Towards a Socle for Lie Algebras

Antonio Fernández López [1], Esther García [2], and Miguel Gómez Lozano [3]

[1]*Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, 29071 Málaga, Spain.*
E-mail: `emalfer@agt.cie.uma.es`
[2]*Departamento de Álgebra, Universidad Complutense de Madrid, 28040 Madrid, Spain.*
E-mail: `egarciag@mat.ucm.es`
[3]*Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, 29071 Málaga, Spain.*
E-mail: `magomez@agt.cie.uma.es`

ABSTRACT: A notion of socle is introduced for 3-graded Lie algebras (over a ring of scalars Φ containing $\frac{1}{6}$) whose associated Jordan pairs are nondegenerate. This notion of socle does not depend on certain 3-gradings and this allows us to define a Jordan socle for non-necessarily 3-graded Lie algebras. The Jordan socle turns out to be a 3-graded ideal and is the sum of minimal 3-graded inner ideals each of which is a central extension of the TKK-algebra of a division Jordan pair. Nondegenerate Lie algebras having a large Jordan socle are essentially determined by TKK-algebras of simple Jordan pairs with minimal inner ideals and their derivation algebras, which are also 3-graded.

INTRODUCTION

The aim of this paper is to summarize the results contained in our two papers [3, 4]. In the first one, inspired by the Jordan notion of socle, we define a socle for nondegenerate 3-graded Lie algebras, and study its main properties. In the second work, motivated by the dependence of that socle on the considered 3-grading, we introduce the so-called Jordan socle, which extends the notion of socle for 3-graded Lie algebras but makes sense also for non-necessarily 3-graded Lie algebras.

[1, 3] Partially supported by the MCYT and the Fondos FEDER, BFM2001-1938-C02-01, and the Junta de Andalucía FQM264.

[2] Partially supported by the MCYT and the Fondos FEDER, BFM2001-1938-C02-02.

The idea of studying Lie algebras by means of Jordan methods is by no means a novelty; on the contrary, fundamental contributions to this topic can be found in papers like [1, 2, 13, 14]. Let us say that in [13], the most related to our approach of these papers, E. Neher describes Lie algebras graded by a 3-graded root system: a Lie algebra L is graded by a 3-graded root system R if and only if it is a central extension of the Tits-Kantor-Koecher algebra of a Jordan pair V (TKK(V) for short) covered by a grid whose associated 3-graded root system is isomorphic to R. He gives the classification of Jordan pairs covered by a grid and describes their Tits-Kantor-Koecher algebras. We note that any simple Jordan pair covered by a grid with division coordinate algebra coincides with its socle, so in this case the socle theory and the grid theory agree.

Our notion of Jordan socle gives rise to a structure theory for nondegenerate Lie algebras having essential socles. Indeed, these Lie algebras can be described in terms of Tits-Kantor-Koecher algebras of simple Jordan pairs with minimal inner ideals and their derivation algebras. In particular, any simple finite-dimensional Lie algebra over an algebraically closed field of characteristic 0 which is not of type E_8, F_4 or G_2 has a non-trivial 3-grading and coincides with its socle.

1. **3-GRADED LIE ALGEBRAS AND JORDAN PAIRS**

1.1 Throughout this paper, we will be dealing with Lie algebras L and Jordan pairs $V = (V^+, V^-)$ over a ring of scalars Φ containing $\frac{1}{6}$. As usual, $[x, y]$ will denote the Lie product and $\operatorname{ad} x$ the adjoint mapping determined by x. Jordan products will be denoted by $Q_x y$, for any $x \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$, with linearizations $Q_{x,z} y = \{x, y, z\} = D_{x,y} z$. The reader is referred to [9, 11, 13] for basic results, notation and terminology. Nevertheless, we will stress some notions and basic properties for both Jordan pairs and Lie algebras.

1.2 An element $x \in V^\sigma$ is called an *absolute zero divisor* if $Q_x = 0$. Then V is said to be *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $Q_{B^\pm} B^\mp = 0$ implies $B = 0$, and *prime* if $Q_{B^\pm} C^\mp = 0$ implies $B = 0$ or $C = 0$, for $B = (B^+, B^-)$, $C = (C^+, C^-)$ ideals of V. Similarly, $x \in L$ is an *absolute zero divisor* of L if $(\operatorname{ad} x)^2 = 0$, and L is *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $[I, I] = 0$ implies $I = 0$, and *prime* if $[I, J] = 0$ implies $I = 0$ or $J = 0$, for I, J ideals of L. A Jordan pair or Lie algebra is *strongly prime* if it is prime and nondegenerate.

1.3 Given a subset S of L, the *annihilator* $\operatorname{Ann}(S)$ of S in L consists on the elements $x \in L$ such that $[x, S] = 0$, and it is an ideal as soon as S is. Indeed, the annihilator of a nondegenerate ideal I in a Lie algebra L coincides with the set of elements $x \in L$ such that $[x, [x, I]] = 0$ by [4, 2.5].

1.4 Nonzero ideals of nondegenerate Jordan pairs inherit nondegeneracy ([11, JP3]). The same is true for Lie algebras: every nonzero ideal of a nondegenerate Lie algebra is nondegenerate [15, Lemma 4; 6, 0.4]. Conversely, a Lie algebra is nondegenerate as soon as it has a nondegenerate ideal (1.3).

1.5 A 3-*grading* of a Lie algebra L is a decomposition $L = L_1 \oplus L_0 \oplus L_{-1}$, where each L_i is a submodule of L satisfying $[L_i, L_j] \subset L_{i+j}$, and where $L_{i+j} = 0$ if $i+j \neq 0, \pm 1$. A Lie algebra is 3-*graded* if it has a 3-grading. We will write (L, π) to denote the Lie algebra L with the particular 3-grading $\pi = (\pi_1, \pi_0, \pi_{-1})$, where each π_i is the projection of L onto L_i, $i = 0, \pm 1$.

1.6 Given (L, π) we have that $\pi(L) := (L_1, L_{-1})$ is a Jordan pair for the triple products defined by $\{x, y, z\} := [[x,\, y],\, z]$ for all $x, z \in L_\sigma$, $y \in L_{-\sigma}$, $\sigma = \pm 1$, which is called the *associated Jordan pair* of (L, π). We note that if L is nondegenerate, so is $\pi(L)$ [14, Lemma 1.8]. A standard example of a 3-graded Lie algebra is the following one:

1.7 For any Jordan pair V, there exists a 3-graded Lie algebra $\mathrm{TKK}(V) = L_1 \oplus L_0 \oplus L_{-1}$, the *Tits-Kantor-Koecher algebra of* V, uniquely determined by the following conditions (cf. [13, 1.5(6)]):

(TKK1) The associated Jordan pair (L_1, L_{-1}) of L is isomorphic to V.

(TKK2) $[L_1, L_{-1}] = L_0$.

(TKK3) $[x_0, L_1 \oplus L_{-1}] = 0$ implies $x_0 = 0$, for any $x_0 \in L_0$.

1.8 We will denote by $\operatorname{Der} L$ the set of derivations of L. If M is an ideal of L with $\operatorname{Ann}(M) = 0$, then L can be embedded in $\operatorname{Der} M$ via the adjoint mapping: $L \cong \operatorname{ad}_M L \leq \operatorname{Der} M$.

Graded derivations, which appear naturally when dealing with graded structures, generalize the adjoint mappings and are characterized by the way they act on the homogeneous parts (see [12, p. 805]). Moreover,

1.9 COROLLARY [8, 1.9]. *The derivation algebra of any nondegenerate 3-graded Lie algebra is 3-graded itself.*

2. THE SOCLE OF A 3-GRADED LIE ALGEBRA

2.1 The *socle* of a nondegenerate 3-graded Lie algebra (L, π) is defined as the ideal of L generated by the socle of the associated Jordan pair $\pi(L)$. Denoted by $\mathrm{Soc}_\pi(L)$ to show which grading we are considering, we have that $\mathrm{Soc}_\pi(L) = \mathrm{Soc}(\pi_1(L)) \oplus [\mathrm{Soc}(\pi_1(L)), \mathrm{Soc}(\pi_{-1}(L))] \oplus \mathrm{Soc}(\pi_{-1}(L))$, [3, 4.3]. Moreover, $\mathrm{Soc}_\pi(L)$ can be decomposed as a direct sum of simple ideals,

$$\mathrm{Soc}_\pi(L) = \bigoplus S^{(i)} = \bigoplus \mathrm{TKK}(\pi(S^{(i)})),$$

where the $\pi(S^{(i)})$ are the simple components of $\mathrm{Soc}(\pi(L))$. Furthermore, for any 3-graded ideal (I, π_I) of (L, π) whose grading π_I is compatible with π, we have that $\mathrm{Soc}_{\pi_I}(I) = \mathrm{Soc}_\pi(L) \cap I$.

2.2 In [14, §5] Zelmanov introduces a notion of inner ideals for **Z**-graded Lie algebras which in the particular case of a 3-graded Lie algebra reads as follows (cf. [7, 1.1]): We say that a graded Φ-submodule $B = B_1 \oplus B_0 \oplus B_{-1}$ of a 3-graded Lie algebra (L, π) is a 3-*graded inner ideal* if

(i) B is a subalgebra of L,

(ii) $[[L, B_1], B_1] + [[L, B_{-1}], B_{-1}] \subset B$.

Notice that condition (ii) implies that both B_1 and B_{-1} are inner ideals of the associated Jordan pair $\pi(L)$. Conversely, if $(B_1, B_{-1}) \subset \pi(L)$ is a pair of inner ideals of $\pi(L)$, then $B_1 \oplus [B_1, B_{-1}] \oplus B_{-1}$ is a 3-graded inner ideal of L. In particular, a Jordan pair idempotent $e = (e^+, e^-)$ of $\pi(L)$ determines the 3-graded inner ideal $L(e) := V_2(e^+) \oplus [V_2(e^+), V_2(e^-)] \oplus V_2(e^-)$, where $V_2(e^\sigma) = [e^\sigma, [e^\sigma, L]]$ for $\sigma = \pm 1$ (see [11, 5.5]). Moreover, any nondegenerate minimal 3-graded inner ideal of L has the form $L(e)$ for a Jordan pair division idempotent $e \in \pi(L)$ [3, 4.6].

2.3 Proposition [3, 4.7]. *For any 3-graded Lie algebra (L, π) with nondegenerate associated Jordan pair, $\mathrm{Soc}_\pi(L) = \sum_e L(e)$, where the sum is taken over all Jordan pair division idempotents $e \in \pi(L)$.*

In general, the definition of the socle of a nondegenerate 3-graded Lie algebra depends on the 3-grading, as can be seen in the following example:

2.4 Example: Let V and W be two Jordan pairs coinciding with their socles, i.e., $V = \mathrm{Soc}(V)$ and $W = \mathrm{Soc}(W)$. Let L be the Lie algebra built as the direct sum of the TKK-algebras of V and W. Notice that L admits the gradings

$$\begin{array}{ccc} \pi_1(L) = V^+ & \pi_0(L) = [V^+, V^-] \oplus \mathrm{TKK}(W) & \pi_{-1}(L) = V^-, \\ \pi'_1(L) = W^+ & \pi'_0(L) = [W^+, W^-] \oplus \mathrm{TKK}(V) & \pi'_{-1}(L) = W^-, \\ \pi''_1(L) = V^+ \oplus W^+ & \pi''_0(L) = [V^+ \oplus W^+, V^- \oplus W^-] & \pi''_{-1}(L) = V^- \oplus W^-, \end{array}$$

which give three essentially different socles: $\mathrm{Soc}_\pi(L) = \mathrm{TKK}(V)$, while $\mathrm{Soc}_{\pi'}(L) = \mathrm{TKK}(W)$ and $\mathrm{Soc}_{\pi''}(L) = L$.

2.5 We will show that the socle is indeed independent of the grading of L when the grading is *effective* in the sense that there is no nonzero ideal contained in the zero part of L. Notice that this condition is satisfied when (L, π) has (TKK3), and, in particular, when L is graded as the TKK-algebra of a Jordan pair or when L is strongly prime.

2.6 Theorem [4, 3.8]. *Let (L, π) be a nondegenerate 3-graded Lie algebra with an effective 3-grading π, and let (I, π') be an ideal of L which is graded with respect to a 3-grading π'. Then $\mathrm{Soc}_{\pi'}(I) \subset \mathrm{Soc}_\pi(L)$.*

Therefore, as soon as a nondegenerate Lie algebra L has an effective 3-grading (L, π), its socle contains the socle of any other 3-grading of L.

2.7 Corollary [4, 3.9]. *Let L be a nondegenerate Lie algebra admitting an effective 3-grading (L, π). Then $\mathrm{Soc}_\pi(L) \supset \mathrm{Soc}_{\pi'}(L)$ for any other 3-grading (L, π') of L. In particular, the socle of L does not depend on the effective 3-grading considered.*

3. THE JORDAN SOCLE

Motivated by theorem (2.6) we are going to introduce a notion of socle, called the *Jordan socle*, for nondegenerate Lie algebras which are not necessarily 3-graded.

3.1 Given a nondegenerate Lie algebra L, we define its *Jordan socle* as the sum of the socles of (I, π), where I is any 3-graded ideal of L and π denotes any of its possible 3-gradings:

$$\mathrm{JSoc}(L) = \sum_{(I,\pi)} \mathrm{Soc}_\pi(I).$$

3.2 Theorem [4, 4.2]. *The Jordan socle of a nondegenerate Lie algebra L is an ideal of L. If $\mathrm{JSoc}(L) \neq 0$ then it is a direct sum of simple ideals each of which* *is the TKK-algebra of a simple Jordan pair with minimal inner ideals. Therefore,* $\mathrm{JSoc}(L) \cong \mathrm{TKK}(V)$*, where* V *is a nondegenerate Jordan pair coinciding with its socle.*

Thus, the Jordan socle of a nondegenerate Lie algebra is the biggest 3-graded ideal of L that coincides with its socle. Moreover, if (L, π) is a nondegenerate 3-graded Lie algebra with effective 3-grading, then $\mathrm{Soc}_\pi(L) = \mathrm{JSoc}(L)$.

This notion of Jordan socle allows us to give a structure theory for nondegenerate Lie algebras with essential Jordan socle, similar to the ones obtained in the associative and Jordan settings [10, 5]. Recall that an *essential subdirect product* of a family of algebras $\{L_\alpha\}$ is any subdirect product of the L_α containing an essential ideal of the full direct product $\prod A_\alpha$.

3.3 Theorem [4, 4.3]. *For a Lie algebra* L*, the following statements are equivalent:*

(i) L *is nondegenerate and has essential Jordan socle.*

(ii) L *is an essential subdirect product of a family of strongly prime Lie algebras* L_α *having nonzero Jordan socles.*

(iii) *there exists a nondegenerate Jordan pair* V *that coincides with its socle such that* $\mathrm{ad}(\mathrm{TKK}(V)) \leq L \leq \mathrm{Der}(\mathrm{TKK}(V))$.

(iv) $\bigoplus \mathrm{ad}(\mathrm{TKK}(V_\alpha)) \triangleleft L \leq \prod \mathrm{Der}(\mathrm{TKK}(V_\alpha))$*, where each* V_α *is a simple Jordan pair with minimal inner ideals.*

Moreover, in case (iii)*,* $\mathrm{JSoc}(L) = \mathrm{ad}(\mathrm{TKK}(V))$*, and* L *is strongly prime if and only if* $\mathrm{JSoc}(L)$ *is simple, if and only if* V *is simple. We also have that* $\mathrm{Der}(\mathrm{TKK}(V))$ *is the largest strongly prime Lie algebra having Jordan socle equal to* $\mathrm{ad}(\mathrm{TKK}(V))$.

REFERENCES

1. B. N. Allison. "A construction of Lie algebras from J-ternary algebras". *Am. J. Math.* **98** (1976), 285-294.
2. G. Benkart, E. I. Zelmanov. "Lie algebras graded by finite root systems and intersection matrix algebras". *Invent. Math.* **129** (1996), 1-45.
3. A. Fernández López, E. García, M. Gómez Lozano. "3-Graded Lie algebras with Jordan finiteness conditions". *Comm. Algebra*, to appear

4. A. Fernández López, E. García, M. Gómez Lozano. "The Jordan Socle and Finitary Lie Algebras", *(preprint).* Posted on the Jordan Theory Preprint Server http://mathematik.uibk.ac.at/mathematik/jordan/.
5. A. Fernández López, M. Tocón. "Strongly prime Jordan pairs with non-zero socle". *Manuscripta Math.* **111** (2003), 269-283.
6. E. García. "Inheritance of primeness by ideals in Lie algebras". *Int. J. Math. Game Theory Algebra*, to appear.
7. E. García. "Simple, primitive and strongly prime Jordan 3-graded Lie algebras". *J. Algebra*, to appear.
8. E. García, M. Gómez Lozano. "Jordan Systems of Martindale-like Quotients". *J. Pure Appl. Algebra*, to appear.
9. N. Jacobson. *Lie Algebras.* Interscience Publishers; New York, 1962.
10. N. Jacobson. *Structure of Rings.* AMS Colloquium Publications, 37; Providence, USA, 1968.
11. O. Loos. *Jordan Pairs.* Lecture Notes in Mathematics, 460, Springer-Verlag; New York, 1975.
12. C. Martínez. "The ring of fractions of a Jordan algebra". *J. Algebra* **237** (2001), 798-812.
13. E. Neher. "Lie algebras graded by 3-graded root systems and Jordan pairs covered by grids". *Amer. J. Math.* **118(2)** (1996), 439-491.
14. E. I. Zelmanov. "Lie algebras with a finite grading". *Math. USSR Sbornik* **52(2)** (1985), 537-552.
15. E. I. Zelmanov. "Lie algebras with an algebraic adjoint representation". *Math. USSR Sbornik* **49(2)** (1984), 537-552.