CONSTRUCTING THE MAXIMAL LEFT QUOTIENT RING OF AN ALTERNATIVE RING.

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ABSTRACT

We study the notion of general left quotient ring of an alternative ring and show the existence of a maximal left quotient ring for every alternative ring that is a left quotient ring of itself.

1. Introduction.

The theory of rings of quotients has its origins between 1930 and 1940, in the works of O. Ore and K. Osano on the construction of the total ring of fractions. In that decade, Ore gave a necessary and sufficient condition for a ring R to have a (left) classical ring of quotients (left Ore condition). At the end of the 50's, Goldie, Lesieur and Croisot characterized the associative rings that are classical left orders in semiprime and left artinian rings [7, Chapter IV] (result known as Goldie's Theorem).

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Later on in 1956, Y. Utumi introduced the notion of a general ring of left quotients [8] and proved that the rings without right zero divisors are precisely those which have maximal left quotient rings.

Following Goldie's idea of characterizing certain types of rings via a suitable envelope, R.E. Johnson characterized those rings R whose maximal left quotient rings are von Neumann regular, see [7, (13.36)], and P. Gabriel specialized it further by giving characterizations for those rings whose maximal left quotient rings are semisimple, i.e., isomorphic to a finite direct product of rings of the form $End_{\Delta}(V)$ for suitable finite dimensional left vector spaces V over a division ring Δ, see [7, (13.40)].

It is natural to ask whether similar notions and results can be obtained for alternative rings.

In [4] H. Essannouni and A. Kaidi gave a version for alternative rings of the classical Goldie theory and in [6] the authors, dealing with orders in no necessarily unital alternative rings (see [1] and [5] for associative rings) extended to alternative rings the results and notions by Fountain and Gould, and Áhn and Márki.

In 1989, K.L. Beidar and A.V. Mikhalev, interested in the structure of nondegenerate and purely alternative algebras, introduced what they referred to as the almost classical localization of an algebra and described, using the theory of orthogonally complete algebraic systems, the structure of this type of algebras, see [3, Theorem 2.12]. This construction, that only works when the associative center coincides with the center (which is a property of nondegenerate and purely alternative rings), coincides with our maximal left quotient ring in these particular conditions, see (2.16)(5).

The notion of left quotient ring for alternative rings was introduced in [6] in order to relate, in the most general way, properties between a ring and its rings of quotients. We want to point out that, following [4], "denominators" were taken in the associative center of the alternative ring.

In this paper we give the ideas that gave rise to the construction of the maximal left quotient ring of an alternative ring R, see [2], in the most general case, i.e., when R is a left quotient ring of itself, equivalently when R is a faithful $N(R)$-module. In general, the maximal left quotient ring is a non-associative ring whose associator is an alternating function of its arguments, see Theorem 3.8. Furthermore, it is an alternative ring when $D(R)$ is semiprime or 2-torsion free.

2. Preliminaries.

2.1 The following three basic central subsets can be considered in an alternative ring R: the *associative center* $N(R)$, the *commutative center* $K(R)$, and the *center* $Z = Z(R)$, defined by:

$$N(R) = \{x \in R \mid (x, R, R) = (R, x, R) = (R, R, x) = 0\},$$
$$K(R) = \{x \in R \mid [x, R] = 0\},$$
$$Z(R) = N(R) \cap K(R),$$

where $[x, y] = xy - yx$ denotes the *commutator* of two elements $x, y \in R$ and $(x, y, z) = (xy)z - x(yz)$ is the *associator* of three elements x, y, z in R.

2.2 The defining axioms for an *alternative* ring R are the *left* and the *right alternative laws*:

$$(x, x, y) = 0 = (y, x, x)$$

for every $x, y \in R$. As a consequence we have the fact that the associator is an alternating function of its arguments. The standard reference for alternative rings is [9].

2.3 The associative nucleus and the associator ideal of an alternative ring will be very important notions in this theory. Given a ring R, every ideal contained in the associative center of R will be called *nuclear ideal.* The largest nuclear ideal of R will be the *associative nucleus*, denoted by $U(R)$. By $D(R)$ we will mean the *associator ideal*, i.e., the ideal of R generated by the set (R, R, R) of all associators.

2.4 From now on, for an alternative ring R, R^1 will denote its unitization, that is, R if the ring is unital, or $R \times \mathbb{Z}$ with product $(x, m)(y, n) := (xy + nx + my, mn)$ if R has no unity.

2.5 An alternative ring without nonzero *trivial ideals* (i.e., ideals with zero multiplication) is called *semiprime.* By [9, Exercise 9.1.8], every semiprime alternative ring does not contain nonzero trivial left (right) ideals. An element a of an alternative ring R is called an *absolute zero divisor* if $aRa = \{0\}$. An alternative ring R is called *nondegenerate* (or *strongly semiprime*) if R does not contain nonzero absolute zero divisors.

2.6 The notion of left quotient ring in the setting of alternative rings was introduced in [6], where the relationship among classical, Fountain and Gould and this type of rings of quotients was established.

Let R be a subring of an alternative ring Q. We recall that Q is a *left quotient ring* of R, denoted by $R \leq_q Q$, if:

(1) $N(R) \subset N(Q)$ and

(2) for every $p, q \in Q$, with $p \neq 0$, there exists $r \in N(R)$ such that $rp \neq 0$ and $rq \in R$.

Note that R and Q can be seen as left $N(R)$-modules and that condition (2) of the previous definition means that R is a dense left $N(R)$-submodule of Q, see [7, (8.2)].

The next proposition gives us the common denominator property for left quotients rings.

2.7 PROPOSITION [2, Proposition 1.12]. *Let R be a subring of an alternative ring Q.*

(i) If $R \leq_q Q$ and we take $q_1, q_2, \ldots, q_n \in Q$, with $q_1 \neq 0$, then there exists $r \in N(R)$ such that $rq_1 \neq 0$ and $rq_i \in R$ for $i = 1, 2, \ldots, n$.

(ii) Let $R \subset S \subset Q$ be three alternative rings. Then $R \leq_q Q$ if and only if $R \leq_q S$ and $S \leq_q Q$.

2.8 The notion of maximal left quotient ring, in the setting of associative rings, was studied by Y. Utumi in [8], where he proved that rings which are left quotient rings of themselves (equivalently rings without total right zero divisors) have a unique maximal left quotient ring. Following the categorical definition given in [8], we define the notion of maximal left quotient ring of an alternative ring:

2.9 DEFINITION. We will say that an alternative ring R has a *maximal left quotient ring* if there exists a ring Q such that:

(i) Q is a left quotient ring of R and

(ii) if S is a left quotient ring of R then there exists a unique monomorphism of rings $f : S \to Q$ with $f(r) = r$ for every $r \in R$.

Clearly, this definition implies that the maximal left quotient ring of a ring R, if it exists, is unique up to isomorphisms. We will denote it by $Q^l_{max}(R)$.

3. Construction of the maximal left quotient ring of an alternative ring.

Let R be an alternative ring. If we suppose that R has a maximal left quotient ring, by (2.7(ii)) it is a left quotient ring of itself. Like in the associative case, we are going to prove the converse of this fact, i.e., given an alternative ring R which is a left quotient ring of itself we are going to prove that R has a maximal left quotient ring.

Our construction of $Q^l_{max}(R)$ (with R an alternative ring) follows, surprisingly, the associative construction, see [7], where the maximal left quotient ring of an associative ring R' is a quotient, under a suitable equivalence relation, in the set of couples (I, f), where I a dense left ideal of R' and f is a homomorphism of left R'-modules from I to R', and, in essence, the sum and the product is defined as the sum and the composition of maps.

3.1 DEFINITION. We will say that a left ideal I of an alternative ring R is *dense* if for every $p, q \in R$, with $p \neq 0$, there exists $a \in N(R)$ such that $ap \neq 0$ and $aq \in I$.

This notion has similar properties to the associative one.

3.2 LEMMA [2, Lema 2.3]. *A left ideal I of an alternative ring R is dense if and only if R is a left quotient ring of I.*

A fundamental result in our work is that the notion of dense ideal can be characterized, essentially, by properties of the associative center of the ideal.

3.3 We will denote by $\mathcal{F}^*$ the family of all left ideals A of $N(R)$ with the property that for every $0 \neq x \in R$ and $\mu \in N(R)$ there exists $\lambda \in N(R)$ such that $\lambda x \neq 0$ and $\lambda\mu \in A$. We note that λ can be taken in A, and that the intersection of a finite family of elements of $\mathcal{F}^*$ is an element of $\mathcal{F}^*$.

3.4 PROPOSITION [2, Proposition 2.6]. *Let R be an alternative ring and let I be an ideal of R. Then I is a dense ideal of R if and only if $N(I) \in \mathcal{F}^*$.*

3.5 Let R be an alternative ring. Let us construct the set that will be the maximal left quotient ring of R. First we define a filter of dense left ideals: $\mathcal{F} := \{R^1 A \mid A \in \mathcal{F}^*\}$. Second we consider the set

$$S := \{(I, f) \mid I \in \mathcal{F} \text{ and } f \in Hom^*_{N(R)}(I, R)\}$$

where $Hom^*_{N(R)}(I,R)$ denotes the set of homomorphisms $f \in Hom_{N(R)}(I,R)$ such that for every $x \in R$ and λ, $\mu \in N(I)$ we have:

(i) $(x\lambda)f = x(\lambda)f$

(ii) $([\lambda,\mu])f \in N(R)$

Remark: When we define a product in $Q^l_{max}(R)$, (i) and (ii) will mean respectively that the associative center of R will be contained in the associative center of $Q^l_{max}(R)$ and the commutator of two elements of $Q^l_{max}(R)$ belongs to its associative nucleus (which is a general property of alternative rings).

3.6 We define on S the equivalence relation:

$$(I,f) \approx (I',f') \;\Leftrightarrow\; \text{there exists } I'' \in \mathcal{F} \text{ such that } I'' \subset I \cap I' \text{ and } f\mid_{I''} = f'\mid_{I''}$$

We denote by $[I,f]$ the equivalence class of the element (I,f) and consider the quotient

$$Q := S/\approx$$

Abusing notation, given an element $q \in Q$ we will denote by A_q any element of $\mathcal{F}^*$ and by f_q any element of $Hom^*_{N(R)}(R^1A_q,R)$ such that $q = [R^1A_q, f_q]$.

3.7 Let us define a ring structure over Q: If $q,q' \in Q$, we define:

⋆ the sum:

$$q + q' := [R^1(A_q \cap A_{q'}),\, f_q + f_{q'}],$$

⋆ the product:

$$qq' := [R^1A_{qq'},\, f_{qq'}]$$

where

$$A_{qq'} := \{\lambda \in A_q \mid (\lambda)f_q \in R^1A_{q'}\} (\in \mathcal{F}^*), \text{ and}$$

$$(\sum x_ia_i)f_{qq'} := \sum x_i((a_i)f_q)f'_q.$$

Remark: The product is well defined and $(Q,+,\cdot)$ is a (non associative) ring. Moreover, Q is unital, with $1_Q = [R^1N(R), Id_R]$ (see [2]).

3.8 THEOREM [2, Theorem 2.11]. *Let Q be as above. Then:*

(i) R is a subring of Q. Moreover, R is a dense left $N(R)$-submodule of Q.

(ii) $N(R) \subset N(Q)$.

(iii) *For every* $q \in Q$ *and* $\lambda \in N(R)$, $[\lambda, q] \in N(Q)$.

(iv) *The associator on* Q *is an alternate trilinear map of its arguments over* $N(Q)$.

Moreover, if $D(R)$ *is 2-torsion free or semiprime,*

(v) Q *is an alternative ring.*

(vi) Q *is the maximal left quotient ring of* R.

Now, by (2.7(ii)) and the above theorem we obtain the main theorem of the paper:

3.9 Theorem. *Let* R *be an alternative ring such that* $D(R)$ *is 2-torsion free or semiprime. Then* R *is a left quotient ring of itself if and only if the maximal left quotient ring of* R *exists.*

The next proposition, which is [6, Lemma 5.7 and Proposition 6.7 (i)], shows that the maximal ring of quotients gives us an appropriate framework to settle the different left quotient rings that have been investigated (Fountain and Gould and classical), see [4] and [6] for definitions. This fact was used by P. Ahn and L. Marki to give a general theory of Fountain and Gould left orden in the setting of associative rings.

3.10 Proposition [6,(5.7) and (6.7)(i)]. *Let* R *be an alternative ring. If* R *is a classical (Fountain and Gould) left order in an alternative ring* S*, then* S *is a left quotient ring of* R*. Therefore* S *is a subring of* $Q^l_{max}(R)$.

Some examples of maximal left quotient rings are the following ones:

3.11 Examples:

(1) The maximal left quotient ring of an associative ring is its maximal left quotient ring as an alternative ring.

(2) Let Q be a Cayley-Dickson algebra over its center. Then $Q^l_{max}(Q) = Q$.

(3) If R is a Cayley-Dickson ring, its maximal left quotient ring is a Cayley-Dickson algebra.

(4) Let us consider a family $\{R_\alpha\}$ of alternative rings such that for every α there exists the maximal left quotient ring of R_α, which we denote by Q_α. Then, $Q^l_{max}(\oplus R_\alpha)$ exists and it is equal to ΠQ_α, the direct product of the Q_α.

(5) Let R be a nondegenerate and purely alternative ring. Then the nearly

classical localization of R, given in [3, section 1] is the maximal left quotient ring of R.

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