

Linear Algebra and Its Applications

Decompositions of Matrices Into a Sum of Torsion Matrices and Matrices of Fixed Nilpotence

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11 **DECOMPOSITIONS OF MATRICES INTO A SUM OF**
12 **TORSION MATRICES AND MATRICES OF FIXED**
13 **NILPOTENCE**
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21 **ABSTRACT.** For $n \geq 2$ and fixed $k \geq 1$, we study when a square matrix A over
22 an arbitrary field \mathbb{F} can be decomposed as $T + N$ where T is a torsion matrix
23 and N is a nilpotent matrix with $N^k = 0$. For fields of prime characteristic, we
24 show that this decomposition holds as soon as the characteristic polynomial of
25 $A \in \mathbb{M}_n(\mathbb{F})$ is algebraic over its base field and the rank of A is at least $\frac{n}{k}$, and
26 we present several examples that show that the decomposition does not hold
27 in general. Furthermore, we completely solve this decomposition problem for
28 $k = 2$ and nilpotent matrices over arbitrary fields (even over division rings).
29 This somewhat continues our recent publications in *Lin. & Multilin. Algebra*
30 (2023) and *Internat. J. Algebra & Computat.* (2022) as well as it strengthens
31 results due to Calugăreanu-Lam in *J. Algebra & Appl.* (2016).

32 *Key words:* (torsion) matrices, nilpotents, zero-square elements, polynomials

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35 **1. INTRODUCTION AND FUNDAMENTALS**

36 The decomposition of matrices over an arbitrary field into the sum of some special
37 elements, like nilpotent elements, idempotent elements, potent elements, units, etc.,
38 was in the focus of so many researchers for a long time (see, e.g., [1], [2], [3], [4], [5],
39 [6], [20], [22] and [23] and the bibliography cited therewith). Specifically, concerning
40 our own work on the subject, in [14] we found some necessary (and sufficient)
41 conditions when any square matrix over a field (finite or infinite) is expressible as
42 a sum of a diagonalizable matrix and a nilpotent matrix of index less than or equal
43 to two. In particular, we also obtained some results on the expression of square
44 matrices into the sum of a potent matrix and a square-zero matrix over finite fields.
45 Nevertheless, such a decomposition does not hold for fields of zero characteristic
46 (see [14, Example 4.3]). Further insight in that matter over some special finite rings
47 was achieved by us in [13]. We also refer the interested reader to [11] for some other
48 aspects of the realization of matrices into the sum of specific elements over certain
49 fields.
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51 By combining the notions of invertibility and nilpotence, Călugăreanu and Lam
52 introduced in 2016 the notion of fine rings [7]: those in which every nonzero element
53 can be written as the sum of an invertible element and a nilpotent one, proving in
54 that work that every nonzero square matrix over a division ring is the sum of an
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invertible matrix and a nilpotent matrix. The rings whose nonzero idempotents are fine turned out to be an interesting class of indecomposable rings and were studied in [8] by Călugăreanu and Zhou. In 2021, the same authors focused on rings in which every nonzero nilpotent element is fine, which they called NF rings, and showed that for a commutative ring R and $n \geq 2$, the matrix ring $\mathbb{M}_n(R)$ is NF if and only if R is a field; see [9]. A slightly more general class of rings than fine rings was defined in [12] under the name nil-good rings (every element a can be expressed the sum $a = n + u$ where n is nilpotent and u is either zero or a unit); in [18] it is shown that the matrix ring $\mathbb{M}_n(D)$ over a division ring D is nil-good. In general, no restriction in the index of nilpotence is required in these decompositions.

In our work [15] we considered the ring of matrices $\mathbb{M}_n(\mathbb{F})$ over an arbitrary field \mathbb{F} , we fixed a bound k for the index of nilpotence, and studied when a matrix A in $\mathbb{M}_n(\mathbb{F})$ could be expressed as the sum of an invertible matrix U and a nilpotent matrix N with $N^k = 0$. Here we will continue our study in this branch by replacing the invertibility condition of U by being a torsion matrix. Recall that a torsion matrix T is the one for which there is a positive integer s such that T^s is the identity matrix. One elementarily sees that such a matrix is necessarily invertible as well as that it is $s + 1$ -potent, i.e., $T^{s+1} = T$. Canonical forms of torsion matrices were studied by D. Sjerve and a full classification over the rational numbers is presented in his paper [21].

The paper is organized as follows: in the first section we will show that the desired decomposition holds as soon as the characteristic polynomial of A is algebraic over its base field and its rank satisfies a certain bound, and we present several examples that show that the decomposition does not hold in general. In the second section, we focus on nilpotent matrices and deal with the problem of finding a necessary and sufficient condition to decompose such matrices as the sum of a torsion matrix and a zero-square matrix (fixed nilpotence $k \leq 2$). Since we solve this problem by dealing with the Jordan canonical blocks of the considered nilpotent matrix, our result also holds for nilpotent matrices over division rings.

2. DECOMPOSING MATRICES INTO A SUM OF TORSION MATRICES AND MATRICES OF FIXED NILPOTENCE

As usual, for convenience of the presentation, let us specify that the letter \mathbb{F} will stand an arbitrary field unless it is not specified something else, and the symbol $\mathbb{M}_n(\mathbb{F})$ is reserved for the matrix ring over \mathbb{F} . All other unexplained explicitly notations are standard and will be in an agreement with the book [19].

In our work [15] we showed:

Theorem 2.1. [15, Theorem 2.7] *Let $n \geq 2$. Let \mathbb{F} be a field, consider the ring $\mathbb{M}_n(\mathbb{F})$ and let us fix $k \geq 1$. Given a nonzero matrix $A \in \mathbb{M}_n(\mathbb{F})$, there exists an invertible matrix $U \in \mathbb{M}_n(\mathbb{F})$ and a nilpotent matrix $N \in \mathbb{M}_n(\mathbb{F})$ with $N^k = 0$ such that $A = U + N$ if and only if the rank of A is greater than or equal to $\frac{n}{k}$*

In this section, we will address the following query:

Problem: *Given a fixed $k \geq 1$, find necessary and sufficient conditions to decompose any non-zero square matrix A over a field \mathbb{F} as a sum of a torsion matrix T and a nilpotent matrix N with $N^k = 0$.*

9 Notice that the proposed Problem is already solved for matrices over finite fields
10 by using Theorem 2.1 and the obvious fact that the unit group of finite rings is
11 always torsion. Nevertheless, the rank condition is not enough to guarantee this
12 decomposition when working over infinite fields. In the rest of this section, we
13 will show some cases when this decomposition holds, and some counterexamples
14 showing that the decomposition does not hold in general.

15 *Remark 2.2.* Let $A \in \mathbb{M}_n(\mathbb{F})$. If there exists a nilpotent matrix $N \in \mathbb{M}_n(\mathbb{F})$ ($N^k =$
16 0) and $T = A - N$ satisfies $T^s = \text{Id}$ for some $s \in \mathbb{N}$, then the following three points
17 are fulfilled:

- 18 • the trace of A coincides with the trace of T ;
- 19 • the minimal polynomial of T divides $X^s - 1$ and therefore the eigenvalues
20 of T (in some extension of \mathbb{F}) are s -roots of the unity. Moreover, if $X^s - 1$
21 is separable, T is diagonalizable;
- 22 • the trace of T coincides with the sum of its eigenvalues, so it is an algebraic
23 number over the base field of \mathbb{F} .

24 For example, a matrix $A \in \mathbb{M}_n(\mathbb{F})$, even of full rank, and whose trace is transc-
25 scendent over its base field can never be decomposed into the sum $T + N$, where T
26 is a torsion matrix and N is a nilpotent matrix.

27 Let $n \geq 2$. Recall that the trace of a polynomial $p(x) = x^n + b_{n-1}x^{n-1} + \dots + b_0 \in$
28 $\mathbb{F}[x]$ is the scalar $-b_{n-1}$ and coincides with the trace of the companion matrix
29 $C(p(x)) \in \mathbb{M}_n(\mathbb{F})$. Notice that the rank of a companion matrix is always greater
30 than or equal to $n - 1$.

31 We can now give a partial solution to the proposed above Problem. Concretely,
32 the following statements hold.

33 **Proposition 2.3.** *Let $n \geq 2$, let $p(x) \in \mathbb{F}[x]$ be a polynomial of degree n and let*
34 *$C(p(x)) \in \mathbb{M}_n(\mathbb{F})$ be its companion matrix. If the trace of $p(x)$ can be expressed*
35 *as the sum of n different roots of the unity in some extension of \mathbb{F} , then A can be*
36 *decomposed (in some extension of \mathbb{F}) into $T + N$, where T is a torsion matrix and*
37 *$N^2 = 0$. In particular, this always holds if the trace of $p(x)$ is either 1, or -1 , or*
38 *0.*

39 *Proof.* By hypothesis, the trace of $p(x) = x^n + b_{n-1}x^{n-1} + \dots + b_0$ can be expressed
40 as $\alpha_1 + \dots + \alpha_n$ for some different roots of unity $\alpha_1, \dots, \alpha_n$ in some extension of \mathbb{F} .
41 Let us consider the polynomial $q(x) = (x - \alpha_1) \dots (x - \alpha_n) = x^n + a_{n-1}x^{n-1} + \dots + a_0$.
42 Thus, we have

$$\begin{aligned}
 43 \quad C(p(x)) &= \begin{pmatrix} 0 & 0 & \dots & -b_0 \\ 1 & 0 & & \vdots \\ & \ddots & \ddots & \\ 44 \quad & 0 & & 1 & -b_{n-1} \end{pmatrix} \\
 45 \quad &= \underbrace{\begin{pmatrix} 0 & 0 & \dots & -a_0 \\ 1 & 0 & & \vdots \\ & \ddots & \ddots & \\ 46 \quad & 0 & & 1 & -a_{n-1} \end{pmatrix}}_{C(q(x))} + \underbrace{\begin{pmatrix} 0 & 0 & \dots & a_0 - b_0 \\ 0 & 0 & & \vdots \\ & \ddots & \ddots & \\ 47 \quad & 0 & & 0 & a_{n-1} - b_{n-1} \end{pmatrix}}_N, \\
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 \end{aligned}$$

where $T := C(q(x))$ is a diagonalizable torsion matrix and $N^2 = 0$, because $a_{n-1} - b_{n-1} = 0$, as required. In view of these arguments, the last claim follows now at once. \square

As an immediate consequence, we obtain:

Corollary 2.4. *Let $A \in \mathbb{M}_n(\mathbb{F})$ and let $p_1(x), \dots, p_k(x)$ be the elementary divisors of A (respectively, the invariant factors of A). If each $p_i(x)$ has degree n_i and its trace is a sum of n_i 's different roots of the unity in some extension of \mathbb{F} , then A can be decomposed (in some extension of \mathbb{F}) into $T + N$, where T is a torsion matrix and $N^2 = 0$.*

Proof. If $p_1(x), \dots, p_k(x)$ are the elementary divisors (respectively, the invariant factors of A), then the matrix A is similar to a direct sum of companion matrices of each $p_i(x)$. However, utilizing Proposition 2.3, we can express every companion matrix of $p_i(x)$ into the sum $T_i + N_i$, where T_i is a torsion matrix and $N_i^2 = 0$, as needed. \square

The next two curious comments are worthwhile.

Remark 2.5. Not every matrix $A \in \mathbb{M}_n(\mathbb{F})$ whose trace is a sum of n roots of unity can be written as $T + N$, where T is a torsion matrix and $N^2 = 0$ (even if those roots are different and A satisfies the rank condition). Indeed, let $n \geq 2$ and let us consider the element $a \in \mathbb{F}$ such that na is a sum of roots of unity, but a itself is not a root of unity (notice that such an element a always exists and is easy to be constructed, so we leave out the details). Then the matrix $A = a \text{Id} \in \mathbb{M}_n(\mathbb{F})$ cannot be written as a sum $T + N$; otherwise there would exist $m \in \mathbb{N}$ such that $T^m = \text{Id}$; but then $\text{Id} = T^m = (A - N)^m = a^m \text{Id} - ma^{m-1}N$, so that $(a^m - 1)^2 \text{Id} = ((a^m - 1) \text{Id})^2 = (ma^{m-1}N)^2 = 0$, a contradiction.

For example, the matrix $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ cannot be expressed as $T + N$ even if its trace is the sum of two (different) 6th-roots of unity: $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

Remark 2.6. Let \mathbb{F} be a field of characteristic 0. When $n \geq 3$, if $p(x)$ is a polynomial of degree n and the trace of $p(x)$ is the sum of n equal roots of unity, then the matrix $C(p(x))$ can never be written as $T + N$. Indeed, let α be a root of unity, and consider a degree n polynomial $p(x)$ whose trace is $n\alpha$ and its companion matrix $C(p(x))$. Suppose now that $C(p(x)) = T + N$ where T satisfies $T^m = \text{Id}$ for some $m \in \mathbb{N}$ and $N^2 = 0$. Since the minimal polynomial of T divides $X^m - 1$ and this polynomial has no multiple roots, then T is diagonalizable and its eigenvalues are all roots of unity whose sum coincides with the trace of T (which, on the other side, coincides with the trace of $p(x)$), so it is exactly $n\alpha$. The only solution to $n\alpha = \alpha_1 + \dots + \alpha_n$, $\alpha, \alpha_1, \dots, \alpha_n$ being roots of unity, is $\alpha = \alpha_1 = \dots = \alpha_n$. Therefore, the eigenvalues of T are all equal to α and thus $T = \alpha \text{Id}$. But then $N = C(p(x)) - \alpha \text{Id}$ should have zero square, which is manifestly untrue (the rank of $C(p(x)) - \alpha \text{Id}$ is at least $n - 1$).

For example, the matrix

$$C((x-1)^3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{pmatrix}$$

cannot be expressed as $T + N$, where T is a torsion matrix and $N^2 = 0$.

8 Nevertheless, when we focus on fields of prime characteristic, we can partially
9 solve the Problem. Let \mathbb{F} be a field of prime characteristic p and let us denote by
10 \mathbb{F}_p its base field.

11 **Lemma 2.7.** *Let $n \geq 2$. If \mathbb{F} is a field of characteristic p and we fix $k \geq 1$, then
12 for every matrix $A \in \mathbb{M}_n(\mathbb{F})$ of rank greater than or equal to $\frac{n}{k}$ and whose entries
13 are algebraic over \mathbb{F}_p there exists a matrix N with $N^k = 0$ such that $A - N$ is a
14 torsion matrix.
15*

16 *Proof.* The subfield \mathbb{K} of \mathbb{F} generated by the base field \mathbb{F}_p and by the entries of
17 matrix A is a finite field, and $A \in \mathbb{M}_n(\mathbb{K})$. Apply Theorem 2.1 to decompose A as
18 $U + N$, where $U \in \mathbb{M}_n(\mathbb{K})$ is invertible and $N \in \mathbb{M}_n(\mathbb{K})$ satisfies $N^k = 0$. Since U
19 is an invertible matrix over a finite field, being invertible is equivalent to being a
20 torsion matrix, as wanted. \square
21

22 The conditions on the entries of the matrix can be translated to the coefficients
23 of the characteristic polynomial of the matrix A . We will say that a polynomial is
24 algebraic over \mathbb{F}_p if all its coefficients are algebraic over \mathbb{F}_p .
25

26 **Theorem 2.8.** *Let \mathbb{F} be a field of characteristic p , let us fix an index of nilpotence
27 $k \geq 1$ and let $A \in \mathbb{M}_n(\mathbb{F})$ of rank greater than or equal to $\frac{n}{k}$. If the characteristic
28 polynomial of A is algebraic over \mathbb{F}_p , then A can be written as $T + N$, where T is
29 a torsion matrix and $N^k = 0$. In particular, this decomposition always holds for
30 nilpotent matrices of rank greater than or equal to $\frac{n}{k}$.
31*

32 *Proof.* Let us consider the primary rational canonical form C of A , whose character-
33 istic polynomial is algebraic over \mathbb{F}_p . The eigenvalues of C (roots in some extension
34 of \mathbb{F} of the characteristic polynomial) are algebraic over \mathbb{F}_p and, therefore, all the el-
35 ementary divisors of A are algebraic polynomials over \mathbb{F}_p . Consequently, the entries
36 of C are all algebraic over \mathbb{F}_p and we can apply Lemma 2.7 to get the proof. \square
37
38

39 **Open Question 1:** Given a fixed index of nilpotence $k \geq 1$, find a suitable criterion
40 for the decomposition of an arbitrary matrix over a field of zero characteristic into
41 the sum of a torsion matrix and a nilpotent matrix of index of nilpotence $\leq k$.
42

43 In the following section we will answer this question for $k \leq 2$ and nilpotent
44 matrices of rank at least $\frac{n}{2}$. Since our arguments are quite technical, we leave open
45 the question of decomposing nilpotent matrices of rank at least $\frac{n}{k}$ into torsion and
46 nilpotent matrices of index less than or equal to k .
47

48 3. DECOMPOSING NILPOTENT MATRICES INTO A SUM OF TORSION AND 49 SQUARE-ZERO MATRICES 50

51 The goal of this section is to show that, for any field \mathbb{F} , every nilpotent matrix in
52 $\mathbb{M}_n(\mathbb{F})$ whose rank is at least $\frac{n}{2}$ can always be decomposed as the sum of a torsion
53 matrix and a square-zero matrix. Recall that every nilpotent matrix is similar to a
54 direct sum of its Jordan blocks – all of them associated to the eigenvalue 0.
55

56 As in the previous section, the main difficulties arise when dealing with Jordan
57 blocks of size 1. The condition on the rank guarantees that they can always be
58 combined with Jordan blocks of bigger size. Let us consider the following two
59 examples.
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61
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Example 3.1. Let A be the nilpotent matrix

$$A = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right),$$

which is the smallest nilpotent matrix decomposed into Jordan blocks with one Jordan block of size 1 and rank of A greater than or equal to $\frac{n}{2}$ ($n = 4$). Let us define

$$N = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and let $T = A - N$. Then $N^2 = 0$ and the characteristic polynomial of T is $x^4 - 1$, so $A = T + N$ with $T^4 = \text{Id}$ and $N^2 = 0$.

Example 3.2. In some situations, more than one Jordan block of size 1 has to be combined with a Jordan block of bigger size. Consider for example

$$A = \left(\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right),$$

which again is the smallest nilpotent matrix expressed in Jordan blocks having two blocks of size 1 and satisfying $\text{rank}(A) \geq \frac{n}{2}$ ($n = 6$ here). Let us consider the zero-square matrix

$$N = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Then $T = A - N$ has characteristic polynomial equal to $x^6 - 1$, so $A = T + N$ with $T^6 = \text{Id}$ and $N^2 = 0$.

The construction of the zero-square matrices above depend on how many Jordan blocks of size 1 we have to combine with a Jordan block of bigger size. Let s be the number of Jordan blocks of size 1 (recall that the condition on the rank requires that $2(s + 1) \leq n$). The construction of N also depends on a certain parameter, called r , that must satisfy $1 + 2s + r < n$ and that it is zero in the above examples. With such ingredients, we are going to define a family of zero-square matrices called $N_{s,r}$ and an auxiliary family of zero-square matrices denoted $N'_{s,r}$.

Definition 3.3. Let $s, r \in \mathbb{N} \cup \{0\}$ such that $2(s+1) \leq n$ and $1+2s+r < n$. Let $A \in \mathbb{M}_n(\mathbb{F})$ be a nilpotent matrix consisting on a single Jordan block and s Jordan blocks of size 1 (notice that the condition $2(s+1) \leq n$ is equivalent to A having rank greater or equal to $\frac{n}{2}$). Let us define the matrices

$$\begin{aligned}
 N_{0,r} &:= -e_{1,n}, \\
 N_{s,r} &:= \sum_{i=n-2s-r}^{n-s-r-2} (e_{i+1,i} + e_{i+1+s+r+1,i}) + e_{1,n-s-r-1} \\
 &\quad - \sum_{i=n-s}^{n-1} (e_{i+1,i} + e_{i+1-s-r-1,i}) - e_{1,n}, \text{ if } s \geq 1, \\
 N'_{0,r} &:= -e_{1,n}, \\
 N'_{s,r} &:= Q_{n-2s-r,n-s-r-1} N_{s,r} Q_{n-2s-r,n-s-r-1} = \\
 &\quad = \sum_{i=n-2s-r}^{n-s-r-2} (e_{i+1,i} - e_{i+1+s+r+1,i}) - e_{1,n-s-r-1} \\
 &\quad - \sum_{i=n-s}^{n-1} (e_{i+1,i} - e_{i+1-s-r-1,i}) - e_{1,n}, \text{ if } s \geq 1, \\
 T_{s,r} &:= A - N_{s,r}, \text{ for } s \geq 0, \\
 T'_{s,r} &:= A - N'_{s,r} \text{ for } s \geq 0.
 \end{aligned}$$

Now we are going to show that the matrices $N_{s,r}$ and $N'_{s,r}$ are zero-square matrices. Let us recall the elementary matrices of $\mathbb{M}_n(\mathbb{F})$:

$$\begin{aligned}
 P_{i,j}(t) &:= \text{Id} + te_{i,j}, t \in \mathbb{F}. \\
 P_{i,j} &:= \text{Id} + e_{i,j} + e_{j,i} - e_{i,i} - e_{j,j} \\
 Q_i(s) &:= \text{Id} + (s-1)e_{i,i}, 0 \neq s \in \mathbb{F}.
 \end{aligned}$$

All of them are invertible with inverses $(P_{i,j}(t))^{-1} = P_{i,j}(-t)$, $(P_{i,j})^{-1} = P_{i,j}$ and $(Q_i(s))^{-1} = Q_i(1/s)$. Let us define $Q_{i,j} := \sum_{r=i}^j Q_r(-1)$.

Proposition 3.4. Let \mathbb{F} be a field and take $n \in \mathbb{N}$ and $s, r \in \mathbb{N} \cup \{0\}$ such that $2(s+1) \leq n$ and $1+2s+r < n$. Then, the matrices $N_{s,r}$ and $N'_{s,r}$ in $\mathbb{M}_n(\mathbb{F})$ are nilpotent of index 2.

Proof. Along this proof, let us identify matrices with the endomorphisms acting on the canonical basis $\{e_1, \dots, e_n\}$ of \mathbb{F}^n . Clearly $N_{0,r}^2 = (N'_{0,r})^2 = 0$. Suppose that $s \geq 1$ and let us denoted by N the matrix $N_{s,r}$. For every $i = 1 \dots, n$ we have:

- (1) if $i = 1, \dots, n-2s-r-1$, $N(e_i) = 0$,
- (2) if $i = n-2s-r, \dots, n-s-r-2$, $N(e_i) = e_{i+1} + e_{i+1+s+r+1}$,
- (3) $N(e_{n-s-r-1}) = e_1$,
- (4) if $i = n-s-r, \dots, n-s-1$, $N(e_i) = 0$,
- (5) if $i = n-s, \dots, n-1$, $N(e_i) = -e_{i+1} - e_{i+1-s-r-1}$,
- (6) $N(e_n) = -e_1$

Therefore, the image of N is spanned by

$$\{e_1\} \cup \{e_{n-2s-r+1} + e_{n-s+2}, \dots, e_{n-s-r-1} + e_n\} \cup \{e_{n-s+1} + e_{n-2s-r}, \dots, e_n + e_{n-s-r-1}\},$$

i.e., the image of N is the subspace generated by

$$S = \{e_1\} \cup \{e_{n-s+i} + e_{n-2s-r-1+i} \mid i = 1, \dots, s\}.$$

Let us see that $N(S) = 0$. Clearly, $N(e_1) = 0$; moreover, if $i = 1, \dots, s-1$, we have

$$\begin{aligned} N(e_{n-s+i} + e_{n-2s-r-1+i}) &= \\ &= -e_{n-s+i+1} - e_{n-2s-r+i} + e_{n-2s-r+i} + e_{n-s+i+1} = 0 \end{aligned}$$

and $N(e_n + e_{n-s-r-1}) = -e_1 + e_1 = 0$. Therefore, $N^2 = 0$.

Since the matrix $Q_{n-2s-r, n-s-r-1}$ satisfies $Q_{n-2s-r, n-s-r-1}^2 = \text{Id}$, we deduce that $N'_{s,r}$ is also a zero-square matrix, as required. \square

In Theorem 3.10 we will calculate the characteristic polynomials of the matrices

$$T_{s,r} := A - N_{s,r} \quad \text{and} \quad T'_{s,r} := A - N'_{s,r},$$

defined in 3.3. Since characteristic polynomials are invariant under similarity, our argument will consist on transforming by similarity the matrices $T_{s,r}$ and $T'_{s,r}$ in a finite number of steps into matrices of the form $T_{s',r'}$ or $T'_{s',r'}$, $s' = 0, 1$ or 2 . We will first prove two descending lemmas. In each of these descending lemmas, we will decrease the first subscript of $T_{s,r}$ or of $T'_{s,r}$ by three units and increase the second subscript by three units. Afterwards, it will only remain to explicitly calculate the characteristic polynomial of the matrices $T_{s',r'}$ and $T'_{s',r'}$, $s' = 0, 1, 2$.

Lemma 3.5 (From $T_{s,r}$ to $T'_{s-3,r+3}$). *Let \mathbb{F} be a field and let $A \in \mathbb{M}_n(\mathbb{F})$ be a nilpotent matrix consisting on a single Jordan block of rank bigger than or equal to $\frac{n}{2}$ and s Jordan blocks of size 1. Suppose that $s \geq 3$ and let $r \in \mathbb{N} \cup \{0\}$ such that $1 + 2s + r < n$. Let $T_{s,r}$ and $T'_{s,r}$ be the matrices defined in 3.3. Then, the matrix $T_{s,r}$ is similar to the matrix $T'_{s-3,r+3}$.*

Proof. For $v = s + r + 1$ and

$$\begin{aligned} T_1 &:= P_{n-s+3, n-s+3-v}(-1) \cdot T_{s,r} \cdot P_{n-s+3, n-s+3-v}(1) \\ T_2 &:= P_{n-s+2, n-s+2-v} \cdot T_1 \cdot P_{n-s+2, n-s+2-v} \\ T_3 &:= P_{n-s+1-v, n-s+1}(-1) \cdot T_2 \cdot P_{n-s+1-v, n-s+1}(1), \\ T_4 &:= Q_{n-s+2-v, n-s+2} \cdot T_3 \cdot Q_{n-s+2-v, n-s+2}, \end{aligned}$$

we will show that $T_4 = T'_{s-3,r+3}$.

Along this proof, let us identify matrices with the endomorphisms acting on the canonical basis $\{e_1, \dots, e_n\}$ of \mathbb{F}^n . Let us denote by P the following matrix

$$P := P_{n-s+3, n-s+3-v}(1) \cdot P_{n-s+2, n-s+2-v} \cdot P_{n-s+1-v, n-s+1}(1) \cdot Q_{n-s+2-v, n-s+2-v}.$$

With this notation in hand, we have $T_4 = P^{-1}T_{s,r}P$. Let us calculate $P(e_i)$, $i = 1, \dots, n$:

$$\begin{aligned} P(e_i) &= e_i \text{ for } i = 1, 2, \dots, n-s-v, \\ P(e_{n-s-v+1}) &= e_{n-s-v+1}, \\ P(e_{n-s-v+2}) &= -e_{n-s+2}, \\ P(e_{n-s-v+3}) &= -e_{n-s-v+3} - e_{n-s+3}, \\ P(e_i) &= -e_i \text{ for } i = n-s-v+4, \dots, n-v \quad (\text{if } n-s-v+4 \leq n-v), \\ P(e_i) &= -e_i \text{ for } i = n-v+1, \dots, n-s-1 \quad (\text{if } n-v+1 \leq n-s-1), \\ P(e_{n-s}) &= -e_{n-s} \quad (\text{if } s \neq 3), \\ P(e_{n-s+1}) &= -e_{n-s+1} - e_{n-s+1-v}, \end{aligned}$$

$$\begin{aligned}
 P(e_{n-s+2}) &= -e_{n-s+2-v} \text{ and} \\
 P(e_i) &= e_i \text{ for } i = n-s+3, \dots, n.
 \end{aligned}$$

Let us see how $T_4 = P^{-1}T_{s,r}P$ acts on the canonical basis: to do so let us divide the set $\{1, 2, \dots, n\}$ into the following subsets relative to the definition of $T'_{s-3,r+3}$:

$$\begin{aligned}
 \Delta'_1 &:= \{1, \dots, n-s-v+3\} & \Delta'_2 &:= \{n-s-v+4, \dots, n-v-1\} \\
 \Delta'_3 &:= \{n-v, \text{ if } s \neq 3\} & \Delta'_4 &:= \{n-v+1, \dots, n-s+2\} \\
 \Delta'_5 &:= \{n-s+3, \dots, n-1\} & \Delta'_6 &:= \{n\}
 \end{aligned}$$

(notice that if $s = 3$, then $\Delta'_2 = \emptyset$, $\Delta'_3 = \emptyset$ and $\Delta'_5 = \emptyset$; and if $s = 4$, then $\Delta'_2 = \emptyset$).

• T_4 acting on elements of Δ'_1 :

- (1) $T_4(e_i) = P^{-1}T_{s,r}P(e_i) = P^{-1}T_{s,r}(e_i) = P^{-1}(e_{i+1}) = e_{i+1}$ for $i = 1, 2, \dots, n-s-v$,
- (2) $T_4(e_{n-s-v+1}) = P^{-1}T_{s,r}P(e_{n-s-v+1}) = P^{-1}T_{s,r}(e_{n-s-v+1}) = P^{-1}(-e_{n-s+2}) = e_{n-s-v+2}$,
- (3) $T_4(e_{n-s-v+2}) = P^{-1}T_{s,r}P(e_{n-s-v+2}) = P^{-1}T_{s,r}(-e_{n-s+2}) = P^{-1}(-e_{n-s+3} - e_{n-s-v+3}) = e_{n-s-v+3}$,
- (4) $T_4(e_{n-s-v+3}) = P^{-1}T_{s,r}P(e_{n-s-v+3}) = P^{-1}T_{s,r}(-e_{n-s+3} - e_{n-s-v+3}) = P^{-1}(-e_{n-s+4} - e_{n-s-v+4} + e_{n-s+4}) = P^{-1}(-e_{n-s-v+4}) = e_{n-s-v+4}$.

• T_4 acting on elements of Δ'_2 (when $\Delta'_2 \neq \emptyset$):

- (5) $T_4(e_i) = P^{-1}T_{s,r}P(e_i) = P^{-1}T_{s,r}(-e_i) = P^{-1}(e_{i+1+v}) = e_{i+1+v}$ for $i = n-s-v+4, \dots, n-v-1$.

• T_4 acting on elements of Δ'_3 (when $\Delta'_3 \neq \emptyset$):

- (6) $T_4(e_{n-v}) = P^{-1}T_{s,r}P(e_{n-v}) = P^{-1}T_{s,r}(-e_{n-v}) = P^{-1}(e_1 - e_{n-v+1}) = e_1 + e_{n-v+1}$.

• T_4 acting on elements of Δ'_4 :

- (7) $T_4(e_i) = P^{-1}T_{s,r}P(e_i) = P^{-1}T_{s,r}(-e_i) = P^{-1}(-e_{i+1}) = e_{i+1}$ for $i = n-v+1, \dots, n-s-1$,
- (8) $T_4(e_{n-s}) = P^{-1}T_{s,r}P(e_{n-s}) = P^{-1}T_{s,r}(-e_{n-s}) = P^{-1}(-e_{n-s+1} - e_{n-s+1-v}) = e_{n-s+1}$,
- (9) $T_4(e_{n-s+1}) = P^{-1}T_{s,r}P(e_{n-s+1}) = P^{-1}T_{s,r}(-e_{n-s+1} - e_{n-s+1-v}) = P^{-1}(-e_{n-s+2} - e_{n-s+2-v} + e_{n-s+2}) = P^{-1}(-e_{n-s+2-v}) = e_{n-s+2}$,
- (10) $T_4(e_{n-s+2}) = P^{-1}T_{s,r}P(e_{n-s+2}) = P^{-1}T_{s,r}(-e_{n-s+2-v}) = P^{-1}(e_{n-s+3}) = e_{n-s+3}$.

• T_4 acting on elements of Δ'_5 (when $\Delta'_5 \neq \emptyset$):

- (11) $T_4(e_{n-s+3}) = P^{-1}T_{s,r}P(e_{n-s+3}) = P^{-1}T_{s,r}(e_{n-s+3}) = P^{-1}(e_{n-s+4} + e_{n-s+4-v}) = e_{n-s+4} - e_{n-s+4-v}$,
- (12) $T_4(e_i) = P^{-1}T_{s,r}P(e_i) = P^{-1}T_{s,r}(e_i) = P^{-1}(e_{i+1} + e_{i+1-v}) = e'_{i+1} - e'_{i+1-v}$ for $i = n-s+4, \dots, n-1$.

• T_4 acting on elements of Δ'_6 :

- (13) $T_4(e_n) = P^{-1}T_{s,r}P(e_n) = P^{-1}T_{s,r}(e_n) = P^{-1}(e_1) = e_1$,

which proves that $T_4 = P^{-1}T_{s,r}P$ behaves on the canonical basis in the same way as $T'_{s-3,r+3}$ so $T_4 = T'_{s-3,r+3}$. \square

Following the same arguments as in the above proof we also obtain the following descending lemma.

Lemma 3.6 (From $T'_{s,r}$ to $T_{s-3,r+3}$). *Let \mathbb{F} be a field and let $A \in \mathbb{M}_n(\mathbb{F})$ be a nilpotent matrix consisting on a single Jordan block of rank bigger than or equal to $\frac{n}{2}$ and s Jordan blocks of size 1. Suppose that $s \geq 3$ and let $r \in \mathbb{N} \cup \{0\}$ such that $1 + 2s + r < n$. Let $T_{s,r}$ and $T'_{s,r}$ be the matrices defined in 3.3. Then the matrix $T'_{s,r}$ is similar to the matrix $T_{s-3,r+3}$. Indeed, if $v = s + r + 1$ and*

$$\begin{aligned} T'_1 &:= P_{n-s+3,n-s+3-v}(1) \cdot T'_{s,r} \cdot P_{n-s+3,n-s+3-v}(-1) \\ T'_2 &:= P_{n-s+2,n-s+2-v} \cdot T'_1 \cdot P_{n-s+2,n-s+2-v} \\ T'_3 &:= P_{n-s+1-v,n-s+1}(1) \cdot T'_2 \cdot P_{n-s+1-v,n-s+1}(-1), \\ T'_4 &:= Q_{n-v-s+3,n-s+1} \cdot T'_3 \cdot Q_{n-v-s+3,n-s+1}, \end{aligned}$$

we obtain that $T'_4 = T_{s-3,r+3}$.

After these two descending lemmas, let us compute the characteristic polynomials of $T_{s,r}$ and $T'_{s,r}$ when $s = 0, 1$ or 2 .

Remark 3.7 (Case $s = 0$). Let \mathbb{F} be a field and let $A \in \mathbb{M}_n(\mathbb{F})$ be a nilpotent matrix consisting on a single Jordan block of rank bigger than or equal to $\frac{n}{2}$ and no Jordan blocks of size 1 ($s = 0$). For any $r < n - 1$ we have that $T_{0,r} = T'_{0,r} = \sum_{t=1}^{n-1} e_{t+1,t} + e_{1,n}$, and its characteristic polynomial is exactly $x^n - 1$.

Lemma 3.8 (Case $s = 1$). *Let \mathbb{F} be a field and let $A \in \mathbb{M}_n(\mathbb{F})$ be a nilpotent matrix consisting on a single Jordan block of rank bigger than or equal to $\frac{n}{2}$ and one Jordan block of size 1. Let $r \geq 0$ be such that $r + 3 < n$. Let us consider the matrices $T_{1,r} = A - N_{1,r}$ and $T'_{1,r} = A - N'_{1,r}$. Then, the characteristic polynomials are*

$$\begin{aligned} \bullet \quad p_{T_{1,r}}(x) &= x^n + x^{r+2} - x^{n-r-2} - 1 = (x^{r+2} - 1)(x^{n-r-2} + 1) \\ \bullet \quad p_{T'_{1,r}}(x) &= x^n - x^{r+2} + x^{n-r-2} - 1 = (x^{r+2} + 1)(x^{n-r-2} - 1). \end{aligned}$$

Proof. We will use the following general remark: suppose we are computing the determinant of a matrix of the form

$$B = \begin{pmatrix} x & 0 & \cdots & 0 & 1 \\ -1 & x & \cdots & & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & x & 0 \\ 0 & 0 & \cdots & -1 & x \end{pmatrix} + a e_{i,j}$$

where a is a non-zero element at position (i, j) , $j > i$. If we multiply B on the left by $P_{i,j+1}(a)$ (we add to row i the row $j + 1$ multiplied by a), we get a new matrix that has a zero at position (i, j) and has ax at position $(i, j + 1)$. Since

$|P_{i,j+1}(a)| = 1$ we get

$$\det \left(\left(\begin{array}{ccccc} x & 0 & \cdots & 0 & 1 \\ -1 & x & \cdots & & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & x & 0 \\ 0 & 0 & \cdots & -1 & x \end{array} \right) + a e_{i,j} \right) =$$

$$\det \left(\left(\begin{array}{ccccc} x & 0 & \cdots & 0 & 1 \\ -1 & x & \cdots & & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & x & 0 \\ 0 & 0 & \cdots & -1 & x \end{array} \right) + ax e_{i,j+1} \right)$$

i.e., when computing the determinant of B , nonzero elements at positions (i, j) , $j > i$, can be moved to the right by multiplying by x at each step.

By a similar argument, multiplying B on the right by the elementary matrix $P_{i-1,j}(a)$ (adding to column j the column $i - 1$ multiplied by a), we also have that

$$\det \left(\left(\begin{array}{ccccc} x & 0 & \cdots & 0 & 1 \\ -1 & x & \cdots & & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & x & 0 \\ 0 & 0 & \cdots & -1 & x \end{array} \right) + a e_{i,j} \right) =$$

$$\det \left(\left(\begin{array}{ccccc} x & 0 & \cdots & 0 & 1 \\ -1 & x & \cdots & & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & x & 0 \\ 0 & 0 & \cdots & -1 & x \end{array} \right) + ax e_{i-1,j} \right)$$

i.e., when computing the determinant of B , nonzero elements at positions (i, j) , $j > i$, can be moved upwards to position $(i - 1, j)$ by multiplying at each step by x .

Notice that

$$T_{1,r} = \sum_{t=1}^{n-1} e_{t+1,t} + e_{1,n} + e_{n-r-2,n-1} - e_{1,n-r-2}.$$

Applying the argument above to the determinant of the matrix $x \text{Id} - T_{1,r}$, we can move the nonzero element at position $(n - r - 2, n - 1)$ towards position $(1, n)$ obtaining $-x^{n-r-2}$ and we can move the non-zero element at position $(1, n - r - 2)$ towards position $(1, n)$ obtaining x^{r+2} , getting at the end the determinant of a matrix of the form

$$\begin{pmatrix} x & 0 & 0 & \cdots & 0 & -x^{n-r-2} + x^{r+2} - 1 \\ -1 & x & 0 & \cdots & & 0 \\ 0 & -1 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & x & 0 \\ 0 & 0 & 0 & \cdots & -1 & x \end{pmatrix}.$$

Finally, by the Laplace expansion of the determinant along the first row we get that the characteristic polynomial of $T_{1,r}$ is given by $x^n - x^{n-r-2} + x^{r+2} - 1$.

Similarly, since

$$T'_{1,r} = \sum_{t=1}^{n-1} e_{t+1,t} + e_{1,n} - e_{n-r-2,n-1} + e_{1,n-r-2},$$

its characteristic polynomial is $x^n + x^{n-r-2} - x^{r+2} - 1$. \square

We continue our work with the next technical assertion.

Lemma 3.9 (Case $s = 2$). *Let \mathbb{F} be a field and let $A \in \mathbb{M}_n(\mathbb{F})$ be a nilpotent matrix consisting on a single Jordan block of rank bigger than or equal to $\frac{n}{2}$ and two Jordan blocks of size 1. Let $r \geq 0$ be such that $r + 5 < n$. Let us consider the matrices $T_{2,r} = A - N_{2,r}$ and $T'_{2,r} = A - N'_{2,r}$. Then, the characteristic polynomials are*

- $p_{T_{2,r}}(x) = x^n + x^{r+3} - x^{n-r-3} - 1 = (x^{n-r-3} + 1)(x^{r+3} - 1)$
- $p_{T'_{2,r}}(x) = x^n - x^{r+3} + x^{n-r-3} - 1 = (x^{n-r-3} - 1)(x^{r+3} + 1)$.

Proof. Let $\{e_1, \dots, e_n\}$ be the canonical basis of \mathbb{F}^n . Let

$$P := P_{n,n-r-3} \cdot Q_{n-r-3,n-r-3}$$

and consider the new basis

$$\{e'_i, | i = 1, \dots, n\}, \text{ where each } e'_i := P(e_i).$$

This new basis is just a reordering of the canonical basis and a change of sign of one of its vectors:

$$\{e'_i, | i = 1, \dots, n\} = \{e_1, e_2, \dots, e_{n-r-4}, -e_n\} \cup \{e_{n-r-3}, \dots, e_{n-1}\}.$$

Recall that

$$\begin{aligned} T_{2,r} &= \sum_{t=1}^{n-r-5} e_{t+1,t} - e_{n,n-r-4} + e_{1,n} \\ &+ \sum_{t=n-r-3}^{n-1} e_{t+1,t} - e_{1,n-r-3} + e_{n-r-4,n-2} + e_{n-r-1,n-1}. \end{aligned}$$

Then, the matrix $P^{-1} \cdot T_{2,r} \cdot P$ is of the form

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & -1 & * & * & * & * \\ 1 & 0 & 0 & 0 & * & * & * & * \\ 0 & \ddots & 0 & 0 & * & * & * & * \\ 0 & 0 & 1 & 0 & * & * & * & * \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right)$$

and its characteristic polynomial, which coincides with the characteristic polynomial of $T_{2,r}$, is the product of the characteristic polynomials of the two diagonal blocks, i.e., it is precisely $(x^{n-r-3} + 1)(x^{r+3} - 1)$.

Similarly, if we consider $P' := P_{n,n-r-3} \cdot Q_{n,n}$ and the new basis

$$\{e''_i, | i = 1, \dots, n\}, \text{ where each } e''_i := P'(e_i),$$

then,

$$\{e''_i, | i = 1, \dots, n\} = \{e_1, e_2, \dots, e_{n-r-4}, e_n\} \cup \{e_{n-r-3}, \dots, -e_{n-1}\}.$$

Thus, the matrix $(P')^{-1} \cdot T'_{2,r} \cdot P'$ is of the form

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & * & * & * & * \\ 1 & 0 & 0 & 0 & * & * & * & * \\ 0 & \ddots & 0 & 0 & * & * & * & * \\ 0 & 0 & 1 & 0 & * & * & * & * \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right)$$

and its characteristic polynomial, which coincides with the characteristic polynomial of $T'_{2,r}$, is the product of the characteristic polynomials of the two diagonal blocks, i.e., it is precisely $(x^{n-r-3} - 1)(x^{r+3} + 1)$. \square

We thus arrive at the following result in which we compute the characteristic polynomials of the matrices $T_{s,r} = A - N_{s,r}$ and $T'_{s,r} = A - N'_{s,r}$. They depend on the equivalence class of s modulo 3.

Theorem 3.10. *Let \mathbb{F} be a field, $n \in \mathbb{N}$, and let $s \in \mathbb{N} \cup \{0\}$ such that $2(s+1) \leq n$. Let $A \in \mathbb{M}_n(\mathbb{F})$ be a nilpotent matrix consisting on a single Jordan block and s Jordan blocks of size 1. Then,*

- (1) *If $s = 3\alpha$ for some $\alpha \geq 0$ and we take any r such that $1 + 2s + r < n$, then the characteristic polynomial of $T_{s,r} = A - N_{s,r}$ is*

$$p_{T_{3\alpha,r}}(x) = x^n - 1.$$

(2.1) If $s = 3\alpha + 1$ for some even $\alpha \geq 0$ and we take any r such that $1 + 2s + r < n$ then the characteristic polynomial of $T_{s,r} = A - N_{s,r}$ is given by

$$p_{T_{3\alpha+1,r}}(x) = p_{T_{1,r+3\alpha}}(x) = (x^{r+3\alpha+2} - 1)(x^{n-r-3\alpha-2} + 1),$$

and the characteristic polynomial of $T'_{s,r} = A - N'_{s,r}$ is given by

$$p_{T'_{3\alpha+1,r}}(x) = p_{T'_{1,r+3\alpha}}(x) = (x^{r+3\alpha+2} + 1)(x^{n-r-3\alpha-2} - 1).$$

(2.2) If $s = 3\alpha + 1$ for some odd $\alpha \geq 1$, and we take any r such that $1 + 2s + r < n$ then the characteristic polynomial of $T_{s,r} = A - N_{s,r}$ is given by

$$p_{T_{3\alpha+1,r}}(x) = p_{T_{1,r+3\alpha}}(x) = (x^{r+3\alpha+2} + 1)(x^{n-r-3\alpha-2} - 1),$$

and the characteristic polynomial of $T'_{s,r} = A - N'_{s,r}$ is given by

$$p_{T'_{3\alpha+1,r}}(x) = p_{T'_{1,r+3\alpha}}(x) = (x^{r+3\alpha+2} - 1)(x^{n-r-3\alpha-2} + 1).$$

(3.1) If $s = 3\alpha + 2$ for some even $\alpha \geq 0$ and we take any r such that $1 + 2s + r < n$, then the characteristic polynomial of $T_{s,r} = A - N_{s,r}$ is given by

$$p_{T_{3\alpha+2,r}}(x) = p_{T_{2,r+3\alpha}}(x) = (x^{n-r-3\alpha-3} + 1)(x^{r+3\alpha+3} - 1),$$

and the characteristic polynomial of $T'_{s,r} = A - N'_{s,r}$ is given by

$$p_{T'_{3\alpha+2,r}}(x) = p_{T'_{2,r+3\alpha}}(x) = (x^{n-r-3\alpha-3} - 1)(x^{r+3\alpha+3} + 1).$$

(3.2) If $s = 3\alpha + 2$ for some odd $\alpha \geq 0$ and we take any r such that $1 + 2s + r < n$, then the characteristic polynomial of $T_{s,r} = A - N_{s,r}$ is given by

$$p_{T_{3\alpha+2,r}}(x) = p_{T_{2,r+3\alpha}}(x) = (x^{n-r-3\alpha-3} - 1)(x^{r+3\alpha+3} + 1),$$

and the characteristic polynomial of $T'_{s,r} = A - N'_{s,r}$ is given by

$$p_{T'_{3\alpha+2,r}}(x) = p_{T'_{2,r+3\alpha}}(x) = (x^{n-r-3\alpha-3} + 1)(x^{r+3\alpha+3} - 1).$$

Proof. Notice that the condition $2(s + 1) \leq n$ is just equivalent to A having rank bigger or equal to $\frac{n}{2}$.

(1) Suppose that $s = 3\alpha$ for some $\alpha \geq 0$. Let us consider $T_{3\alpha,r} = A - N_{3\alpha,r}$. If $\alpha \geq 1$, applying Lemma 3.5 to $T_{3\alpha,r}$ we obtain the matrix $T'_{3(\alpha-1),r+3}$, which is similar to $T_{3\alpha,r}$. If $\alpha - 1 \geq 1$ then use Lemma 3.6 to get another similar matrix $T_{3(\alpha-2),r+6}$, and repeating this process, in α steps we will end with either $T_{0,r+3\alpha}$ or with $T'_{0,r+3\alpha}$, which are both equal to the matrix $\sum_{t=1}^{n-1} e_{t+1,t} + e_{1,n}$ (see Remark 3.7), whose characteristic polynomial is $x^n - 1$. Thus the characteristic polynomial of $T_{s,r}$ is $x^n - 1$.

(2.1) Suppose that $s = 3\alpha + 1$ for some even $\alpha \geq 0$. Let us consider $T_{s,r}$. If $\alpha > 1$, as in (1) we can use Lemma 3.5 and Lemma 3.6 several times to obtain, in α steps, a similar matrix $T_{1,r+3\alpha}$; therefore, the characteristic polynomial of $T_{s,r}$ coincides with the characteristic polynomial of $T_{1,r+3\alpha}$, which is $(x^{r+3\alpha+2} - 1)(x^{n-r-3\alpha-2} + 1)$ by Lemma 3.8. Similarly, if we begin with $T'_{s,r}$ and we use Lemmas 3.6 and 3.5, we will end up in α steps with the similar matrix $T'_{1,r+3\alpha}$, whose characteristic polynomial is $(x^{r+3\alpha+2} + 1)(x^{n-r-3\alpha-2} - 1)$ by Lemma 3.8.

(2.2) Suppose that $s = 3\alpha + 1$ for some odd $\alpha \geq 1$. If we start with $T_{s,r}$ and apply Lemmas 3.5 and 3.6, after α steps we will end up with the similar matrix $T'_{1,r+3\alpha}$, and if we begin with $T'_{s,r}$ we will end up with $T_{1,r+3\alpha}$.

(3.1) and (3.2) follow as (1) and (2.1) and (2.2) above, taking into account the characteristic polynomials of $T_{2,r+3\alpha}$ and $T'_{2,r+3\alpha}$ calculated in Lemma 3.9. \square

As three important consequences, we derive the following:

Corollary 3.11. *Let \mathbb{F} be a field, suppose that $n \in \mathbb{N}$ is even, and let $s \in \mathbb{N} \cup \{0\}$ such that $2(s+1) \leq n$. Let $A \in \mathbb{M}_n(\mathbb{F})$ be a nilpotent matrix consisting on a single Jordan block and s Jordan blocks of size 1. If we take $r = \frac{n}{2} - s - 1$, the characteristic polynomial of $T_{s,r} = A - N_{s,r}$ and the characteristic polynomial of $T'_{s,r} = A - N'_{s,r}$ coincide and they are $x^n - 1$. In particular, $T_{s,r}^n = \text{Id}$ so that A decomposes as the sum of a torsion matrix $T_{s,r}$ and a zero-square matrix $N_{s,r}$.*

Proof. If $s \equiv_3 0$, since the characteristic polynomial of $T_{s,r}$ is always $x^n - 1$ for any r such that $1 + 2s + r < n$, this also is true if we take $r = \frac{n}{2} - s - 1$.

If $s \equiv_3 1$ or $s \equiv_3 2$ and we take $r = \frac{n}{2} - s - 1$, we obtain that the characteristic polynomial of $T_{s,r}$ is $(x^{\frac{n}{2}} - 1)(x^{\frac{n}{2}} + 1) = x^n - 1$. \square

Corollary 3.12. *Let \mathbb{F} be a field, suppose that $n \in \mathbb{N}$ is odd, and let $s \in \mathbb{N} \cup \{0\}$ such that $2(s+1) \leq n$. Let $A \in \mathbb{M}_n(\mathbb{F})$ be a nilpotent matrix consisting on a single Jordan block and s Jordan blocks of size 1.*

- (1) *If $s \equiv_3 0$ and we take any r such that $1 + 2s + r < n$, then $N_{s,r}$ such that the characteristic polynomial of $T_{s,r} = A - N_{s,r}$ is $x^n - 1$.*
- (2) *If $s \equiv_3 1$ or $s \equiv_3 2$, and:*
 - (a) *if $n = 4m + 1$, α is even, $s = 3\alpha + 1$ or $s = 3\alpha + 2$, and we take $r = 2m - s$, then $N_{s,r}$ is such that the characteristic polynomial of $T_{s,r} = A - N_{s,r}$ is $(x^{2m} + 1)(x^{2m+1} - 1)$;*
 - (b) *if $n = 4m + 1$, α is odd, $s = 3\alpha + 1$ or $s = 3\alpha + 2$, and we take $r = 2m - s$, then $N'_{s,r}$ is such that the characteristic polynomial of $T'_{s,r} = A - N'_{s,r}$ is $(x^{2m} + 1)(x^{2m+1} - 1)$;*
 - (c) *if $n = 4m + 3$, α is even, $s = 3\alpha + 1$ or $s = 3\alpha + 2$, and we take $r = 2m - s$, then $N_{s,r}$ is such that the characteristic polynomial of $T_{s,r} = A - N_{s,r}$ is $(x^{2m+2} + 1)(x^{2m+1} - 1)$;*
 - (d) *if $n = 4m + 3$, α is odd, $s = 3\alpha + 1$ or $s = 3\alpha + 2$, and we take $r = 2m - s$, then $N'_{s,r}$ is such that the characteristic polynomial of $T'_{s,r} = A - N'_{s,r}$ is $(x^{2m+2} + 1)(x^{2m+1} - 1)$.*

Proof. It is an automatic consequence of Theorem 3.10. \square

Corollary 3.13. *Let \mathbb{F} be a field of zero characteristic, suppose that $n \in \mathbb{N}$ is odd, and let $s \in \mathbb{N} \cup \{0\}$ such that $2(s+1) \leq n$. Let $A \in \mathbb{M}_n(\mathbb{F})$ be a nilpotent matrix consisting on a single Jordan block and s Jordan blocks of size 1. Then, there exists a zero-square matrix N and $d \geq 1$ such that $(A - N)^d = \text{Id}$, i.e., A decomposes as the sum of a torsion matrix and a zero-square matrix.*

Proof. Following the cases of the above corollary, we have:

- if we are in case (1), take $N = N_{s,r}$ and $d = n$; clearly $(A - N)^n = \text{Id}$;
- if we are in case (2)(a), take $r = 2m - s$ and $N = N_{s,r}$; then if $d = \text{lcm}(4m, 2m+1)$, we obtain $(A - N)^d = \text{Id}$, because the polynomials $x^{2m} + 1$ and $x^{2m+1} - 1$ have no common roots, so the matrix $A - N$ has n different eigenvalues (in the algebraic closure of \mathbb{F}), all of them roots of unity ξ with $\xi^{4m} = 1$ or $\xi^{2m+1} = 1$;
- if we are in case (2)(b), take $r = 2m - s$ and $N = N'_{s,r}$; then if $d = \text{lcm}(4m, 2m+1)$, we obtain $(A - N)^d = \text{Id}$ arguing as in case (2)(a);

- if we are in case (2)(c), take $r = 2m - s$ and $N = N_{s,r}$; then if $d = \text{lcm}(4m + 2, 2m + 1)$, we obtain $(A - N)^d = \text{Id}$ arguing as in case (2)(a);
- if we are in case (2)(d), take $r = 2m - s$ and $N = N'_{s,r}$; then if $d = \text{lcm}(4m + 2, 2m + 1)$, we obtain $(A - N)^d = \text{Id}$ arguing as in case (2)(a).

□

We now have all the ingredients necessary to establish our chief result in this section.

Theorem 3.14. *Any nilpotent matrix in $\mathbb{M}_n(\mathbb{F})$ can be written as the sum of a torsion matrix plus a square-zero matrix if, and only if, its rank is at least $\frac{n}{2}$.*

Proof. According to the above Theorem 2.1, a necessary condition to express $A \in \mathbb{M}_n(\mathbb{F})$ as the sum $T + N$, where T is a torsion matrix and $N^2 = 0$, is that the rank of A is greater than or equal to $\frac{n}{2}$, because torsion matrices must be invertible. Conversely, to prove the sufficiency, suppose that the rank of $A \in \mathbb{M}_n(\mathbb{F})$ is no less than $\frac{n}{2}$.

If \mathbb{F} is a field of prime characteristic, then the result follows by Corollary 2.8 given above. Suppose from now on that \mathbb{F} is a field of characteristic zero. By the primary rational canonical decomposition of any nilpotent matrix, we can assume without loss of generality that A is the direct sum of its Jordan blocks.

If A only contains Jordan blocks of size $r > 1$, each of them can be decomposed separately in the following way

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & & \vdots \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 0 & & \vdots \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \dots & \dots & -1 \\ \vdots & & & \vdots \\ \vdots & & & \\ 0 & \dots & \dots & 0 \end{pmatrix}$$

where the first matrix is a torsion matrix and the second one is zero-square.

Otherwise, since the rank of A is greater than or equal to $\frac{n}{2}$, we can reorder the blocks of matrix A – which just corresponds to a reordering of the elementary divisors of A – as follows: we follow each Jordan block of size $r > 1$ by s ($s \leq r - 2$) blocks of size 1. If $r + s$ is even, the combined block of size $r + s$ can be decomposed as the sum of a torsion matrix and a zero-square matrix by virtue of Corollary 3.11. If, however, $r + s$ is odd, we elementarily see that the combined block of size $r + s$ can be decomposed as the sum of a torsion matrix and a zero-square matrix in view of Corollary 3.13. □

The next commentaries are worthwhile to show that the above theorem is somewhat true in more general settings.

Remark 3.15. It is well known (see, for example, [10], [16] or [17]) that every nilpotent matrix over a division ring is still similar to its Jordan form, i.e., it is similar to a direct sum of Jordan blocks, all of them associated to 0. Thus, Theorem 3.14 is still valid for nilpotent matrices over division rings.

The above result suggests the following question:

Open Question 2: Given a fixed index of nilpotence $k \geq 2$, is it true that any nilpotent matrix in $\mathbb{M}_n(\mathbb{F})$ can be written as the sum of a torsion matrix and a nilpotent matrix of index less than or equal to k if, and only if, its rank is at least $\frac{n}{k}$?

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