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On the Topology of Stable Causality

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A topological description of the stable causality condition on a spacetime is given. The structure of the subsets that, for a given point, control the fulfillment of strong and stable causality conditions at that point are studied. Separation properties of the relevant topologies are also analyzed.

1. INTRODUCTION

This paper gives a topological description of the stable causality condition on a spacetime—i.e., a time-orientable and time-oriented, connected, Lorentzian (metric with signature $-, +, ..., +$) manifold of dimension $\geqslant 2$—, using the notion of stably C-convex sets (as defined later).

In Section 2, we comment on the standard notion of *C (causally)-convex (CC) sets* and the (in a natural way) associated topology τ^{CC}. We remark that, if some point of a spacetime has arbitrarily small neighborhoods (in the manifold topology τ^{M}) that are CC sets, then the point itself is a CC set; and we recall some well-known results about the *causality* and *strong causality* conditions on a spacetime.

In Section 3, we introduce the notion of *S (stably)-C-convex (SCC) sets* and the associated topology τ^{SCC}. After some technical lemmas, we prove (Theorem 3.1) that a point of a spacetime is a SCC set if and only if it has arbitrarily small neighborhoods that are SCC sets and also (Theorem 3.2) that the last condition holds for every point if and only if the stable causality condition holds on the spacetime.

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In Section 4, we introduce the notion of *returning sets* in a spacetime, we analyze the limits of certain sequences of returning sets (Theorems 4.1 and 4.2), and we use these limits in order to show (Theorem 4.1.ii and Theorem 4.3.iv) the structure of the subsets that, for a given point, control the fulfillment of the strong and stable causality conditions at that point.

In Section 5, we show (Lemma 5.1) the separation properties of those topologies, in particular, that τ^{SCC} is T^1 if and only if stable causality holds on the spacetime.

Throughout this paper, we use the following notational conventions:

(M, g): spacetime
Lor (M): set of Lorentzian metrics on M
τ^M (or τ^D): manifold—(or discrete)—topology on M
$I^{\pm}(A)$: chronological future or past of a subset A in (M, g)
$J^{\pm}(A)$: causal future or past of a subset A in (M, g)
$J_h^{\pm}(A, W)$: causal future or past of a subset $A \subset W$ in the spacetime $(W, h|_W)$ [for any open subset $W \subset M$ and $h \in \text{Lor}(M)$].
$J_h^{\pm}(A) := J_h^{\pm}(A, M)$; thus, $J_g^{\pm}(A) = J^{\pm}(A)$
$\dot{A} := \bar{A} \cap \overline{(M - A)}$ (for any subset $A \subset M$)
f.s.n. at p: fundamental system of (τ^M-)neighborhoods at a point $p \in M$.

2. C-CONVEX SETS

We follow the standard definitions in causality theory (see, e.g., [1–4]), in particular the notions of timelike and causal (piecewise C^∞) curves on (M, g), the chronological ($\ll$) and causal ($\leqslant$) precedence relations between points of (M, g), and the chronological future or past $I^{\pm}(A)$ and causal future or past $J^{\pm}(A)$ of any subset $A \subset M$.

Let us begin by quoting the following well-known results:

(1) (i) $p \ll q \Rightarrow p \leqslant q, \quad \forall p, q \in M$
(ii) $\ll \circ \ll \, = \, \ll$ and $\leqslant \circ \leqslant \, = \, \leqslant$, where $\circ$ denotes the composition of relations
(iii) $I^+(p)$ and $I^-(p)$ are τ^M-open sets, $\forall p \in M$
(iv) $\ll \circ \leqslant \, = \, \ll$ and $\leqslant \circ \ll \, = \, \ll$

(2) The collection $\{I^+(p) \cap I^-(q) \mid p, q \in M\}$ is a basis for a topology —called the Alexandrov topology τ^A—on (M, g), and $\tau^A < \tau^M$ holds.

Remark. The notation $\tau < \tau'$—both τ and τ' being topologies on M—means that τ' is finer than τ.

Definition 2.1. A subset A of (M, g) is said to be *C-convex* (CC) iff every causal curve between points of A lies entirely in A.

Remark 1. $I^+(p)$ and $I^-(p)$ are CC, $\forall p \in M$. On the other hand, if A is CC and $p, q \in A$, then $I^+(p) \cap I^-(q) \subset A$.

Remark 2. The collection of all C-convex sets of (M, g) is a basis for a topology τ^{CC} on M, and $\tau^A < \tau^{CC}$ holds.

Remark 3. A τ^{CC}-open set is in general neither τ^M-open nor CC (the union of two CC sets needs not be CC).

Remark 4. Consider (M, g) given. If $p \in M$ has arbitrarily small (τ^M-)neighborhoods that are CC, then p is itself CC. The failure of the converse can be easily seen from the following example [2]. Delete the closed half-lines $(t = 1, x \geqslant -1)$ and $(t = -1, x \leqslant 1)$ from Minkowski 2-space; then identify $(t = -2, x)$ with $(t = 2, x)$. Every point on the line segment $L = (t = -1 + a, x = 1 - a)$, $a \in (0, 2) \subset R$, in the resulting spacetime (M, g) is CC, but it does not admit arbitrarily small CC neighborhoods.

The next definition is equivalent to the usual one.

Definition 2.2. (M, g) is said to be

(i) *Causal* iff $\tau^{CC} = \tau^D$

(ii) *Strongly causal* iff every point of M has arbitrarily small (τ^M-) neighborhoods that are CC

Remark 1. Remark 4 to Definition 2.1 implies: strong causality $\Rightarrow$ causality.

Remark 2. Any g-normal convex neighborhood, regarded itself as a spacetime, is causal. Moreover, it can be proved ([1], Proposition 4.10) that any relatively compact g-normal convex neighborhood whose closure is contained in another g-normal convex neighborhood is, regarded as a spacetime, strongly causal.

Remark 3. (M, g) is strongly causal iff $\tau^A = \tau^M$.

3. STABLY C-CONVEX SETS

Now we introduce the notion of stably C-convex sets in the following way:

Definition 3.1. A subset of (M, g) is said to be *stably C-convex* (SCC) iff there exists $h \in \mathrm{Lor}(M)$, $h > g$, such that A is CC in (M, h).

Remark 1. Given $g, g' \in \mathrm{Lor}(M)$, the notation $g' > g$—terminology: g' is wider than g—means that $\forall v(\neq 0) \in TM$, $g(v, v) \leqslant 0 \Rightarrow g'(v, v) < 0$. It is very easy to see that, given $g_1, g_2 \in \mathrm{Lor}(M)$ with $g_1 < g_2$, $\exists g_3 \in \mathrm{Lor}(M)$ with $g_1 < g_3 < g_2$.

Remark 2. The collection of all stably C-convex sets of (M, g) is a basis for a topology τ^{SCC} on M, and $\tau^{\mathrm{SCC}} < \tau^{\mathrm{CC}}$ holds.

Remark 3. As for CC sets, a τ^{SCC}-open set is in general neither τ^M-open nor SCC.

We need the following two technical lemmas:

Lemma 3.1. Given (M, g), consider some sequence $\{h_n\} \subset \mathrm{Lor}(M)$ with $h_1 > \cdots > h_n > \cdots > g$ and $h_n \to g$ (on each tangent space), and take $p \in M$. Then $\exists(\tau^M$-)neighborhood W of p, such that, if $\{q_n\}$ and $\{q'_n\}$ are point sequences in M, the following implication holds:

$$\left.\begin{array}{r} q_n \to q \in W \\ q'_n \to q' \in W \\ q'_n \in J^{\pm}_{h_n}(q_n, W), \quad \forall n \end{array}\right\} \Rightarrow q' \in J^{\pm}_{g}(q, W)$$

Remark. This lemma says that locally the causality of $\{h_n\}$ converges to the causality of g as n tends to infinity. *We avoid saying that the sequence $\{h_n\}$ converges to g*; in fact, if we give $\mathrm{Lor}(M)$ the so-called interval topology [5]—whose defining basis can be taken as the collection [for all pairs $h_1, h_2 \in \mathrm{Lor}(M)$ with $h_1 < h_2$] of subsets of $\mathrm{Lor}(M)$ of the form $\{h \in \mathrm{Lor}(M) \mid h_1 < h < h_2\}$, and that coincides [6] with the Whitney fine C^0 topology on $\mathrm{Lor}(M)$—convergence of a sequence $\{h_n\}$ in that topology would require that, for some compact $K \subset M$ and for sufficiently large $m, n \in N$, $h_m = h_n$ outside of K, thus contradicting the hypotheses that we have made about the sequence $\{h_n\}$.

Proof. See [7], Lemma 1, with slight modifications. ∎

Lemma 3.2. Given (M, g), consider some sequence $\{h_n\} \subset \mathrm{Lor}(M)$ with $h_1 > \cdots > h_n > \cdots > g$ and $h_n \to g$ (on each tangent space), and take $p \in M$. Let W be some $(\tau^M$-)neighborhood of p as in Lemma 3.1 and let W_1 be some relatively compact, g-normal convex neighborhood of p, $\overline{W}_1 \subset W$. Then, $\exists$ neighborhood $\mathscr{V}$ of p, $\mathscr{V} \subset W_1$, and $\exists\, s(>1) \in N$ such that

$$F^{\pm}_{h_1}(p) \supset F^{\pm}_{h_s}(\mathscr{V})$$

($F^{\pm}_{\bar h_n}(A)$ being defined, for every subset $A \subset W_1$, as the intersection $J^{\pm}_{\bar h_n}(A, W) \cap \dot W_1$) (see Fig. 1).

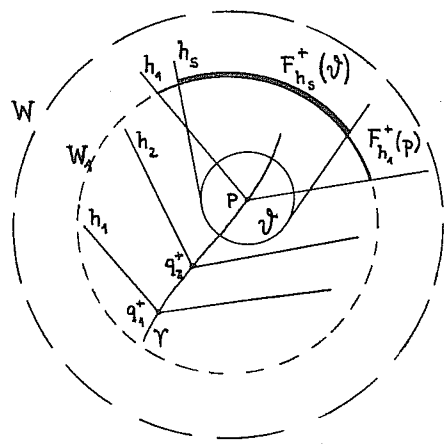


Fig. 1.

Proof. Obviously $F^{\pm}_{\bar h_1}(p) \supset F^{\pm}_{g}(p)$ holds. Take some future g-timelike curve $\gamma: [-1, 1] \to W_1$, $\gamma(0) = p$, and choose two sequences: $\{t_n^+\} \subset [-1, 0)$ with $t_n^+ < t_{n+1}^+\ \forall n$, $t_n^+ \to 0^-$, and $\{t_n^-\} \subset (0, 1]$ with $t_n^- > t_{n+1}^-\ \forall n$, $t_n^- \to 0^+$. Consider the set sequences $\{A_n^{\pm}\}$ defined by $A_n^{\pm} := F^{\pm}_{\bar h_n}(q_n^{\pm})$, $q_n^{\pm} \equiv \gamma(t_n^{\pm})$; because $A_n^{\pm} \supset A_{n+1}^{\pm}$, $\forall n$, upper and lower limits[3] coincide, thus write simply $\lim A_n^{\pm}$. Clearly $A_n^{\pm} \supset F_g^{\pm}(p)$, $\forall n$ $[\Rightarrow \lim A_n^{\pm} \supset F_g^{\pm}(p)]$ and Lemma 3.1 tells us that $\lim A_n^{\pm} = F_g^{\pm}(p)$; $\Rightarrow \exists s(>1) \in N$ such that $F^{\pm}_{\bar h_1}(p) \supset F^{\pm}_{\bar h_s}(q_s^{\pm})$; $\Rightarrow$ (because p belongs to both $I^{\pm}_{\bar h_s}(q_s^{\pm})$, open sets) $\exists$ neighborhood $\mathscr{V}$ of p, $\mathscr{V} \subset W_1$, satisfying $F^{\pm}_{\bar h_1}(p) \supset F^{\pm}_{\bar h_s}(\mathscr{V})$, as required. ∎

Theorem 3.1. Consider (M, g) given. Then $p \in M$ has arbitrarily small (τ^M-)neighborhoods that are SCC iff p is itself SCC.

[3] Given a sequence $\{A_n\}_{n \in N}$ of subsets of a manifold M, the following limits are defined:

$\limsup A_n := \{q \in M \mid \forall$ neighborhood $\mathscr{V}$ of q, $\exists$ subsequence $\{A_m\}$ such that $A_m \cap \mathscr{V} \neq \phi,\ \forall m\}$

$\liminf A_n := \{q \in M \mid \forall$ neighborhood $\mathscr{V}$ of q, $\exists\, n_0 \in N$ such that $A_n \cap \mathscr{V} \neq \phi,\ \forall n \geqslant n_0\}$

Both $\limsup A_n$ and $\liminf A_n$ exist always, are closed in M, can be void, and need not be contained in some A_n; moreover: $\liminf A_n \subset \limsup A_n$.

If $A_n \supset A_{n+1} (\forall n)$, then one sees immediately that both limits coincide—notation in that case $\lim A_n$—and it holds: $\lim A_n = \bigcap_n \bar A_n$ (thus, $\bigcap_n A_n \subset \lim A_n$); but even in that case, $\lim A_n$ needs not be contained in some A_n (thus, $\bigcap_n A_n \neq \lim A_n$, in general).

Proof. (*If*) By assumption, $\exists\, h \in \mathrm{Lor}(M)$, $h > g$, such that p is CC in (M, h). Take some sequence $\{h_n\} \subset \mathrm{Lor}(M)$ with $h = h_1 > \cdots > h_n > \cdots > g$ and $h_n \to g$ (on each tangent space). Let W be some neighborhood of p as in Lemma 3.1 and let W_1 be some relatively compact, g-normal convex neighborhood of p, $\bar{W}_1 \subset W$; let us impose moreover the further condition that $(W_1, h_1|_{W_1})$ is, as a spacetime, strongly causal (this is always possible; see Remark 2 to Definition 2.2). Lemma 3.2 tells us that $\exists$ neighborhood $\mathscr{V}$ of p, $\mathscr{V} \subset W_1$, and $\exists\, s(>1) \in N$ such that $F^{\pm}_{h_1}(p) \supset F^{\pm}_{h_s}(\mathscr{V})$; being $(W_1, h_1|_{W_1})$ strongly causal, we can—without loss of generality; see Definition 2.2.ii—choose $\mathscr{V}$ to be arbitrarily small and CC in $(W_1, h_1|_{W_1})$. Now every h_s-causal curve γ between points of $\mathscr{V}$ must lie inside of $\mathscr{V}$; otherwise, γ should either remain inside of W_1—which is forbidden, because $\mathscr{V}$ is CC in $(W_1, h_1|_{W_1})$—or escape from W_1—that would mean $\exists\, q^{\pm} \in \mathrm{Im}\,\gamma \cap F^{\pm}_{h_s}(\mathscr{V}) \subset J^{\pm}_{h_1}(p)$ and moreover $q^{-} \in J^{+}_{h_s}(q^{+}) \subset J^{+}_{h_1}(q^{+})$, which is also forbidden because p is assumed to be CC in (M, h_1). Thus, $\mathscr{V}$ must be CC in (M, h_s) and this completes the proof.

(*Only if*) Let be $h \in \mathrm{Lor}(M)$, $h > g$, and let $\mathscr{U}$ be some h-normal convex neighborhood of p; thus (Remark 2 to Definition 2.2), $(\mathscr{U}, h|_{\mathscr{U}})$ is, as a spacetime, causal. By assumption, $\exists$ some neighborhood $\mathscr{V}$ of p, $\mathscr{V} \subset \mathscr{U}$, and $\exists\, k \in \mathrm{Lor}(M)$, $h > k > g$, such that $\mathscr{V}$ is CC in (M, k); therefore, any (assumed existing) closed k-causal—thus h-causal—curve through p should lie entirely in $\mathscr{V}(\subset \mathscr{U})$, which is forbidden, because $(\mathscr{U}, h|_{\mathscr{U}})$ is causal. Thus, p is CC in (M, k), and this completes the proof. ∎

Definition 3.2. (M, g) is said to be *stably causal* iff $\exists\, h \in \mathrm{Lor}(M)$, $h > g$, such that (M, h) is causal.

Remark. It follows trivially: stable causality $\Rightarrow \tau^{\mathrm{SCC}} = \tau^{D}$.

The following theorem reformulates the notion of stable causality—concerning the spacetime as a whole—in terms of a global condition for small regions around any point (as Definition 2.2 does for strong causality).

Theorem 3.2. (M, g) is stably causal iff every point of M has arbitrarily small (τ^{M}-)neighborhoods that are SCC.

Proof. (*If*) Take any $h \in \mathrm{Lor}(M)$, $h > g$. By assumption, we can find two countable locally finite (τ^{M}-)open coverings of M, $\{\mathscr{V}_n\}$ and $\{\mathscr{U}_n\}$, such that, $\forall n \in N$: (i) $\mathscr{U}_n$ is relatively compact with $\bar{\mathscr{U}}_n \subset \mathscr{V}_n$, and (ii) $\mathscr{V}_n$ is a relatively compact SCC set contained in some h-normal convex neighborhood—thus (Remark 2 to Definition 2.2), every $(\mathscr{V}_n, h|_{\mathscr{V}_n})$, regarded as a spacetime, is causal; thus, $\forall n \in N$, $\exists\, h_n \in \mathrm{Lor}(M)$, $h_n > g$, such that $\mathscr{V}_n$ is CC in (M, h_n)—without loss of generality, we can take $h_n < h$ and therefore $(\mathscr{V}_n, h_n|_{\mathscr{V}_n})$ is causal. Let us consider some countable set $\{g_1, g_2, \ldots\} \subset \mathrm{Lor}(M)$ constructed in the following way: $\forall n \in N$, g_n satisfies (we take $g_0 \equiv g$)

$$\begin{aligned} &g_n = g_{n-1}, && \text{on} \quad M - \mathscr{V}_n \\ &h_m > g_n \geqslant g_{n-1}, && \text{on} \quad \mathscr{V}_n \quad (\forall m \text{ such that } \mathscr{V}_n \cap \mathscr{V}_m \neq \phi) \\ &g_n > g_{n-1}, && \text{on} \quad \bar{\mathscr{U}}_n \end{aligned}$$

Note that, because both $\mathscr{V}_n$ and $\mathscr{U}_n$ are contained in compact sets, the locally finite character of both coverings implies that every $\mathscr{V}_n$ (or $\mathscr{U}_n$) only intersects a finite number of $\mathscr{V}_m$'s (or $\mathscr{U}_m$'s). Because $\mathscr{V}_n$ is CC in (M, h_n) and $(\mathscr{V}_n, h_n|_{\mathscr{V}_n})$ is causal $(\forall n \in N)$, the nonexistence of closed g_{n-1}-causal curves on M—by assumption this property holds for $n = 1$—is not altered by taking g_n instead of g_{n-1}. Now the fact that $\forall n \in N$,

$$g_n > g \qquad \text{on} \quad \bigcup_{k=1}^{n} \mathscr{U}_k$$

allows us to construct some $g_\infty \in \mathrm{Lor}(M)$, $g_\infty > g$, such that (M, g_∞) is causal—for assume $\exists$ some closed g_∞-causal curve γ; because Im_γ is compact, one sees immediately that $\exists\, k \in \bar{N}$ such that $g_k|_\gamma = g_\infty|_\gamma$, thus (M, g_k) would not be causal (contradiction). This completes the proof.

(*Only if*) It follows trivially from Definition 3.2 and Theorem 3.1. ∎

Remark. This theorem, together with Definition 2.2.ii implies: stable causality $\Rightarrow$ strong causality.

The circular reasoning: (M, g) is stably causal $\Rightarrow$ (Definition 3.2) $\tau^{\mathrm{SCC}} = \tau^D \Leftrightarrow$ (Theorem 3.1) every point in M has arbitrarily small $(\tau^M$-)neighborhoods that are SCC $\Rightarrow$ (part *If* of Theorem 3.2) (M, g) is stably causal, leads to the following:

Corollary. (M, g) is stably causal iff $\tau^{\mathrm{SCC}} = \tau^D$.

Remark. This corollary can be also proved by the following circular reasoning: (M, g) is stably causal $\Rightarrow$ (Definition 3.2) $\tau^{\mathrm{SCC}} = \tau^D \Rightarrow$ (trivially) $\forall p \in M$, $\bigcap_{h>g} R_h(p) = \{p\}$ [for the definition of the set $R_h(p)$, see later] $\Leftrightarrow$ ([8], Lemma 1) (M, g) is stably causal.

The following diagram summarizes the relations between the above topologies on a generic spacetime:

$$\begin{array}{ccccc} & & \tau^D & & \\ & \swarrow & & \searrow & \\ & \tau^{CC} & & & \tau^M \\ \swarrow & & \searrow & \swarrow & \\ \tau^{SCC} & & & \tau^A & \end{array}$$

with the following properties: $\tau^{SCC}=\tau^D$ (stable causality) $\Rightarrow \tau^A=\tau^M$ (strong causality) $\Rightarrow \tau^{CC}=\tau^D$ (causality).

4. RETURNING SETS

Definition 4.1. Let (M, g) be a spacetime. For every subset $A\subset M$ and for every $h\in \mathrm{Lor}(M)$—such that (M, h) remains a spacetime—we define the *returning-set of A with respect to h* as the set

$$R_h(A) := J_h^+(A)\cap J_h^-(A)$$

Remark 1. $\forall A\subset M$ and $\forall h\in \mathrm{Lor}(M)$ as above, $R_h(A)\supset A$. Moreover, (i) $A\supset B\Rightarrow R_h(A)\supset R_h(B)$, and (ii) $h>h'\Rightarrow R_h(A)\supset R_{h'}(A)$.

Remark 2. Given (M, g) and a subset $A\subset M$, it follows immediately that (i) A is CC $\Leftrightarrow R_g(A)=A$, and (ii) A is SCC $\Leftrightarrow \exists h\in \mathrm{Lor}(M)$, $h>g$, such that $R_h(A)=A$.

Let us analyze the limit of a sequence of returning sets obtained by letting A run over some particular sequence of (τ^M-)neighborhoods of a given point:

Theorem 4.1. Consider a spacetime (M, g) and a point $p\in M$. Let $\{V_n\}$ be any countable f.s.n. at p such that, $\forall n$, $V_n\supset V_{n+1}$. Then we have

(i) $\lim R_g(V_n)[=\bigcap_n \overline{R_g(V_n)}]$ becomes independent of the chosen sequence $\{V_n\}$.

(ii) $\lim R_g(V_n)=\{p\}$ iff p has arbitrarily small (τ^M-)neighborhoods that are CC.

Proof. (i) is trivial.

(ii) p has arbitrarily small neighborhoods that are CC sets $\Leftrightarrow$ (Definition 4.1) $\exists$ some sequence $\{W_n\}$ as in the statement, such that $\forall n\in N$, $R_g(W_n)=W_n$, therefore $\lim R_g(W_n)=\{p\}$; $\Rightarrow$ [Part (i)] $\forall$ sequence $\{V_n\}$ as in the statement, $\lim R_g(V_n)=\{p\}$.

Thus, the only thing remaining to be proved is the converse of last implication. So take some sequence $\{V_n\}$ as above with $\lim R_g(V_n) = \{p\}$. Choose W and W' to be g-normal convex neighborhoods of p, with compact $\bar{W} \subset W'$; thus $(W, g|_W)$ is, as a spacetime, strongly causal. By assumption, no point $q \in \dot{W}$ belongs to $\lim R_g(V_n)$; $\Leftrightarrow \forall q \in \dot{W}$, $\exists$ neighborhood $\mathscr{V}_q$ of q and $\exists\, n_q \in N$ such that $R_g(V_{n_q}) \cap \mathscr{V}_q = \phi$; $\Rightarrow$ (choosing from the covering $\{\mathscr{V}_q\}_{q \in \dot{W}}$ of $\dot{W}$ a finite subcovering) $\exists\, m \in N$ such that $R_g(V_m) \subset W$; $\Rightarrow$ (because $(W, g|_W)$ is strongly causal) $\exists$ neighborhood W_1 of p, $W_1 \subset V_m$, such that W_1 is CC in $(W, g|_W)$, and therefore $R_g(W_1) = W_1$; proceeding further, we arrive the desired result. ∎

Remark 1. In general, $\lim R_g(V_n) \subset R_g(V_n)(\forall n)$ does not hold. In Fig. 2 (corresponding to the example in the Remark 4 to Definition 2.1), the whole (g-null) segment L belongs to $\lim R_g(V_n)$; however, once a future g-causal curve coming from V_n meets $L - V_n$, it cannot reach V_n again; thus $\forall q \in L - V_n$, $q \notin R_g(V_n)$, and, in fact, no point in $L - \{p\}$ belongs to $\bigcap_n R_g(V_n)$.

Remark 2. Part (ii) says that the set $\lim R_g(V_n)$ controls the fulfillment of strong causality at p.

We are now going to see what happens when h runs over some particular sequence of Lorentzian metrics on M. For that purpose, we need

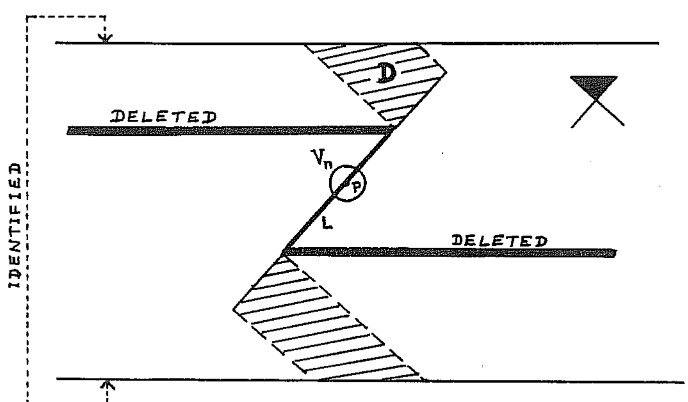


Fig. 2.

$$\bigcap_{\mathscr{V} \in \mathscr{N}_p} R_g(\mathscr{V}) = \{p\} \cup D$$

$$\lim R_g(V_n) = \bigcap_{h > g} R_h(p) = \lim R_{h_n}(p) = L \cup \bar{D}$$

the following technical lemma (which is a consequence of Lemmas 3.1 and 3.2):

Lemma 4.1. Consider a spacetime (M, g) and some sequence $\{h_n\} \subset \mathrm{Lor}(M)$ with $h_1 > \cdots > h_n > \cdots > g$ and $h_n \to g$ (on each tangent space). Let P be some point and let A be some subset of M such that, for some (τ^M-)neighborhood $\mathscr{U}$ of P, $\mathscr{U} \cap A = \phi$. Let $\{V_n\}$ be any countable f.s.n. at P such that $V_n \supset V_{n+1}$, $\forall n \in N$. Then, $\exists$ some h_n-causal curve between points of V_n through $A (\forall n)$ iff $\exists$ some closed h_n-causal curve through P and A $(\forall n)$.

Proof. The direction "if" being trivial, we prove the converse. Let W be some neighborhood of P as in Lemma 3.1 and let W_1 be some relatively compact, g-normal convex neighborhood of P, $\bar{W}_1 \subset W$. Then (Lemma 3.2), $\exists\, \bar{n} \in N$ such that $V_{\bar{n}} \subset W_1$ and $F^{\pm}_{h_1}(P) \supset F^{\pm}_{h_{\bar{n}}}(V_{\bar{n}})$. Take some $h_{\bar{n}}$-causal curve between points of $V_{\bar{n}}$ through A; $\Rightarrow \exists s_1^{\pm} \in \dot{W}_1 (s_1^+ \neq s_1^-)$ such that $s_1^- \leqslant V_{\bar{n}} \leqslant s_1^+ \leqslant s_1^-$ ($\leqslant$ denotes here $h_{\bar{n}}$-causal precedence; moreover, the last precedence relation is through A); thus $s_1^- \leqslant p \leqslant s_1^+ \leqslant s_1^-$ (with $\leqslant$ denoting here h_1-causal precedence and, as before, the last precedence relation is through A); $\Rightarrow \exists$ some closed h_1-causal curve through P and A.
We repeat this construction and the desired result follows. ∎

Theorem 4.2. Consider a spacetime (M, g) and some sequence $\{h_n\} \subset \mathrm{Lor}(M)$ with $h_1 > \cdots > h_n > \cdots > g$ and $h_n \to g$ (on each tangent space). Take $p \in M$. Then,

(i) $\lim R_{h_n}(p) (= \bigcap_n \overline{R_{h_n}(p)}) \subset R_{h_n}(p)$, $\quad \forall n \in N$
[thus, $\lim R_{h_n}(p) = \bigcap_n R_{h_n}(p)$]

(ii) $\lim R_{h_n}(p) = \{p\}$ iff p itself is $\mathrm{SCC}_{\{h_n\}}$
[we define a subset $A \subset M$ to be $\mathrm{SCC}_{\{h_n\}}$ iff $\exists\, m \in N$ such that A is CC in (M, h_m)]

(iii) Let $\{V_n\}$ be any countable f.s.n. at p such that $V_n \supset V_{n+1}$, $\forall n$; it holds: $\lim R_g(V_n) \subset \lim R_{h_n}(p)$

Proof. (i) and (iii). Let be $q \in \lim R_g(V_n)$; $\Leftrightarrow$ either $q = p$ or q has arbitrarily small neighborhoods $\mathscr{V}$ such that, $\forall n \in N$, $\exists$ some g-causal—thus h_n-causal—curve between points of V_n through $\mathscr{V}$; $\Rightarrow$ (Lemma 4.1 with $P \equiv p$ and $A \equiv \mathscr{V}$) either $q = p$ or q has arbitrarily small neighborhoods $\mathscr{V}$ such that $\forall n \in N$, $\exists$ some closed h_n-causal curve through p and $\mathscr{V}$ [that ends the proof of Part (iii)]; $\Rightarrow$ (Lemma 4.1 with $P \equiv q$ and $A \equiv p$) either $q = p$ or $\forall n \in N$, $\exists$ some closed h_n-causal curve through q and p [that ends the proof of Part (i)].

(ii) p is $\mathrm{SCC}_{\{h_n\}}$; $\Leftrightarrow$ (Definition 4.1) $\exists\, n \in N$ such that $R_{h_n}(p) = \{p\}$; $\Rightarrow$ [Part (i)] $\lim R_{h_n}(p) = \{p\}$.

Thus, it remains only to prove the converse of last implication. So choose W to be an h_1-normal convex neighborhood of p, with compact $\bar{W}$; thus, $(W, h_1|_W)$ is, as a spacetime, causal. By assumption, no point $q \in \dot{W}$ belongs to $\lim R_{h_n}(p)$; proceeding in a similar way as in the proof of Theorem 4.1.ii, we arrive at the desired result. ∎

Remark 1. Figure 3 shows clearly that the inclusion in (iii) need not be an equality. Moreover, it follows from (i) and (iii) that, $\forall$ countable f.s.n. $\{V_n\}$ of p such that $V_n \supset V_{n+1} (\forall n)$ and $\forall h \in \mathrm{Lor}(M)$, $h > g$:

$$\lim R_g(V_n) \subset R_h(p)$$

Remark 2. After (ii), the set $\lim R_{h_n}(p)$ does not control the fulfillment of stable causality at p; the reason is that the converse of the obvious implication p is $\mathrm{SCC}_{\{h_n\}} \Rightarrow p$ is SCC needs not hold in general. A sufficient condition would be that $\exists\, \bar{n} \in N$ such that, for some compact subset $K \subset M$, $R_{h_{\bar{n}}}(p) \subset K$—for, in that case, $\forall h \in \mathrm{Lor}(M)$, $h > g$, $\exists\, \bar{m} \geqslant \bar{n}$ such that $h_{\bar{m}}|_K < h|_K$, thus $R_{h_{\bar{m}}}(p) \subset R_h(p)$, and therefore

$$\lim R_{h_n}(p) \neq \{p\} \Rightarrow \forall h > g, \qquad R_h(p) \neq \{p\}$$

However, the existence of some $k \in \mathrm{Lor}(M)$, $k > g$, such that $\{p\} \neq R_k(p) \subset K$, is impossible if (M, g) is strongly causal (see Remark 3

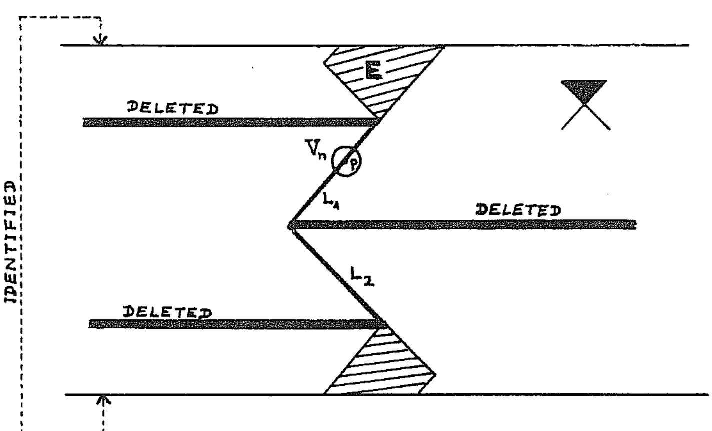


Fig. 3.

$$\bigcap_{\mathcal{V} \in \mathcal{N}_p} R_g(\mathcal{V}) = \lim R_g(V_n) = \{p\}$$

$$\bigcap_{h > g} R_h(p) = \lim R_{h_n}(p) = L_1 \cup L_2 \cup E$$

to Theorem 4.3); on the other hand, in the nonstrong causal case, such metric k may or may not exist [see Fig. 2, where $L \subset R_k(p)(\forall k)$ and $\nexists$ compact $K \subset M$ such that $L \subset K$]. Let us mention, by the way, that p is $\mathrm{SCC}_{\{h_n\}}$ iff p has arbitrarily small (τ^M-)neighborhoods that are $\mathrm{SCC}_{\{h_n\}}$ (the proof follows the same steps as the proof of Theorem 3.1).

Theorem 4.3. Consider a spacetime (M, g) and a point $p \in M$. Let $\{V_n\}$ be any countable f.s.n. at p such that, $\forall n$, $V_n \supset V_{n+1}$. Let us denote by $\mathcal{N}_p$ the set of all (τ^M-)neighborhoods of p. Then,

(i) $\bigcap_{\mathscr{V} \in \mathcal{N}_p} R_g(\mathscr{V}) = \bigcap_n R_g(V_n)$

(ii) $\bigcap_{\mathscr{V} \in \mathcal{N}_p} R_g(\mathscr{V}) = \{p\}$ only if p is CC

(iii) $\bigcap_{h > g} R_h(p)$ is (τ^M-)closed

(iv) $\bigcap_{h > g} R_h(p) = \{p\}$ iff p is SCC

Remark 1. Each part—except (ii), which needs (i)—is proved independently. Part (iii) was proved in [Lemma 1(1) of Ref. 8]; we give here another proof. Concerning Part (iv), the result $\bigcap_{h > g} R_h(p) = \{p\}$ $(\forall p)$ iff p is a SCC set $(\forall p)$ can also be obtained by the circular reasoning mentioned in (Remark to the corollary of Theorem 3.2).

Proof. (i) is trivial.

(ii) $p = \bigcap_{\mathscr{V} \in \mathcal{N}_p} R_g(\mathscr{V}) \Rightarrow$ [because $\{p\} \in R_g(p) = R_g(\bigcap_n V_n) \subset \bigcap_{\mathscr{V} \in \mathcal{N}_p} R_g(\mathscr{V})$] $\overline{\{p\} = R_g(p)}$.

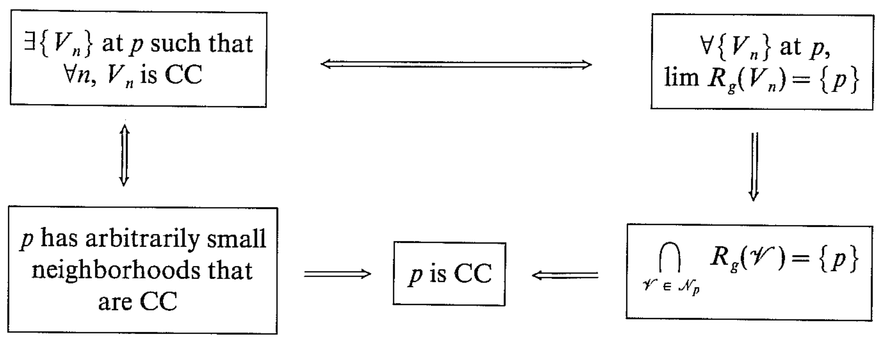
(iii) $q \in \bigcap_{h > g} R_h(p) \Rightarrow \forall \{W_n\}$, $\forall \{h_n\}$, and $\forall n$, $\exists$ some (closed) h_n-causal curve between points of W_n through p, $\{W_n\}$ being a countable f.s.n. at q with $W_n \supset W_{n+1} (\forall n)$ and $\{h_n\} \subset \mathrm{Lor}(M)$ being such that $h_1 > \cdots > h_n > \cdots > g$, and $h_n \to g$ (on each tangent space); $\Leftrightarrow$ (Lemma 4.1) $q \in \bigcap_{h > g} R_h(p)$.

(iv) p is SCC; $\Leftrightarrow$ (Definition 4.1) $\exists\, h > g$ such that $R_h(p) = \{p\}$; $\Rightarrow$ (trivial) $\bigcap_{h > g} R_h(p) = \{p\}$. Now we prove the converse of last implication. Take $h_0 \in \mathrm{Lor}(M)$, $h_0 > g$, and choose W to be some h_0-normal convex neighborhood of p with compact $\bar{W}$; thus $(W, h_0|_W)$ is, as a spacetime, causal. By assumption, no point $q \in \dot{W}$ belongs to $\bigcap_{h > g} R_h(p)$. Proceeding as in the proof of Theorem 4.2.ii, we arrive at the desired result. ∎

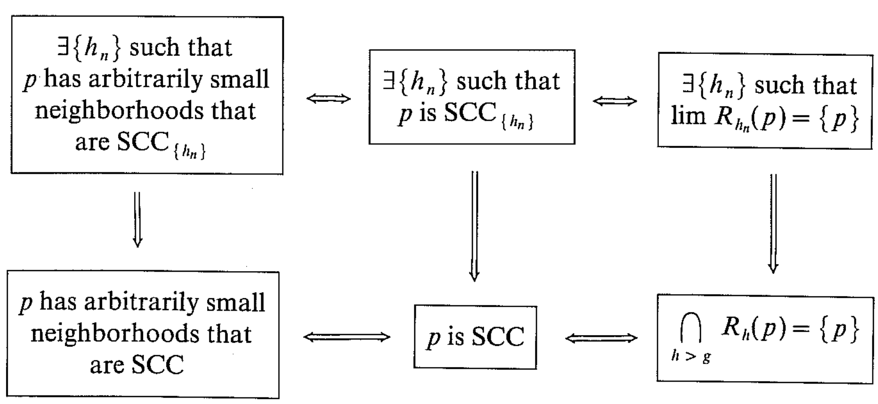
Remark 2. This theorem shows that the set $\bigcap_{h > g} R_h(p)$ controls the fulfillment of stable causality at p. On the other hand, the set $\bigcap_{\mathscr{V} \in \mathcal{N}_p} R_g(\mathscr{V})$ yields no useful information even on the fulfillment of causality at p (see Fig. 2, which also shows that this set—being equal to the dashed region D together with the point p itself—need not be either open or closed).

Remark 3. The set $\bigcap_{h>g} R_h(p)$ is closed but, if nontrivial [and provided that (M, g) is strongly causal], it cannot be compact. Indeed (see [7], Corollary 3), every open relatively compact subset of a strongly causal spacetime is, as a spacetime, stably causal.

The following diagram summarizes the results of Sections 3 and 4 (with notations as above):



$$\bigcap_{\mathscr{V}\in\mathscr{N}_p} R_g(\mathscr{V}) = \bigcap_n R_g(V_n) \subset \lim R_g(V_n) \subset \bigcap_{h>g} R_h(p) \subset \bigcap_n R_{h_n}(p) = \lim R_{h_n}(p)$$



5. SEPARATION PROPERTIES

Concerning the separation properties of the above topologies, we have the following lemma [Part (ii) is well known; see Ref. 1, 4.16 and Theorem 4.24; we give here a simpler proof, showing the similarities with Part (iii)]:

Lemma 5.1. Let (M, g) be a spacetime. With the above notations, the following holds:

(i) τ^{CC} is T^1 iff $\tau^{CC} = \tau^D$

(ii) τ^A is T^2 iff $\tau^A = \tau^M$

(iii) τ^{SCC} is T^1 iff $\tau^{SCC} = \tau^D$

Proof. (i) is trivial.

(ii) Prove "only if." Assume (Definition 2.2) (M, g) is not strongly causal; thus, $\exists p \in M$ having not arbitrarily small (τ^M-)neighborhoods that are CC; $\Leftrightarrow$ (Theorem 4.1.ii) the set $A \equiv \lim R_g(V_n) - \{p\}$ is no void—$\{V_n\}$ being any (Theorem 4.1.i) countable f.s.n. at p such that $V_n \supset V_{n+1}$, $\forall n$. Let W^p be any τ^A-neighborhood of p belonging to the defining basis of τ^A; thus, W^p is CC; $\Leftrightarrow$ (Definition 4.1) $R_g(W^p) = W^p$. Now, $\forall$ such W^p, $\forall q \in A$ and $\forall \tau^M$-neighborhood $\mathscr{V}$ of q, it holds: $(W^p \cap \mathscr{V} =)\ R_g(W^p) \cap \mathscr{V} \neq \phi$; in particular, taking as $\mathscr{V}$ any τ^A-neighborhood W^q of q belonging to the defining basis of τ^A, we have $\forall q \in A$ and $\forall$ such W^p and W^q, $W^p \cap W^q \neq \phi$. Thus, no such W^p and W^q can (disjointly) separate p from q and τ^A cannot be T^2.

(iii) Prove "only if." Assume $\exists p \in M$ that is not SCC; $\Leftrightarrow$ (Theorem 4.3.iv) the set $A \equiv \bigcap_{h > g} R_h(p) - \{p\}$ is no void. Let W^p be any τ^{SCC}-neighborhood of p belonging to the defining basis of τ^{SCC}; thus, W^p is SCC; $\Leftrightarrow$ (Definition 4.1) $\exists k \in \mathrm{Lor}(M)$, $k > g$, such that $R_k(W^p) = W^p$. Now, $\forall$ such W^p and $\forall q \in A$ (thus, $q \neq p$), it holds: $q \in R_k(p) \subset R_k(W^p) = W^p$. Thus, no such W^p can separate p from q and τ^{SCC} cannot be T^1. ∎

Remark. In the general case, τ^A need not be even T^1; in fact, it can be easily seen that τ^A being T^1 implies that (M, g) is either *past* or *future distinguishing*, i.e., $\forall p, q \in M$, either $I^-(p) = I^-(q) \Rightarrow p = q$ or $I^+(p) = I^+(q) \Rightarrow p = q$.

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