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DECOMPOSITIONS OF MATRICES INTO A SUM OF INVERTIBLE MATRICES AND MATRICES OF FIXED NILPOTENCE

PETER DANCHEV, ESTHER GARCÍA, AND MIGUEL GÓMEZ LOZANO

ABSTRACT. For any $n \geq 2$ and fixed $k \geq 1$, we give necessary and sufficient conditions for an arbitrary nonzero square matrix in the matrix ring $\mathbb{M}_n(\mathbb{F})$ to be written as a sum of an invertible matrix U and a nilpotent matrix N with $N^k = 0$ over an arbitrary field \mathbb{F} .

Key words: matrices, nilpotents, units, ranks

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1. INTRODUCTION

In 1977, when studying lifting properties of idempotents, Nicholson defined an element a in a ring R to be *clean* if it can be written in the form $e + u$, where e is an idempotent and u is a unit (i.e., an invertible); see [9]. If every element in a ring R is clean, the ring is called clean. Inspired by these definitions, in 2013 Diesl [6] defined a ring element $b \in R$ to be *nil-clean* if it can be expressed as a sum of an idempotent and a nilpotent element, and the ring R is nil-clean if every element in R is so.

By combining the notions of invertibility and nilpotence, Călugăreanu and Lam introduced in 2016 the notion of *fine* rings: those in which every nonzero element can be written as the sum of an invertible element and a nilpotent one; see [1]. One of the main results of the paper [1] is the fact that every nonzero square matrix over a division ring is the sum of an invertible matrix and a nilpotent matrix. Indeed, they proved that as soon as the division ring has more than two elements, every nonzero square matrix over such division ring is similar to what they call a matrix in *good form*, i.e., a matrix with all diagonal entries nonzero. By decomposing this last matrix into its (invertible) lower part and its strictly (nilpotent) upper part, one concludes matrix rings over division rings with more than two elements are fine. Moreover, they separately proved that nonzero matrices over \mathbb{F}_2 are also clean, reaching to the desired result.

In the same paper (see the Acknowledgements section), the authors remarked that there was no previous reference to the fact that every square nonzero matrix (even over the complex field) could be expressed as the sum of a nilpotent matrix and an invertible one. Notice that the nilpotent matrices in Călugăreanu and

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Lam decomposition have high indices of nilpotence because they correspond to the strictly upper part of a matrix in good form.

The rings whose nonzero idempotents are fine turned out to be an interesting class of indecomposable rings and were studied in [2] by Călugăreanu and Zhou. In 2021, the same authors focused on rings in which every nonzero nilpotent element is fine, which they called NF rings, and showed that for a commutative ring R and $n \geq 2$, the matrix ring $\mathbb{M}_n(R)$ is NF if and only if R is a field; see [3].

On the same vein, a slightly more general class of rings than the aforementioned class of fine rings was defined in [5] under the name *nil-good rings* and some their characteristic properties, including the behaviour of the matrix ring over a nil-good ring, were explored in [4] and [7], respectively.

In our work, we begin by fixing a bound $k \geq 1$ for the index of nilpotence of the nilpotent part and pose the following problem for matrices over fields:

Problem: *Given $k \geq 1$, find necessary and sufficient conditions to decompose any nonzero square matrix A over a field \mathbb{F} as a sum of an invertible matrix U and a nilpotent matrix N with $N^k = 0$.*

Remark 1.1. Notice that the problem of decomposing a matrix as the sum of a unit matrix and a nilpotent matrix of index at most k is not true in general. In fact, invertible square matrices have full rank, and the rank of a nilpotent matrix of index k is the sum of the rank of every nilpotent block of index k_i (whose rank is $k_i - 1$) in the Jordan canonical form of N . Therefore, in the matrix ring $\mathbb{M}_n(\mathbb{F})$ over the field \mathbb{F} , if we decompose $n = ck + d$ where $c, d \in \mathbb{N}$ and $0 \leq d < k$, the rank of every nilpotent matrix is less than or equal to

$$\begin{cases} c(k-1) = n - \frac{n}{k}, & \text{if } d = 0; \\ c(k-1) + d - 1 = n - \frac{n}{k} + \frac{d}{k} - 1 \leq n - \frac{n}{k}, & \text{if } d > 0 \end{cases}$$

(c blocks of index k , and one block of index d when $d > 0$, in its Jordan canonical form), so a necessary condition for this decomposition to hold is that the rank of the original matrix must be greater than or equal to $\frac{n}{k}$. To illustrate this more concretely, let $k \geq 2$, suppose $n \geq k + 2$ and let $A = e_{12} \in \mathbb{M}_n(\mathbb{F})$ be the standard matrix. If we assume in a way of contradiction that $A = U + N$, where U is an invertible matrix and $N^k = 0$, then one may write that $U = A - N$. But the rank of an invertible matrix is always maximal (that is, exactly n in this case), whereas the rank of A is one and the rank of N is $\leq n - \frac{n}{k}$, so it cannot be recovered a rank n matrix from a matrix of rank 1 and a matrix of rank at most $n - \frac{n}{k}$.

In this paper we completely solve this problem for matrices over arbitrary fields, proving that following result:

Theorem. *Let \mathbb{F} be a field, let $n \geq 2$, and let us fix $k \geq 1$. Given a nonzero matrix $A \in \mathbb{M}_n(\mathbb{F})$, there exists an invertible matrix $U \in \mathbb{M}_n(\mathbb{F})$ and a nilpotent matrix $N \in \mathbb{M}_n(\mathbb{F})$ with $N^k = 0$ such that $A = U + N$ if, and only if, the rank of A is greater than or equal to $\frac{n}{k}$.*

Since the properties invertibility and nilpotence are both invariant conditions under similarity, we will use the primary rational canonical form of a matrix ([8, VII.Corollary 4.7(ii)]), which states that every matrix $A \in \mathbb{M}_n(\mathbb{F})$, where \mathbb{F} is a field, is similar to a direct sum of companion matrices of prime power polynomials $p_1^{m_{11}}, \dots, p_s^{m_{ss}} \in \mathbb{F}[x]$ where each p_i is prime (irreducible) in $\mathbb{F}[x]$. The matrix

A is uniquely determined except for the order of these companion matrices. The polynomials $p_1^{m_{11}}, \dots, p_s^{m_{sk}}$ are called *the elementary divisors* of the matrix A .

2. DECOMPOSING MATRICES INTO A SUM OF INVERTIBLE AND NILPOTENT MATRICES

In our argument we will separate the elementary divisors $q(x)$ of a matrix $A \in \mathbb{M}_n(\mathbb{F})$ between those that satisfy $q(0) \neq 0$ and those with $q(0) = 0$, i.e., $q(x) = x^m$, $m \geq 1$. Among these last ones, we will also distinguish between those of degree 1 and those of degree bigger than 1:

- (i) Any elementary divisor $q(x) = x^m + a_{m-1}x^{m-1} + \dots + a_0$ with $q(0) = a_0 \neq 0$ gives rise to an invertible companion matrix

$$C(q(x)) = \begin{pmatrix} 0 & 0 & \dots & -a_0 \\ 1 & 0 & & \vdots \\ & \ddots & \ddots & \\ 0 & & 1 & -a_{m-1} \end{pmatrix} \in \mathbb{M}_m(\mathbb{F}).$$

- (ii) Any elementary divisor of the form $q(x) = x$ gives rise to the 1×1 companion matrix $C(x) = (0)$.
- (iii) Any elementary divisor of the form $q(x) = x^m$, $m > 1$, gives rise to a companion of the form

$$C(x^m) = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & & \vdots \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix} = \sum_{i=1}^{m-1} e_{i+1,i} \in \mathbb{M}_m(\mathbb{F}).$$

Definition 2.1. Let \mathbb{F} be a field, let $n \geq 2$, and let us fix an index of nilpotence k with $2 \leq k \leq n$. For each $1 \leq r < s \leq n$ such that $s + k - 2 \leq n$, we define the following matrices, which will be the ingredients of our main result:

$$N_{r,s,k} := e_{r,r} + e_{s,r} - e_{r,s+k-2} - \sum_{i=0}^{k-2} e_{s,s+i} + \sum_{i=0}^{k-3} e_{s+i+1,s+i} \in \mathbb{M}_n(\mathbb{F}).$$

We begin our work with a series of technicalities, which we need to establish our chief result.

Lemma 2.2. *Let \mathbb{F} be a field, let $n \geq 2$, and let us fix an index of nilpotence k with $2 \leq k \leq n$. For each $1 \leq r, s \leq n$ such that $s + k - 2 \leq n$, the matrices $N_{r,s,k}$ have rank equal to $k - 1$ and are nilpotent of index k .*

Proof. Let $N = \sum_{i=1}^{k-1} e_{i+1,i} \in \mathbb{M}_n(\mathbb{F})$ be the matrix consisting on a single nilpotent Jordan block of size k and $n - k$ blocks of size 1. By construction, N is nilpotent of index k .

We claim that each $N_{r,s,k}$ can be obtained from the matrix N by an appropriate change of basis. If we denote by $\{e_1, e_2, \dots, e_k, e_{k+1}, \dots\}$ the canonical basis, the

matrix

$$N_{1,2,k} = \left(\begin{array}{cccccc|ccc} 1 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 1 & -1 & -1 & \cdots & -1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{array} \right)$$

is just the same operator N represented on the basis

$$\mathcal{B}_{1,2,k} = \{v_1 = e_1, v_2 = e_2 - e_1, \dots, v_k = e_k - e_1, v_{k+1} = e_{k+1}, \dots\}.$$

Similarly, if we consider the basis $\mathcal{B}_{r,s,k} = \{v_1, \dots, v_n\}$ where

$$\begin{aligned} v_r &= e_1 \\ v_s &= e_2 - e_1, \\ v_{s+1} &= e_3 - e_1, \\ &\vdots \\ v_{s+k-2} &= e_k - e_1, \end{aligned}$$

and the rest of the vectors v_i of $\mathcal{B}_{r,s,k}$ are any reordering of the vectors e_{k+1}, \dots, e_n , the matrix $N_{r,s,k}$ is the representation of the operator N on the basis $\mathcal{B}_{r,s,k}$.

The rank and the index of nilpotence of the matrices $N_{r,s,k}$ is a direct consequence of the rank and the index of nilpotence of the original matrix N , as claimed. \square

Proposition 2.3. *Let \mathbb{F} be a field, let $n \geq 2$, and let us fix $k \geq 1$. Also, let $B \in \mathbb{M}_n(\mathbb{F})$ be a matrix consisting of a single invertible block of type (i) and size t , and also of $r = n - t$ nilpotent blocks of type (ii). If $r \leq t(k-1)$ (or, equivalently, $\text{rank}(B) = t \geq n/k$), then there exists a nilpotent matrix N_B with $N_B^k = 0$ such that $B + N_B$ is invertible.*

Proof. By hypothesis, the matrix B consists of an invertible block of the form $C(q(x))$, for some polynomial $q(x)$ of degree t with $q(0) \neq 0$, and r nilpotent blocks of type (ii). If $r = 0$, we just take $N_B = 0$ since B itself is invertible. For the rest of the proof, suppose that $r > 0$.

Let us use the classical division theorem to write $r = c(k-1) + d$ with $0 \leq d < k-1$ (c represents the number of nilpotent matrices of type $N_{r,s,k}$ that we will use in our argument and d , if nonzero, means an extra nilpotent matrix of type $N_{r,s,d+1}$). The condition $r \leq t(k-1)$ means that

$$\begin{cases} c \leq t, & \text{if } d = 0; \\ c + 1 \leq t, & \text{if } d > 0. \end{cases}$$

– If $c > 0$, we consider the matrix

$$N_{1,t+1,k} + N_{2,t+1+(k-1),k} + \cdots + N_{c,t+1+(c-1)(k-1),k} = \sum_{i=1}^c N_{i,t+1+(i-1)(k-1),k}.$$

By construction, $(\sum_{i=1}^c N_{i,t+1+(i-1)(k-1),k})^k = 0$ (the matrices $N_{r,s,k}$ appearing in this sum are nilpotent of index k by Lemma 2.2 and they are two-by-two orthogonal).

– If $d > 0$, we consider $N_{c+1,t+1+c(k-1),d+1}$ which satisfies $(N_{c+1,t+1+c(k-1),d+1})^k = 0$ because $d+1 < k$; $N_{c+1,t+1+c(k-1),d+1}$ is orthogonal to $\sum_{i=1}^c N_{i,t+1+(i-1)(k-1),k}$. Define the nilpotent matrix

$$N_B = \underbrace{\sum_{i=1}^c N_{i,t+1+(i-1)(k-1),k}}_{\text{if } c > 0} + \underbrace{N_{c+1,t+1+c(k-1),d+1}}_{\text{if } d > 0}, \quad N_B^k = 0.$$

We assert that the matrix $U_B = B + N_B$ is invertible. Indeed, since the determinant of a matrix remains the same if we replace some columns by the original columns to which we add some other columns,

- we add to the first column of U_B the one in position $t+k-1$,
- we add to the second column of U_B the one in position $t+2k-2$,
- \vdots
- we add to the c -column of U_B the one in position $t+c(k-1)$,
- if $d > 0$, we add to the $c+1$ -column of U_B the one in position $t+c(k-1)+d$.

The condition $c \leq t$ if $d = 0$ and $c+1 \leq t$ if $d > 0$ assures that these sums of columns in U_B only affects, at most, to the first t -columns of U_B . We end up with a matrix of the form

$$\left(\begin{array}{c|c|c|c|c} C(q(x)) & * & * & * & * \\ \hline 0 & J_{k-1} & * & * & * \\ \hline 0 & 0 & J_{k-1} & * & * \\ \hline 0 & 0 & 0 & \ddots & * \\ \hline 0 & 0 & 0 & 0 & J_d \end{array} \right)$$

where

$$J_r = \begin{pmatrix} -1 & -1 & \dots & -1 \\ 1 & 0 & \dots & 0 \\ 0 & \ddots & & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathbb{M}_r(\mathbb{F}), \quad r = k-1 \text{ or } d.$$

Since $\det(J_r) = (-1)^r$, $r = k-1$ or d , the determinant of U_B coincides with \pm the determinant of the companion matrix $C(q(x))$, which by hypothesis is nonzero, as required. \square

The above proposition can be substantiate by the following concrete construction.

is invertible because, if we add the column 6 to column 1 and add column 9 to column 2, it would follow that

$$\left(\begin{array}{cc|ccc|ccc} 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right) = \left(\begin{array}{c|c|c} C(x^2 - x - 1) & * & * \\ \hline 0 & J_4 & * \\ \hline 0 & 0 & J_3 \end{array} \right),$$

which is clearly invertible, as expected.

Proposition 2.5. *Let \mathbb{F} be a field, let $n \geq 2$, and let us fix $k \geq 2$. Let $C = \sum_{i=1}^{t-1} e_{i+1,i} \in \mathbb{M}_n(\mathbb{F})$ be a matrix consisting of a nilpotent block of type (iii) and size t , and $r = n - t$ nilpotent blocks of type (ii). If $r \leq k - 2 + (t - 1)(k - 1)$ (or, equivalently, $\text{rank}(C) = t - 1 \geq \frac{n}{k}$), then there exists a nilpotent matrix N_C with $N_C^k = 0$ and such that $C + N_C$ is invertible.*

Proof. If $r < k - 2$, we take the nilpotent matrix $N_C = N_{1,t,r+2} = e_{1,1} + e_{t,1} - e_{1,t+r} - \sum_{i=0}^r e_{t,t+i} + \sum_{i=0}^{r-1} e_{t+i+1,t+i}$; then

$$U_C = C + N_C = \sum_{i=1}^{r-1} e_{i+1,i} + e_{1,1} + e_{t,1} - e_{1,t+r} - \sum_{i=0}^r e_{t,t+i}.$$

Adding the column in position $t + r$ to the first column of U_C and replacing row t by the sum of that row and the rest of the rows below, we obtain the matrix $C(x^n + x^t + 1)$, which is the companion matrix of the polynomial $p(x) = x^n + x^t + 1$, $p(0) \neq 0$, so U_C is invertible. Moreover, since $r + 2 < k$, we have $N_C^{r+2} = N_C^k = 0$.

If $r \geq k - 2$, arguing as in the proof of Proposition 2.3 but beginning at position (2,2), let us use the classical division theorem to write $r - k + 2 = c(k - 1) + d$ with $0 \leq d < k - 1$. Define

$$\begin{aligned} N_C &= N_{1,t,k} \\ &+ N_{2,t+(k-1),k} + N_{3,t+2(k-1),k} + \cdots + N_{c+1,t+c(k-1),k} \quad \text{if } c > 0 \\ &+ N_{c+2,t+(c+1)(k-1),d+1} \quad \text{if } d > 0 \\ &= N_{1,t,k} + \underbrace{\sum_{i=2}^{c+1} N_{i,t+(i-1)(k-1),k}}_{\text{if } c > 0} + \underbrace{N_{c+2,t+(c+1)(k-1),d+1}}_{\text{if } d > 0}. \end{aligned}$$

The matrix N_C satisfies $N_C^k = 0$, because it consists of orthogonal nilpotent matrices of the form $N_{r,s,j}$, $j \leq k$, all of them satisfying $N_{r,s,j}^k = 0$.

In order to see that $U_C = C + N_C$ is invertible, if

- we add to the first column of U_C the column in position $t + k - 2$,
- we add to the second column of U_C the one in position $t + 2(k - 1) - 1$,
- \vdots
- we add to the c -column of U_C the one in position $t + (c + 1)(k - 1) - 1$,

Moreover, since $r = 5$ and $k = 4$, we can get $c = 1$ and $d = 0$ in the formula $r - k + 2 = c(k - 1) + d$, so we also consider the matrix

$$N_{2,7,4} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus $N_C = N_{1,4,4} + N_{2,7,4}$ satisfies $N_C^4 = 0$ and

$$C + N_C = \left(\begin{array}{cccccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right)$$

is invertible because, if we add the column 4 to column 1, add column 9 to column 2 and add rows 5 and 6 to row 4, we will receive

$$\left(\begin{array}{cccccc|ccc} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right) = \left(\begin{array}{c|c} C(x^6 + x^3 + 1) & * \\ \hline 0 & J_3 \end{array} \right),$$

which is clearly invertible, as promised.

Combining the previous two propositions we reach the main result which motivates writing of this article.

Theorem 2.7. *Let \mathbb{F} be a field, let $n \geq 2$, and let us fix $k \geq 1$. Given a nonzero matrix $A \in \mathbb{M}_n(\mathbb{F})$, there exists an invertible matrix $U \in \mathbb{M}_n(\mathbb{F})$ and a nilpotent matrix $N \in \mathbb{M}_n(\mathbb{F})$ with $N^k = 0$ such that $A = U + N$ if, and only if, the rank of A is greater than or equal to $\frac{n}{k}$.*

Proof. As already mentioned in Remark 1.1, a necessary condition to express A as the sum $U + N$, where U is invertible and $N^k = 0$, is that the rank of A is greater than or equal to $\frac{n}{k}$ because $n = \text{rank}(U) = \text{rank}(A - N) \leq \text{rank}(A) + \text{rank}(N) \leq \text{rank}(A) + n - \frac{n}{k}$.

Conversely, suppose that the rank of $A \in \mathbb{M}_n(\mathbb{F})$ is no less than $\frac{n}{k}$. Without loss of generality, we may assume that A is expressed in its primary rational canonical

form, i.e., it is a sum of the companion matrices of its elementary divisors. Let us show that there will exist an invertible matrix U and a nilpotent matrix N , $N^k = 0$, such that $A = U + N$, as pursuing.

Let us reorder the blocks of matrix A – which just corresponds to a reordering of the elementary divisors of A – as follows:

- (1) we follow each $t_i \times t_i$ block B_i of type (i) by s_i blocks of type (ii) ($s_i \leq t_i(k-1)$),
- (2) we follow each $t_j \times t_j$ block C_j of type (iii) by r_j blocks of type (ii) ($r_j \leq k-2 + (t_j-1)(k-1)$).

such that $\sum_i s_i + \sum_j r_j = n - \text{rank}(A)$. The rank of A guarantees that all blocks of type (ii) can be distributed and combined with either blocks of type (i) or with blocks of type (iii) by following points (1) and (2).

We, thereby, have the following two cases:

- In accordance with Proposition 2.3, for every invertible block B_i of rank t_i which is followed by s_i blocks of type (ii), $s_i \leq t_i(k-1)$, there exists a nilpotent matrix N_{B_i} such that $N_{B_i}^k = 0$ such that $B_i + N_{B_i}$ is an invertible matrix of rank $t_i + s_i$.
- In accordance with Proposition 2.5, for every nilpotent block C_j of index t_j and rank $t_j - 1$ which is followed by r_j blocks of type (ii), $r_j \leq k-2 + (t_j-2)(k-1)$, there exists a nilpotent matrix N_{C_j} such that $N_{C_j}^k = 0$ such that $C_j + N_{C_j}$ is an invertible matrix of rank $t_j + r_j$.

Now, define $N = -\sum_i N_{B_i} - \sum_j N_{C_j}$. Since the nilpotent matrices that we add are mutually orthogonal, we therefore can get a nilpotent matrix N with $N^k = 0$ and such that $U = A - N$ is invertible:

$$A - N = \left(\begin{array}{c|c|c|c|c|c} B_1 + N_{B_1} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & B_2 + N_{B_2} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C_1 + N_{C_1} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & C_2 + N_{C_2} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \ddots \end{array} \right).$$

Finally, we decompose $A = U + N$, as stated. \square

In conclusion, it is worthwhile noticing that the key tool in our arguments is the primary rational canonical form of any square matrix, which holds for matrices over arbitrary fields. However, since the mentioned above Calugareanu-Lam's result from [1] about the decomposition of matrices into invertible and nilpotent is true for matrices over division rings [1, Remark 3.12], we can close our work by posing the following query:

Open Problem: Given a fixed bound $k \geq 1$ for the index of nilpotence, find necessary and sufficient conditions to expressed every nonzero square matrix over a division ring as the sum of an invertible matrix and a nilpotent matrix N with $N^k = 0$.

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