q) \leq 1$. Since $q < n$, at least one eigenvalue of D has multiplicity ≥ 2, and we can repeat the above argument to get an eigenvector of A, a contradiction. ■

Remark 2.5: It is worthwhile noticing that Breaz's counterexample [7, Example 6] is a particular case of the above argument.

Theorem 2.6: *Given any field K, all matrices in $\mathbb{M}_2(K)$ admit a decomposition into $D + Q$, where D is a diagonalizable matrix and Q is a matrix such that $Q^2 = 0$.*

Let $n \geq 3$ and let K be a field with $|K| \geq n + 1$. Then every matrix $A \in \mathbb{M}_n(K)$ admits a decomposition into $D + Q$, where D is a diagonalizable matrix and Q is a matrix such that $Q^2 = 0$. In particular, square matrices over infinite fields always admit such decomposition.

Proof: The case $\mathbb{M}_2(K)$ was studied in Section 1.

Suppose that $n \geq 3$ and that $|K| \geq n + 1$. Let A be any matrix in $\mathbb{M}_n(K)$. The matrix A is similar to a direct sum of the companion matrices of the invariant factors of A. Each of these companion matrices is associated to a polynomial of degree $m \leq n$, so it can be decomposed into $D + Q$ because $m + 1 \leq n + 1 \leq |K|$. ■

Since diagonalizable matrices over a finite field of q elements are q-potent, we immediately obtain the following claim.

Corollary 2.7: *Let $\mathbb{F}_q$ be the finite field of q elements, $q > 2$. Then every matrix in $\mathbb{M}_n(\mathbb{F}_q)$ with $n \leq q - 1$ admits a decomposition into $D + Q$, where D is a q-potent matrix and Q is a nilpotent matrix such that $Q^2 = 0$.*

Remark 2.8: Another decomposition based on Jordan blocks can be considered. Let K be a field of characteristic q and let J be a Jordan block in $\mathbb{M}_n(K)$, $n \geq 3$, associated to $a \in K$

$$J = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 1 & a & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & a \end{pmatrix}.$$

Depending on whether the characteristic of K divides or not the order n of J, the decomposition into diagonalizable and nilpotent differs:

(i) Suppose that $\mathrm{char}(K) = q$ does not divide n. If K contains the n (different) roots of the polynomial $x^n - 1 \in K[x]$, then J decomposes as

$$J = \underbrace{(J + e_{1n})}_{D} + \underbrace{(-e_{1n})}_{Q}$$

where e_{1n} denotes the nilpotent matrix with 1 in the $(1n)$-entry and zero in the rest of entries, and the matrix D is diagonalizable since its minimal polynomial $p(x) = (x - a)^n - 1$ has n different roots in K because $p'(x) = n(x-a)^{n-1} \neq 0$ (recall $q \nmid n$).

(ii) Suppose that $\mathrm{char}(K) = q$ divides n. If K contains the $n-1$ (different) roots of the polynomial $x^{n-1} - 1 \in K[X]$, then J decomposes as

$$J = \underbrace{(J + e_{2n})}_{D} + \underbrace{(-e_{2n})}_{Q}$$

where e_{2n} denotes the nilpotent matrix with 1 in the $(2n)$-entry and zero in the rest of entries, and the matrix D is diagonalizable since its minimal polynomial $p(x) = (x-a)^n - (x-a)$ has n roots different roots in K because $p'(x) = -1 \neq 0$ (recall $q|n$).

Remark 2.9: The latter decompositions of Jordan blocks can be applied to showing that every nilpotent matrix over a field can be written as $D + Q$, where D is a potent matrix (i.e. $D^q = D$ for a certain $q \in \mathbb{N}$) and Q is a nilpotent matrix with $Q^2 = 0$. Indeed, every nilpotent matrix A over a field admits a decomposition into Jordan blocks $J_1, \ldots, J_s$ associated to the eigenvalue zero (notice that this can be related to [9, § 2] where the authors gave minimal conditions for a nilpotent element in a ring to admit a decomposition into Jordan blocks). Write any of these Jordan blocks $J_i \in \mathbb{M}_{n_i}(K)$ as $D_i + Q_i$ with $D_i = J_i + e_{2n_i}$ and $Q_i = -e_{2n_i}$ (see Remark 2.8(ii)). It directly follows that $D_i^{n_i} = D_i$ and $Q_i^2 = 0$. Then A can be written as the sum of a potent matrix and a nilpotent matrix of zero square.

This result can be related to [10, Corollary 8] of Breaz and Megiesan, where they decomposed any nilpotent matrix into an idempotent matrix and a nilpotent matrix. Notice that our decomposition fixes the order of nilpotency into two and allows the potency to grow bigger, while Breaz an Megiesan decomposition fixes the potency into two (idempotency).

The next construction sheds some more light on the established above theorem and remarks.

Example 2.10: When K is a finite field with $|K| = q$ then the decomposition of a nilpotent block of maximal size $q \times q$

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = C(p(x)) \in \mathbb{M}_q(K)$$

for $p(x) = x^q$ given in Remark 2.8(ii) coincides with the decomposition of J and labeled with $(*)$ in the proof of Lemma 2.1. Indeed, $J + e_{2n}$ is diagonalizable because its minimal polynomial is $x^q - x$ and its roots coincide with the q different elements of the field K. Notice also that $J + e_{2n} = C(q(x))$ for $q(x) = x^q - x$, so:

$$J = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \vdots \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}}_{C(q(x))} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \vdots \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{Q}.$$

Notice that $C(q(x))^q = C(q(x))$ and this decomposition is indeed a decomposition of J into an q-potent matrix and a square-zero matrix.

For nilpotent blocks of smaller size $m \times m$ $(4 \leq m < q)$ the decomposition of Remark 2.8 may not work (as it might be the case where none of the polynomials $x^m - 1$ and $x^{m-1} - 1$ have their roots in K) while the decomposition of Lemma 2.1 can be used because there exist m different elements in K whose sum is zero.

3. A decomposition into semi-simple and nilpotent of order two

In the previous section we have shown that every matrix in $\mathbb{M}_n(K)$ admits a decomposition into a diagonalizable matrix D and a nilpotent matrix Q of order less than or equal to two as soon as K is an infinite field. If the field is finite, we have shown that such decompositions exist for matrices of size $n \times n$ over fields of at least $n + 1$ elements.

In the following proposition we will show that this last hypothesis is not necessary if we only require that D is semi-simple (diagonalizable in some field extension of K). This decomposition can be related to the Jordan–Chevalley decomposition over perfect fields (in particular, over finite fields), that decomposes all square matrices into their semi-simple and their nilpotent parts with the extra hypothesis of commutation between the semi-simple and the nilpotent parts, see for example [11, 4.2 Proposition] or [12, Theorem 16].

Proposition 3.1: *Let K be a field and $n \in \mathbb{N}$. Then every matrix in $\mathbb{M}_n(K)$ admits a decomposition into a semi-simple matrix (diagonalizable in some extension field of K) and a nilpotent matrix of zero square.*

Proof: If the field is infinite, then every matrix decomposes into a diagonalizable matrix and a nilpotent matrix of order two by Theorem 2.6.

Suppose from now on that the field K is finite. It suffices to show that every companion matrix $C(p(x)) \in \mathbb{M}_n(K)$ admits such a decomposition (here we can suppose that $n \geq 3$ because matrices of $\mathbb{M}_2(K)$ can always be expressed as diagonalizable + nilpotent of order 2). The key point here is the fact that for every finite field K there exist irreducible polynomials of any degree $n \geq 3$ with any trace in the field K (see the Hansen and Mullen conjecture [13, Conjecture B] and its solutions by Wan [14] and Ham and Mullen [15]). To decompose any $C(p(x)) \in \mathbb{M}_n(K)$ pick an irreducible polynomial $q(x) \in K[x]$ of degree n with the same trace as $p(x)$ and repeat the decomposition $(*)$ of Lemma 2.1. Notice that $C(q(x))$ is semi-simple because it has n different roots in its decomposition field, which is an extension of K (every finite field is perfect, so irreducible polynomials have no multiple roots in their decomposition fields). ■

Since every semi-simple matrix over a finite field is r-potent for certain $1 < r \in \mathbb{N}$, we get the following non-trivial property of matrices over finite fields.

Corollary 3.2: *Let K be a finite field and $n \in \mathbb{N}$. Then every matrix in $\mathbb{M}_n(K)$ admits a decomposition into an r-potent matrix, for certain $1 < r \in \mathbb{N}$, and a square-zero matrix.*

The next example sheds some more light on the motivating for us Breaz's example alluded to above.

Example 3.3: Let us look at Breaz's example mentioned in Section 1 ([7, Example 6]): Let $K = \mathbb{F}_3$ and consider the companion matrix of the irreducible polynomial $p(x) = x^3 + 2x^2 + 2x + 2 \in K[x]$

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Since the polynomial $p(x)$ splits in $\mathbb{F}_{3^3}$ with three different roots, the matrix A is diagonalizable in this field. Therefore the matrix A is written as the sum of a 27-potent matrix and a square-zero matrix, which is precisely the zero matrix in this particular case.

4. A decomposition for 4 × 4 matrices over $\mathbb{F}_3$

Although $\mathbb{M}_4(\mathbb{F}_3)$ do not fit the hypothesis of Theorem 2.6 because the cardinality of the field is less than the order of these matrices, in this section we are going to prove that all matrices in $\mathbb{M}_4(\mathbb{F}_3)$ admit a decomposition into a diagonalizable and a square-zero matrix. Let us first look into some examples.

Example 4.1: Let A be the companion matrix of the polynomial $p(x) = x^4 + x + 2$. This polynomial is irreducible in $\mathbb{F}_3[x]$. Consider the following matrix B

$$B = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}}_{Q}.$$

One can check that $Q^2 = 0$ and that the minimal polynomial of B is again $p(x)$, so there exists an invertible P such that

$$A = P^{-1}BP = P^{-1}DP + P^{-1}QP.$$

Furthermore, A does not admit a decomposition $D' + Q'$ where D' is diagonalizable with set of eigenvalues $S_1 = \{1, 1, 2, 2\}$ and $(Q')^2 = 0$: if that were true, $(D')^2 = I$ and from $(A - D')^2 = 0$ we derive that $A^2D' = D'A^2$. We can compute the centralizer of A^2 but none of its elements satisfy $(D')^2 = 0$.

Example 4.2: If we build a 4×4 matrix A with one indecomposable block, it does not mean that A is indecomposable itself. Suppose that

$$A = \left(\begin{array}{c|ccc} 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

where we know that the block $\left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right)$ cannot be decomposed. The minimal polynomial of A coincides with its characteristic polynomial and it is $p(x) = x^4 + x^2 + x + 2 = (x - 2)(x^3 + 2x^2 + 2x + 2)$. Define

$$B = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 2 \end{pmatrix}}_{Q}.$$

One can check that $Q^2 = 0$ and that the minimal polynomial of B coincides with its characteristic polynomial and is again $p(x)$. Therefore, there exists an invertible P such that

$$A = P^{-1}BP = P^{-1}DP + P^{-1}QP.$$

Example 4.3: If the minimal polynomial of A is of the form $p(x) = x^4 + ax^2 + b$, then A admits a decomposition into $D + Q$, where D has set of eigenvalues $S_1 = \{1, 1, 2, 2\}$:

$$A \sim \begin{pmatrix} 0 & 0 & 0 & 2b \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2a \\ 0 & 0 & 1 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & -1 & 0 & 2b \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2a-1 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{Q}$$

Proposition 4.4: *Every matrix in $\mathbb{M}_4(\mathbb{F}_3)$ admits a decomposition into $D + Q$, where D is a diagonalizable matrix (in particular, a 3-potent matrix) and Q is a matrix with $Q^2 = 0$.*

Proof: Let us consider the decomposition of $p_A(x)$ into invariant factors $p_A(x) = f_1(x)f_2(x)\cdots$ with

$$\cdots f_3(x)|f_2(x)|f_1(x)$$

where $f_1(x)$ is the minimal polynomial of A. Then A similar to a direct sum of the companion matrices of $f_1(x), f_2(x)$, etc.

If for every i the degree of $f_i(x)$ is less than or equal to 2, the decomposition is straightforward (see Section 1). If the degree of $f_1(x)$ is 3 and the degree of $f_2(x)$ is one, then $f_2(x) = (x-a)|f_1(x)$, so $f_1(x) = (x-a)^k q(x)$ where $k = 1$, 2 or 3 and $(x-a)$ and $q(x)$ are co-primes. Hence $C(f_1(x)) \sim C((x-a)^k) \oplus C(q(x))$. If $k = 1$ or 2, A is similar to a direct sum of companion matrices of size ≤ 2, hence it is decomposable. If $k = 3$ then $f_1(x) = (x-a)^3$ and it correspond to a zero trace matrix that admits decomposition using three different eigenvalues.

Suppose from now on that $f_1(x) = p_A(x)$, i.e. A is the companion matrix of a polynomial of degree 4.

- First reduction: we can suppose that the trace of A is zero because A decomposes if and only if $A - tr(A)I$ decomposes. From now on $p(x) = x^4 + 2cx^2 + 2bx + 2a$ and

$$A = \begin{pmatrix} 0 & 0 & 0 & a \\ 1 & 0 & 0 & b \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- Second reduction: we can suppose that $|A| \neq 0$; otherwise $a = 0$ in which case, by the decomposition of Lemma 2.1 applied to the 3×3 block of A,

$$A = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & b \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \end{array}\right) = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & C(q(x)) & \\ 0 & & & \end{array}\right) + \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & \xi_1 \\ 0 & 0 & 0 & \xi_2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

where $C(q(x))$ is the companion matrix of $x(x-1)(x-2)$.

- Third reduction: we can suppose that $b \neq 0$. Otherwise, decompose as in Example 4.3.
- Fourth reduction: the companion matrix A of $p(x) = x^4 + 2cx^2 + 2bx + 2a$ admits a decomposition if and only if $2A$ admits a decomposition. The minimal polynomial of $2A$ is $q(x) = x^4 + 2cx^2 + bx + 2a$, so we can suppose that $b = 1$.

The remaining cases are $b = 1$, $a \neq 0$ and any $c = 0, 1, 2$.

$$A = \begin{pmatrix} 0 & 0 & 0 & a \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \end{pmatrix}, \text{i.e.}$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & \boxed{1} \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & \boxed{0} \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & \boxed{1} \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & \boxed{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & \boxed{2} \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & \boxed{0} \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 0 & \boxed{2} \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & \boxed{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} 0 & 0 & 0 & \boxed{1} \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & \boxed{1} \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 0 & 0 & 0 & \boxed{2} \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & \boxed{1} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- Matrix A_1 is similar to that of Example 4.1.
- Matrix A_2 is similar to that of Example 4.2. It is also similar to $2D + Q$ for D and Q given in Example 4.1.
- Matrix A_3: A_3 is similar to the following matrix

$$B = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & 2 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}}_{Q}.$$

Notice that $Q^2 = 0$ and that the minimal polynomial of B is again $p(x) = x^4 + x + 1$ so there exists an invertible P such that

$$A = P^{-1}BP = P^{-1}DP + P^{-1}QP.$$

- Decomposition of A_4: A_4 is similar to the following matrix

$$B = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & 2 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}}_{Q}.$$

Notice that $Q^2 = 0$ and that the (irreducible) characteristic polynomial of B is again $p(x) = x^4 + x^2 + x + 1$ so there exists an invertible P such that

$$A = P^{-1}BP = P^{-1}DP + P^{-1}QP.$$

- Decomposition of A_5: A_5 is similar to the following matrix

$$B = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 2 & 2 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}}_{Q}.$$

Notice that $Q^2 = 0$ and that the minimal polynomial of B is again $p(x) = x^4 + 2x^2 + x + 2$ so there exists an invertible P such that

$$A = P^{-1}BP = P^{-1}DP + P^{-1}QP.$$

- Decomposition of A_6: A_6 is similar to the following matrix

$$B = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 2 & 0 & 2 & 1 \end{pmatrix}}_{Q}.$$

Notice that $Q^2 = 0$ and that the minimal polynomial of B is again $p(x) = x^4 + 2x^2 + x + 1$ so there exists an invertible P such that

$$A = P^{-1}BP = P^{-1}DP + P^{-1}QP.$$

∎

We close our work with the following two still remaining problems of some interest and importance.

Problem 4.1: Decide whether or not the decomposition into a diagonalizable matrix and a square-zero matrix is still true for all $n \times n$ matrices with $n \geq 4$ over fields of at least three elements (excluding the 3×3 matrices over $\mathbb{F}_3$).

In contrast to finite fields, concerning the special case of some finite rings, especially $\mathbb{Z}_4$, in regards to recent results from [16] one can state the following query:

Problem 4.2: Does it follow that, for any $n \geq 1$, each element of $\mathbb{M}_n(\mathbb{Z}_4)$ is a sum of a square-zero nilpotent and a potent?

Notice that a similar representation of such a matrix ring already exists in terms of a nilpotent of order less than or equal to 8 and an idempotent (see, e.g. [3]). Even more generally, it was established in [6, Lemma 1] and [6, Theorem 4] that, for every $m \in \mathbb{N}$, the matrices in $\mathbb{M}_m(\mathbb{Z}_{p^n})$ are presentable as the sum of a nilpotent matrix and a p-potent matrix, whenever p is a prime.

Acknowledgments

The authors express their sincere thanks to the two anonymous expert referees for the careful reading of the manuscript and the competent insightful comments and suggestions made which improve substantially the structural shape of the presentation. The authors are also very grateful to the handling editor, Professor Stephane Gaubert, for his professional editorial management. The first named author is also very thankful to Professor Yaroslav Shitov for their valuable correspondence on the subject, which led to the main Question that motivated the writing of this paper, and on Remark 2.2.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

The first author was partially supported by the Bulgarian National Science Fund under Grant KP-06 N 32/1 of 7 December 2019. The second two authors were partially supported by MTM2017-84194-P (AEI/FEDER, UE), and by the Junta de Andalucía FQM264.

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