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Singularities of Invariant Connections

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A reductive homogeneous space $M = P/G$ is considered, endowed with an invariant connection, i.e., such that all left translations of M induced by members of P preserve it. We study the set of singularities of such connections giving sufficient conditions for it to be empty, or, in other cases, families of b-incomplete curves converging to singularities. A full description of the b-completion of a connection with $M = \mathbb{R}^m$ (or a quotient of it) is given with information on its topology.

1. INTRODUCTION AND STATEMENT OF RESULTS

The definition of singularity of a linear connection was given by Schmidt [5] having in mind as the main example the Levi-Civita connection of a spacetime. Except for a few cases, it is difficult to describe the whole set of singularities or its topology (see Ref. 4, p.284). It is our aim in this paper to consider an example of a manifold and a connection on it with a high degree of symmetry in the hope of getting good information about singularities. In fact, we have chosen as manifold a reductive homogeneous space, so $M = P/G$ where G is a closed Lie subgroup of P and $\mathfrak{g}$ admits an Ad_G-invariant supplementary $\mathfrak{m}$ in $\mathfrak{p}$. The connection is one that is invariant under all translations $uG \to vuG$, $v \in P$. We show that these connections are in bijective correspondence with certain linear maps $\mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ and that if the connection is mapped from a connection

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in P in a natural sense, then it has no singularities. Nevertheless, invariant connections may have singularities and a general theorem showing b-incomplete curves converging to them is proved. We also give conditions for different curves to converge to the same singularity, and, finally, we study a example of a connection on $\mathbb{R}^m$ (or a quotient of it) where a complete description of singularities and its topology may be given. This example generalizes a previous one of Dodson and Sulley [3].

2. GENERALITIES

We briefly summarize the definition of singularity of a linear connection given by Schmidt [2,5]. Let M be an m-dimensional connected manifold and L a connected component of the frame bundle. It is sometimes useful to consider frames as isomorphisms of a fixed space $\mathbb{V}$ into $T_x M$, $x \in M$, although $\mathbb{V} = \mathbb{R}^m$ is the usual choice. In the principal bundle $\Pi : L \to M$ with group $GL^+(\mathbb{V})$ the fundamental form $\Phi \in \Lambda^1(L;\mathbb{V})$ will be considered as well as a connection $\lambda \in \Lambda^1(L;\mathfrak{gl}(\mathbb{V}))$. Choose a Euclidean inner product in $\mathbb{V}$ denoted by $\langle\, . \, , \, . \,\rangle$ and the induced Euclidean inner product in $\mathfrak{gl}(\mathbb{V})$, also denoted by $\langle\, . \, , \, . \,\rangle$, and given by $\langle A, B\rangle = \mathrm{tr}(AB^\top)$, $B^\top =$ transpose of B. There is a natural Euclidean inner product on $\mathbb{V} \times \mathfrak{gl}(\mathbb{V})$, still denoted by $\langle\, . \, , \, . \,\rangle$ and a Riemann metric g^λ on L defined by

$$g^\lambda(\xi,\eta) = \langle (\Phi + \lambda)(\xi), (\Phi + \lambda)(\eta) \rangle \qquad \text{for} \qquad \xi, \eta \in TL.$$

The metric g^λ gives a distance d^λ and the metric space L has a Cauchy completion $\bar{L}$. The natural action of $GL^+(\mathbb{V})$ on L is extendible to $\bar{L}$, and we have an orbit space $\bar{M}$ where M imbeds as a subspace. We call $\bar{M}$ the b-completion of M with connection λ and the points of $\bar{M} - M$ are the singularities of the connection. Obviously $\bar{M}$ depends on λ, but it can be shown to be independent of the initial inner product on $\mathbb{V}$. Hence, an arbitrary basis of $\mathbb{V}$ can be declared to be orthonormal, members of $\mathfrak{gl}(\mathbb{V})$ can be identified with matrices, the transpose becoming the transpose of a matrix, etc., and this will always give the same $\bar{M}$.

Singularities can be obtained in the following way. Consider an inextendible curve $c : [0, a) \to M$, i.e. such that $\lim c(t)$ does not exist for $t \to a$. This curve is called b-incomplete if one of its λ-horizontal lifts to L has finite length for g^λ. This property is independent of the horizontal lift chosen. It is shown that if c is b-incomplete there exists the limit $\lim c(t)$ for $t \to a$ in $\bar{M}$ which is clearly a singularity. If $\bar{M}$ is not Hausdorff this limit may not be unique.

3. REDUCTIVE HOMOGENEOUS SPACES

We fix a connected Lie group P and a closed Lie subgroup G. The quotient space P/G is denoted by M, and the projection of P onto M by π. We have the corresponding Lie algebras $\mathfrak{p}$ and $\mathfrak{g}$. It is well known that P is a principal fibre bundle with base M, group G, and such that the right action of G onto P is the multiplication. Therefore, if $A \in \mathfrak{g} \subseteq \mathfrak{p}$, the fundamental field $A_\#$ in $\mathfrak{X}(P)$ associated to A is just A. This bundle has the important property that P acts on the left on the total space P by left multiplication. The right and left translations induced by $v \in P$ on P are denoted as usual by R_v and L_v and the iffeomorphism on M given by $uG \to vuG$ is denoted by τ_v.

We assume our homogeneous space to be *reductive*. This means there is a supplementary $\mathfrak{m}$ of $\mathfrak{g}$ in $\mathfrak{p}$ such that $Ad_s(\mathfrak{m}) \subseteq \mathfrak{m}$ for all $s \in G$. The linear map $(Ad_s)|_{\mathfrak{m}}R : \mathfrak{m} \to \mathfrak{m}$ is denoted by A. Clearly, $s \to A_s$ is a homomorphism of G into $GL(\mathfrak{m})$, having as induced homomorphism between Lie algebras the map $X \to ad_X|_{\mathfrak{m}}$ from $\mathfrak{g}$ into $\mathfrak{gl}(\mathfrak{m})$. We denote this last map by a_X.

For a diffeomorphism φ, we denote by φ_* the push-forward map acting on fields.

Lemma 1. We have the identities

a) $(R_v)_* X = Ad_v^{-1}(X)$ for $X \in \mathfrak{p}$ and $v \in P$.

b) $T_{us}\pi(X(us)) = T_u\pi(Ad_s X(u))$ for $X \in \mathfrak{p}$, $u \in P$, and $s \in G$.

Proof. The first formula follows from the equalities

$$\begin{aligned}(R_v)_* X(u) &= TR_v(X(uv^{-1})) = TR_v \circ TL_u \circ TLv^{-1}(X(e))\\ &= TL_u \circ TR_v \circ TL_{v^{-1}}(X(e))\\ &= TL_u(Ad_{v^{-1}}(X)(e)) = Ad_v^{-1}(X)(u),\end{aligned}$$

where e is the unit of G. To get (b) we apply $T\pi$ to (a) and we use $\pi \circ R_s = \pi$ and we can write if $v = us^{-1}$

$$T_v\pi\Big(X(v)\Big) = T_u\pi\Big(Ad_{s^{-1}}(X)(u)\Big)$$

and (b) follows after the substitution of s by s^{-1}. ∎

It is convenient to consider the elements of the frame bundle LM of M as linear isomorphisms $r : \mathfrak{m} \to T_x M$. We define a map $j : P \to LM$ as follows:

$$\text{For } u \in P, \quad Z \in \mathfrak{m}, \qquad j(u)(Z) = T_u\pi(Z(u)).$$

Theorem 1. The map $j : P \to LM$ is a principal fibre bundle homomorphism over the identity of M, with associated group homomorphism $s \to A_s$ from G into $GL(\mathfrak{m})$. Therefore, $j(us) = j(u) \circ A_s$ for $u \in P$, $s \in G$.

Proof. If $\Pi : LM \to M$ is the projection, we get easily $\pi \circ j = \pi$. We apply (b) in lemma 1 and we get

$$j(us)(Z) = T_{us}\pi(Z(us)) = T_u\pi(Ad_s Z(u)) = j(u)(A_s Z).$$

We prove the differentiability of j as follows. Consider $q : P \times \mathfrak{m} \to TM$ given by $q(u, Z) = T_u\pi(Z(u))$, which is smooth. There is a right action of G on $P \times \mathfrak{m}$, $(u, Z)s = (us, A_s^{-1}(Z))$. By lemma 1(b) one sees that the orbits of this action are the fibres of q. This implies that the orbit space $E = (P \times \mathfrak{m})/G$, which is an associated bundle of P, is isomorphic to TM. Then each $u \in P$ can be considered as a frame in E, say $\tilde{u}$, by $\tilde{u}(Z) = q(u, Z)$. But the map j is, modulo an isomorphism, just the map $u \to \tilde{u}$, so it must be smooth.

The other statements in the proof are easy. ∎

We give as theorem 2 some simple properties of j. First some notation. The diffeomorphism τ_v of M given above induces naturally another one in the frame bundle LM which we denote by $\dot{v}$. The fundamental form on LM is denoted by $\Phi \in \Lambda^1(LM; \mathfrak{m})$ (Recall our frames are defined on $\mathfrak{m}$!). We define a natural 1-form θ in $\Lambda^1(P, \mathfrak{m})$ as follows. For each $X \in \mathfrak{p}$, we decompose it as $X = Y + Z$, with $Y \in \mathfrak{g}$ and $Z \in \mathfrak{m}$ and put $\theta(X) = Z$. The reader will easily prove

Theorem 2. With the preceding notation we have

a) For all $v \in P$, $j \circ L_v = \dot{v} \circ j$.

b) For each $A \in \mathfrak{g}$, the fields $A_\#$ in $\mathfrak{X}(P)$ and $(a_A)_\#$ in $\mathfrak{X}(LM)$ are j-related.

c) $j^*\Phi = \theta$.

We deal now with connections. We say that a connection $\lambda \in \mathrm{Conn}(LM)$ is mapped from a connection $\omega \in \mathrm{Conn}(P)$ if

$$j^*\lambda(X) = a_{\omega(X)} \qquad \text{for all } X \in \mathfrak{X}(P)$$

and we write briefly $j^*\lambda = a_\omega$. There is a canonical connection ω_0 in P characterized by this condition. If $X \in \mathfrak{p} \subseteq \mathfrak{X}(P)$ is written as $X = Y + Z$ with Y in $\mathfrak{g}$ and Z in $\mathfrak{m}$, then $\omega_0(X) = Y$, so it is, roughly speaking, the $\mathfrak{g}$-component. A connection λ_0 can be defined on LM by defining as horizontal the j-image of $\mathfrak{m}$. More precisely, λ_0 is characterized by the condition that the horizontal space in $j(u)$ is the image under $T_u j$

of $\{Z(u)/Z \in \mathfrak{m}\}$. It is easy to check that ω_0 is mapped onto λ_0. More generally, suppose another supplementary $\mathfrak{m}_1$ of $\mathfrak{g}$ in $\mathfrak{p}$ could be found being invariant under Ad_s for all $s \in G$. Then a new morphism $j_1 : P \to LM$ can be constructed and new j_1-related connections ω_1 and λ_1 as before. It will be seen later that we have just given the family of all connections such that ω_1 is mapped on λ_1 and which are invariant in a sense to be defined just now. Indeed a connection λ in LM is called invariant if all bundle automorphisms $\hat{v}$, for $v \in P$, verify $\hat{v}^*\lambda = \lambda$. Given $\lambda \in \mathrm{Conn}(LM)$ we consider $\mu = j^*\lambda \in \Lambda^1(P, \mathfrak{gl}(\mathfrak{m}))$. One proves easily

Theorem 3. The form μ verifies

a) For $A \in \mathfrak{g}$, the map $\mu(A_\#) : P \to \mathfrak{gl}(\mathfrak{m})$ takes the constant value a_A.

b) For every $s \in G$, we have $R_s^*\mu = A_s^{-1} \circ \mu \circ A_s$. (The right hand side is to be understood as the 1-form sending $\xi \in T_uP$ onto the element of $\mathfrak{gl}(\mathfrak{m})$ obtained by composition of A_s^{-1}, $\mu(\xi)$, and A_s.)

Conversely, any $\mu \in \Lambda^1(P, \mathfrak{gl}(\mathfrak{m}))$ verifying (a) and (b), induces a unique λ in $\mathrm{Conn}(LM)$ such that $j^*\lambda = \mu$.

According to (a) in the preceding theorem, the information carried by μ about the connection λ depends only on its behaviour on fields $Z \in \mathfrak{m}$. With some abuse of notation we still denote by μ the map from $\mathfrak{m}$ into $C^\infty(P, \mathfrak{gl}(\mathfrak{m}))$ given by applying $\mu \in \Lambda^1(P, \mathfrak{gl}(\mathfrak{m}))$ on the field $Z \in \mathfrak{m}$. From theorem 3 there is immediately an alternative way to study $\mathrm{Conn}(LM)$.

Theorem 4. The connections λ on LM are in bijective correspondence with the linear maps $\mu : \mathfrak{m} \to C^\infty(P, \mathfrak{gl}(\mathfrak{m}))$ such that

$$\mu(A_s(Z)(us^{-1})) = A_s \circ (\mu(Z)(u)) \circ A_s^{-1} \quad \text{for all } u \in P,\ s \in G,\ Z \in \mathfrak{m}.$$

The relation between λ and μ is $\mu(Z)(u) = (j^*\lambda)(Z(u))$.

Suppose that λ in theorem 4 were invariant. One checks easily that the map $\mu(Z) \in C^\infty(P, \mathfrak{gl}(\mathfrak{m}))$ must be constant. Hence, if we identify $\mu(Z)$ with its constant value we get

Theorem 5. The invariant connections λ on LM are in bijective correspondence with the linear maps $\mu : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ such that

$$\mu(A_s(Z)) = A_s \circ \mu(Z) \circ A_s^{-1} \qquad \text{for all } s \in G,\ Z \in \mathfrak{m}$$

or the bilinear maps $\mu : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ such that

$$\mu(A_s(X), A_s(Y)) = A_s\Big(\mu(X, Y)\Big) \qquad \text{for all } s \in G,\ X, Y \in \mathfrak{m}.$$

The following theorem is well known and easy to prove.

Theorem 6. If $\lambda \in \mathrm{Conn}(LM)$ is invariant and comes from a connection ω of P, there is a supplementary $\mathfrak{m}_1$ of $\mathfrak{g}$ in $\mathfrak{p}$ which is invariant by all A_s, $s \in G$. In fact $\mathfrak{m}_1 = \ker(\mu)$. Conversely, each invariant supplementary $\mathfrak{m}_1$ as above, induces an invariant connection coming from a connection of P and being determined by $j^*\lambda(Z) = 0$ if $Z \in \mathfrak{m}_1$.

This theorem has been stated for the following reason. Suppose λ is invariant and comes from a connection on P. Then, using the associated subspace $\mathfrak{m}_1$, we could construct the morphism $j_1 : P \to LM$ and $j_1^*\lambda$ would be 0 on the subspace $\mathfrak{m}_1$.

4. SINGULARITIES OF INVARIANT CONNECTIONS

We denote by L a connected component of LM.

Theorem 7. An invariant connection λ on L coming from a connection of P, has no singularities.

Proof. As we remarked after theorem 6, we may assume without losing generality that the morphism $j : P \to L$ verifies $j^*\lambda = \mu$ is null on $\mathfrak{m}$. Suppose there is a singularity. Then, there is (see Ref. 5) a finite length horizontal curve C in L defined on $[0,1)$ and a sequence (t_n) converging to 1 such that $(C(t_n))$ is a non-convergent Cauchy sequence in L. After translation by some $a \in GL(\mathfrak{m})$, we may assume that $C(0) \in j(P)$. Let ω be the connection on P such that $j^*\lambda = a_\omega$. By the uniqueness of horizontal lifts, C can be written as $C = j \circ C_1$ for some curve $C_1 : [0,1) \to P$ which is horizontal for ω. Indeed, C_1 is the horizontal lift $\pi \circ C : [0,1) \to M$ starting at a point of $j^{-1}(C(0))$.

We have Riemann metrics g^λ in L and g^ω in P induced by the connections λ and ω. If $(\,|\,)$ is a Euclidean product in $\mathfrak{p}$ we have $g^\omega = (\theta \oplus \omega | \theta \oplus \omega)$. On the other hand, the Euclidean product $(\,|\,)$ in $\mathfrak{m}$ gives a Euclidean product $\langle\ ,\ \rangle$ in $\mathfrak{gl}(\mathfrak{m})$ by $\langle a, b\rangle = \mathrm{Tr}(ab^T)$. For simplicity $\langle\ ,\ \rangle$ will denote the induced product on $\mathfrak{m} \oplus \mathfrak{gl}(\mathfrak{m})$. The metric g^λ is $\langle \Phi \oplus \lambda, \Phi \oplus \lambda\rangle$. Since λ comes from ω, we have by theorem 2c that C and C_1 have the same length. We have then a curve C_1 with finite length in a Lie group P with a left invariant metric g^ω. Since these metrics are complete there exists $u = \lim(C_1(t))$ for $t \to 1$ and, by continuity of j also $\lim(C(t)) = j(u)$. So $(C(t_n))$ has limit in L and this is a contradiction, ending the proof. ∎

The following lemma is well known.

Lemma 2. Let $\pi : L \to M$ be a principal bundle with group G. Consider a curve $c : I \to M$ and a lift $C_1 : I \to L$. The maps $g : I \to G$ such that $C = C_1 g$ are horizontal for a connection λ are the solutions of the equation

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The following lemma is well known.

Lemma 2. Let $\pi : L \to M$ be a principal bundle with group G. Consider a curve $c : I \to M$ and a lift $C_1 : I \to L$. The maps $g : I \to G$ such that $C = C_1 g$ are horizontal for a connection λ are the solutions of the equation

$$Ad_{g(t)}^{-1}\Big(\lambda(\dot{C}_1(t))\Big) + g^*\Sigma(t) = 0$$

Corollary. Let $k \neq 0$ and $Z \in \mathfrak{m}$ be an eigenvector of $\mu(Z)$ with k as eigenvalue. Take $\epsilon = \pm 1$ with opposite sign to k. The length of $C_{|[0,\infty)}$ is finite.

Proof. Using $\mathrm{EXP}(t\mu(Z))(Z) = e^{kt}Z$ we get

$$\mathrm{Length}(C) = |Z| \int_0^\infty e^{\epsilon k t} dt < \infty$$

because $\epsilon k < 0$. ∎

A curve $c : [0,\infty) \to M$ is inextendible if the limit of $c(t)$ for $t \to \infty$ does not exist. The geodesics of any linear connection are inextendible if defined on $[0,\infty)$. Consider the curve $c(t) = \pi(u \exp(tZ))$. We claim it is inextendible, by showing it is a geodesic for a certain connection. Indeed, let λ be the connection such that $\mu = 0$ in the correspondence of theorem 5. We consider the horizontal lift C given by theorem 8. We shall prove that $\dot{c}(t) = C(t)(Z)$ (this means that the frame $C(t)$ acting on Z gives the tangent vector $c(t)$), so, since Z does not depend on t, the covariant derivative of $\dot{c}$ will be 0. The formula for $\dot{c}(t)$ is easy because the hypothesis $\mu = 0$ gives $C(t)(Z) = j(u \exp(tZ))(Z)$ and this is just $\dot{c}(t)$ by definition of j.

Theorem 9. Consider an invariant connection λ on L given by $\mu : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ and $Z \in \mathfrak{m}$ such that $\mu(Z)(Z) = kZ$ with $k \neq 0$. If $\epsilon = \pm 1$ has opposite sign to k, the curve $c(t) = \pi(u \exp(\epsilon t Z))$, $t \geq 0$, is b-incomplete and thus converges to a singularity.

Proof. Take as an auxiliary connection λ_0 such that the corresponding μ_0 is 0. We have just shown that c is a λ_0-geodesic so it is inextendible. (Perhaps it is not a λ-geodesic, but this is irrelevant.) We proved in the corollary of theorem 8 that it has finite length. It is therefore b-incomplete, and by Ref. 5, section 4, there is a singularity among its endpoints. ∎

Theorem 9 is not the best possible but is stated first for simplicity. Indeed, considering the Jordan normal form of $\mu(Z)$ more general conditions can be given to get $\mathrm{Length}(C) < \infty$.

Suppose M is an endomorphism of the real vector space $\mathbb{E}$. It can then be uniquely written as $M = D + N$, with $DN = ND$, N nilpotent and D diagonalizable if extended to the complexification $\mathbb{E}_c$ of $\mathbb{E}$. In fact, a basis of $\mathbb{E}$, the Jordan basis, can be constructed such that the matrix of

D is zero except for some blocks in diagonal positions of one of the forms

$$\begin{pmatrix} r & & & \\ & \cdot & & \\ & & \cdot & \\ & & & r \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} R & & & \\ & \cdot & & \\ & & \cdot & \\ & & & R \end{pmatrix}$$

$$\text{with } R = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \text{ and } b > 0.$$

The first blocks correspond to the real eigenvalues r of M, whilst the second blocks correspond to the complex eigenvalues $r = a + bi$ with $b > 0$. Clearly, for $\epsilon = \pm 1$ and $t \geq 0$ we have $\mathrm{EXP}(\epsilon t M) = \mathrm{EXP}(\epsilon t D)\mathrm{EXP}(\epsilon t N)$, where the coefficients of $\mathrm{EXP}(\epsilon t N)$ are polynomials in t and $\mathrm{EXP}(\epsilon t D)$ is made of diagonal blocks of one of the forms

$$\begin{pmatrix} e^{\epsilon t r} & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & e^{\epsilon t r} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} e^{R} & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & e^{R} \end{pmatrix}$$

$$\text{with } e^{R} = e^{\epsilon t a} \begin{pmatrix} \cos(\epsilon t b) & -\sin(\epsilon t b) \\ \sin(\epsilon t b) & \cos(\epsilon t b) \end{pmatrix}$$

and $b > 0$. Define for M the subspaces M_c^+ (resp. M_c^-) as the direct sum of the generalized eigenspaces of M_c whose eigenvalues have strictly positive (resp. negative) real part. Define then $M^+ = \mathbb{E} \cap M_c^+$ (resp. $M^- = \mathbb{E} \cap M_c^-$). If $z \in M^+$, taking $\epsilon = -1$ we get that the components in the Jordan basis of

$$\mathrm{EXP}(\epsilon t M)(z) = \mathrm{EXP}(\epsilon t D)\mathrm{EXP}(\epsilon t N)(z)$$

are of the form $e^{-tu}p(t)$ with $u > 0$ and $p(t)$ a polynomial times a bounded function. Therefore,

$$\int_0^\infty |\,\mathrm{EXP}(\epsilon t M)(z)\,|dt < \infty.$$

We also get a finite integral if $z \in M^-$ and we take $\epsilon = +1$. We apply all this for $M = \mu(Z) : \mathfrak{m} \to \mathfrak{m}$, denoting M^+ and M^- by $\mu(Z)^+$ and $\mu(Z)^-$. Then

Theorem 10. With the above notations, if Z belongs to $\mu(Z)^+$, or $\mu(Z)^-$, taking $\epsilon = -1$ or $\epsilon = 1$, we get that the curve $c(t) = \pi(u\exp(\epsilon tZ))$, $t \in [0, \infty)$ is b-incomplete and converges to a singularity.

5. CURVES INDUCING THE SAME SINGULARITY

In this section we consider an arbitrary connected subbundle $\pi : P \to M$ of the frame bundle of M with connected group $G \subseteq GL(m;\mathbb{R})$. The fundamental form is denoted by Φ and λ is a fixed connection on P. The reader may only consider the case $P = L$, a connected component of the frame bundle of M.

Consider curves $C, D : I \to P$ with finite length. We call them *singularity equivalent*, or briefly, *s-equivalent*, if they have endpoints $u, v \in \bar{P}$ such that $\bar{\pi}(u) = \bar{\pi}(v) = x \in \bar{M}$. The condition clearly implies that for all sequences t_n converging to 1, the sequences $\pi \circ C(t_n)$ and $\pi \circ D(t_n)$ converge to x, but they may also converge to a different point in $\bar{M}$, because this space is not Hausdorff in general.

Theorem 11. Consider curves $C, D : [0,1) \to P$ with finite length. The following statements are equivalent.

a) The curves are s-equivalent.

b) There is an increasing sequence $t_n \to 1$ and curves $\Gamma_n^* : [0,1] \to P$ joining $C(t_n)$ to $D(t_n)$ and a constant k such that

$$\int_0^1 |\lambda(\dot{\Gamma}_n^*(t))|dt \le k \quad \text{for all } n, \qquad \int_0^1 |\Phi(\dot{\Gamma}_n^*(t))|dt \to 0.$$

c) There is an increasing sequence $t_n \to 1$ and horizontal curves $\Gamma_n : [0,1] \to P$, joining $C(t_n)$ to the fibre $\pi^{-1}(\pi(D(t_n)))$ such that $\text{Length}(\Gamma_n) \to 0$ and the sequence s_n in G defined by $\Gamma_n(1)s_n = D(t_n)$ converges to some $s_0 \in G$.

Proof. We give circular proof. A lemma is required.

Lemma 3. Consider an interval I with ends a, b (open, closed, or semi-open) and curves $C, C^* : I \to P$ such that $c = \pi \circ C = \pi \circ C^*$. Define $s : I \to G$ by $C^*(t) = C(t)s(t)$ and $c^i, c^{*i} : I \to \mathbb{R}$ by

$$\dot{c}(t) = \sum c^i(t)C_i(t) = \sum c^{*i}(t)C^*{}_i(t)$$

If C is horizontal we have

a) $$\text{Length}(C^*) \geq \max\left\{\int_a^b \left(\sum c^{*i}(t)^2\right)^{1/2} dt,\ \text{Length}(s)\right\} \qquad (1)$$

where the length in G is measured with the left invariant metric induced by the scalar product $\langle\ ,\ \rangle$ in $\mathfrak{g}$.
b) If $\text{Length}(C^*) < \infty$ there is a constant K, such that $\text{Length}(C) \leq K\,\text{Length}(C^*)$. In particular, if a curve c in M has a lift C^* with finite length, then any of its horizontal lifts has finite length.

Proof. We have the formula

$$\dot{C}^*(t) = TR_{s(t)}(\dot{C}(t,) + (s^*\Sigma(t))_{\#}(C^*(t)) \qquad (2)$$

where Σ is the Maurer–Cartan form of G, and $\#$ denotes the vertical field induced by an element of $\mathfrak{g}$. It follows that

$$g^\lambda(\dot{C}^*(t), \dot{C}^*(t)) = |\Phi(\dot{C}^*(t))|^2 + |s^*\Sigma(t)|^2 = \sum (c^{*i}(t))^2 + |s^{-1}(t)\dot{s}(t)|^2$$

and (a) is immediate by integration.

To prove (b) we consider the norm $\|\,.\,\|$ given by the supremum on the unit sphere. If C^* has finite length, we get from (a) that $\text{Im}(s)$ is contained in a bounded subset of G, which must be relatively compact because, with the metric given, G is complete. We take K such that $\|s(t)\| \leq K$ for all $t \in I$. The scalars c^i, c^{*i} are related by $c^i = \sum s^i_j c^{*i}$, so we get

$$\begin{aligned} \text{Length}(C) &= \int_a^b \left(\sum c^i(t)^2\right)^{1/2} dt \\ &\leq K \int_a^b \left(\sum c^{*i}(t)^2\right)^{1/2} dt \leq K\,\text{Length}(C^*) \end{aligned}$$

and this ends the proof. ∎

We start the proof of the theorem. Suppose the curves are s-equivalent. By hypothesis, for some $a \in G$ the curves C and $D' = Da$ have the same endpoint. Choose any increasing sequence $t_n \to 1$ and curves Δ_n defined on $[0,1]$ joining $C(t_n)$ to $D'(t_n)$ with $\text{Length}(\Delta_n) \to 0$. Take also a curve $s : [0,1] \to G$ joining the unit e to a^{-1} such that $\text{Length}(s) \leq 2d(e, a^{-1})$. Finally, define $\Gamma^*_n(t) = \Delta_n(t)s(t)$, which is a curve joining $C(t_n)$ to $D(t_n)$.

We have then the formulas

$$\dot{\Gamma}^*_n(t) = TR_{s(t)}(\dot{\Delta}_n(t)) + ((s^*\Sigma)(t))_{\#}(\Gamma^*_n(t))$$

$$|\lambda(\dot\Gamma_n^*(t))| = |Ad(s^{-1}(t))(\lambda(\dot\Delta_n(t)) + s^*\Sigma(t)|^2$$
$$\leq \left(|Ad(s^{-1}(t))(\lambda(\dot\Delta_n(t))| + |s^*\Sigma(t)|\right)^2.$$

Since $\mathrm{Im}(s)$ is compact, there is a constant H such that $\|Ad(s^{-1}(t))\|$ and $\|s^{-1}(t)\| \leq H$. (The symbol $\|\cdot\|$ was defined in lemma 3.) Therefore we get.

$$\int_0^1 |\lambda(\dot\Gamma_n^*(t))|dt \leq \int_0^1 H|\lambda(\dot\Delta_n(t))|dt + 2d(e, a^{-1}).$$

The right hand side is bounded because

$$\int_0^1 |\lambda(\dot\Delta_n(t))|dt \leq \mathrm{Length}(\Delta_n) \to 0$$

and this implies that the curves Γ_n^* verify the first half of (b). Also,

$$|\Phi(\dot\Gamma_n^*(t))|^2 = |s^{-1}(t)(\Phi(\dot\Delta_n(t))|^2 \leq H^2|\dot\Phi(\Delta_n(t))|^2$$

and, by integration,

$$\int_0^1 |\Phi(\dot\Gamma_n^*(t))|dt \leq H\ \mathrm{Length}(\Delta_n) \to 0$$

and (b) is fully proved.

Assume (b) holds. Define Γ_n as the horizontal lift of $\pi \circ \Gamma_n^*$ starting at $C(t_n)$, and the curves s_n in G by $\Gamma_n^* = \Gamma_n s_n$. The conditions on the Γ_n^* imply that for n big enough all $\mathrm{Length}(\Gamma_n^*) \leq 2k$. Using lemma 3, we get that all points $s_n(t)$ are in a ball centered on e with radius $2k$. Taking subsequences if necessary we may assume that the sequence $s_n(1)$ is convergent, giving part of condition (c). On the other hand, applying Φ to Γ_n^* we get

$$\phi(\dot\Gamma_n(t)) = s_n(t)(\phi(\dot\Gamma_n^*(t))).$$

But all $s_n(t)$ are in a compact ball in G as we said, so we may suppose that for some constant H we have $\|s_n(t)\| \leq H$ and

$$\mathrm{Length}(\Gamma_n) = \int_0^1 |\Phi(\dot\Gamma_n(t))|dt \leq H\int_0^1 |\Phi(\dot\Gamma_n^*(t))|dt \to 0$$

showing that the condition (c) holds.

We show that (c) implies (a). All limits are understood for $n \to \infty$. If $\lim(C(t_n)) = u$, then u is also the limit of $\Gamma_n(1)$. Therefore

$$us_0 = \lim(\Gamma_n(1)s_0) = \lim(\Gamma_n(1)s_n(1)) = \lim(\Gamma_n^*(1)) = \lim(D(t_n))$$

proving that C and D are s-equivalent. ∎

We will apply theorem 11 to show that certain curves give the same singularity. Suppose that P is compact and connected and that for some $Z \in \mathfrak{m}$ we have $\mu(Z) = id$. By theorem 9, the curve $c(t) = \pi(\exp(-tZ))$, $t \in [0, \infty)$, converges to a singularity. We will show that for any $u \in P$ the curve $d(t) = \pi(u \exp(-tZ))$ converges to the same singularity. Since P is compact and connected, exp is onto, so we may suppose that we can write $u = \exp(U)$ for some $U \in \mathfrak{p}$. The curves

$$C(t) = j(\exp(-tZ))e^t, \qquad D(t) = j(\exp(U)\exp(-tZ))e^t, \qquad t \in [0, \infty)$$

give according to theorem 8 horizontal lifts of c and d. (We identify $\mathrm{EXP}(t\mu(Z))$ with the scalar e^t.) The curves

$$\Gamma_t(s) = j(\exp(sU)\exp(-tZ))e^t, \quad s \in [0, 1]$$

join $C(t)$ to $D(t)$. Let

$$\alpha_t(U) = Ad_{\exp(tZ)}(U) \in \mathfrak{p}, \quad \beta_t(U) = pr_{\mathfrak{g}} \circ \alpha_t(U), \quad \gamma_t(U) = pr_{\mathfrak{m}} \circ \alpha_t(U).$$

For the supremum norm, P being compact, a number H can be chosen such that

$$H \geq \max \{ \|\alpha_t\|, \|\beta_t\|, \|\gamma_t\|, \|\alpha\|, \|\mu\| \mid t \in [0, \infty) \}.$$

One gets the formulas

$$\begin{aligned}
\dot{\Gamma}_t(s) &= TR_{(e^t)} \circ Tj\{\alpha_t(U)(\exp(sU)\exp(-tZ))\} \\
\lambda(\dot{\Gamma}_t(s)) &= (j^*\lambda)(\alpha_t(U))(\exp(sU)\exp(-tZ)) \\
&= \alpha(\beta_t(U)) + \mu(\gamma_t(U)) \\
\Phi(\dot{\Gamma}_t(s)) &= e^{-t}\gamma_t(U) \\
\int_0^1 |\lambda(\dot{\Gamma}_t(s))|ds &\leq 2H^2|U| \\
\int_0^1 |\Phi(\dot{\Gamma}_t(s))|ds| &\leq e^{-t}H|U|.
\end{aligned}$$

The last two formulas show that condition (b) in theorem 11 holds and we have therefore proved that c and d converge to the same singularity.

We give a concrete example of the situation described. Take $SO(4;\mathbb{R})$ as P and $SO(2;\mathbb{R})$ as G identifying its matrices with those in P leaving fixed the plane spanned by (0,0,1,0) and (0,0,0,1). It is clear that we may take as m the set of matrices in $\mathfrak{p} = so(4;\mathbb{R})$ of the form

$$Z = \begin{pmatrix} \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} & -x^T \\ x & 0_2 \end{pmatrix}$$

with $a \in \mathbb{R}$, x an arbitrary 2×2 matrix, and 0_2 the zero 2×2 matrix. The notation $Z = (a, x)$ is compact and convenient. One checks easily that

$$\mu : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}, \qquad \mu(\,(a,x),(b,y)\,) = (ab, ay + bx)$$

is $\mathcal{A}_G$-invariant as defined in theorem 5. Therefore, μ gives an invariant connection. Take $Z = (1, 0_2) \in \mathfrak{m}$. The preceding work shows that all curves

$$c(t) = \pi(u\exp(-tZ)\,), \qquad t \in [0,1), \quad u \in SO(4;\mathbb{R}) = P$$

converge to the same singularity.

6. AN EXAMPLE

We take as P the abelian group $\mathbb{R}^m$ and $G = 0$. There are immediate identifications: $P = M = \mathbb{R}^m$, $\mathfrak{p} = \mathfrak{m} = \mathbb{R}^m$, $\pi = id$, and $u\exp(tZ) = u + tZ$. We identify the frame bundle of M with $\mathbb{R}^m \times GL(m;\mathbb{R})$ and L with $\mathbb{R}^m \times GL^+(m;\mathbb{R})$, so L-valued maps will have 2 components. Our connection will be given by the map $\mu : \mathfrak{m} = \mathbb{R}^m \to \mathfrak{gl}(\mathfrak{m})$, $\mu(Z) = f(Z)$ where f is a linear map with values in $\mathbb{R}$ and all $\mu(Z)$ are scalars. To simplify the formulas we may suppose $f(Z) = \langle U, Z\rangle$ with $U = (0, 0, \ldots, 1)$ and $\langle\,,\,\rangle$ the standard inner product.

Consider a curve $c : I \to M$ and let $C : I \to L$ be the horizontal lift with initial condition $C(0) = (x, 1) \in \mathbb{R}^m \times GL^+(m;\mathbb{R})$. If a, b are the ends of I, using theorem 8 we easily get the formulas

$$C(t) = \left(c(t), e^{-[f(C(t)) - f(x)]}\right) \in \mathbb{R}^m \times GL^+(m;\mathbb{R}), \tag{3}$$

$$\text{Length of } C = \int_a^b e^{[f(c(t)) - f(x)]} |\dot{c}(t)|\,dt. \tag{4}$$

The following inequality will be important:

$$\text{Length of } C \geq \left| \left\{ e^{[f(c(t)) - f(x)]} \right\}_{t=a}^{t=b} \right|. \tag{5}$$

To prove it we use $|f(Z)| \leq |Z|$ and

$$\frac{d}{dt}\left[e^{[f(c(t))-f(x)]}\right] = f(\dot{c}(t))e^{[f(c(t))-f(x)]}.$$

Then, by integration,

$$\begin{aligned}\text{Length of } C &\geq \int_a^b \left|\frac{d}{dt}\left[e^{[f(c(t))-f(x)]}\right]\right| dt \\ &\geq \left|\left\{e^{[f(c(t))-f(x)]}\right\}_{t=a}^{t=b}\right|,\end{aligned}$$

getting (5).

Lemma 4. If $c : [0,\infty) \to M = \mathbb{R}^m$ is b-incomplete, then $f(c(t)) \to -\infty$.

Proof. The starting point of the curve is irrelevant, so we assume that $f(c(0)) = 0$. We get from (5) that $e^{f(c(t))}$ is bounded above. If it were also bounded below, say $e^{f(c(t))} \geq e^H$ for some constant H, then

$$\text{Length of } C \geq \int_0^\infty e^H |\dot{c}(t)| dt = e^H \,\text{Length}(c).$$

Therefore, c has finite length in the standard Riemann metric of $\mathbb{R}^m$ and must converge to a point there; so c cannot be b-incomplete. We have proved that for some sequence (t_n), $(f(c(t_n)) \to -\infty$. But we also have the bound

$$\text{Length}(C) \geq \int_0^\infty |e^{f(c(t))} f(\dot{c}(t))| dt$$

and the absolute convergence of the integral gives the existence of the limit of $e^{f(c(t))} - 1$. For at least a sequence (t_n) it is -1, so it is -1 for all sequences and $f(c(t)) \to -\infty$. ∎

Let $\gamma : [0,\infty) \to M = \mathbb{R}^m$, $\gamma(t) = -tU$. Then $\Gamma(t) = (\gamma(t), e^t)$ is the horizontal lift of γ with $\Gamma(0) = (0,1)$ and $\text{Length}(\Gamma) = \int_0^\infty e^{-t} dt < \infty$. Therefore γ is b-incomplete and converges to a singularity, say p^*.

Theorem 12. Let $c : [0,\infty) \to M$ be a b-incomplete curve and C the horizontal lift of c with $C(0) = (c(0), 1)$. Then C and Γ converge to the same point and p^* is the unique singularity of the connection.

Proof. Since $\text{Length}(C) < \infty$, all sequences $(C(t_n))$ with $t_n \to \infty$ are equivalent Cauchy sequences, having the same limit in L. To get the first part of the theorem we only need a sequence $(b_n) \to \infty$ such that

$$d(C(b_n), \Gamma(-f(c(b_n)))) \to 0.$$

We point out first that for $x, y \in M$ such that $f(x) = f(y)$, the curve

$$\sigma : [0,1] \to M, \qquad \sigma(s) = (1-s)x + sy$$

has the curves $\Sigma(s) = (\sigma(s), r)$, $r > 0$, as horizontal lifts by (3). Therefore the points $C(t_n)$ and $\Gamma(-f(c(t_n)))$ can be joined by the horizontal curve

$$\Sigma_n(s) = \big((1-s)\gamma(-f(c(t_n))) + sc(t_n),\ e^{-f(c(t_n))}\big), \qquad s \in [0,1]$$

whose length is less than $e^{f(c(t_n))}|c(t_n)|$. (See formulas (3) and (4).) The first part will be proved if we construct a sequence $(b_n) \to \infty$ such that

$$f_n = f(c(b_n)) \to -\infty, \qquad e^{f_n}|c(b_n)| \to 0.$$

This is done as follows. By lemma 4, taking subsequences if necessary, we have a strictly increasing sequence (a_n) such that $f(c(a_n))$ is strictly decreasing and converges to $-\infty$. We may also suppose that the condition

$$e^{f(c(a_{n+1}))}|c(a_n)| < \frac{1}{n} \qquad \text{for all } n$$

holds. Define now $b_n \in [a_n, a_{n+1}]$ by

$$f(c(b_n)) = \min\{f(c(t))\,|\,a_n \le t \le a_{n+1}\}.$$

The condition $\mathrm{Length}(C) < \infty$ implies that

$$\epsilon_n = \int_{a_n}^{b_n} e^{f(c(t))}|\dot{c}(t)|\,dt \to 0.$$

Hence, for some θ_n in $[a_n, b_n]$ we have

$$\epsilon_n = e^{f(c(\theta_n))}\int_{a_n}^{b_n} |\dot{c}(t)|\,dt \ge e^{f(c(\theta_n))}\big(|c(b_n)| - |c(a_n)|\big).$$

Using that $-f(c(\theta_n)) \le -f(c(b_n))$ we easily get

$$\begin{aligned} e^{f(c(b_n))}|c(b_n)| &\le e^{f(c(b_n))}|c(a_n)| + \epsilon_n \\ &\le e^{f(c(a_{n+1}))}|c(a_n)| + \epsilon_n \le \tfrac{1}{n} + \epsilon_n \to 0. \end{aligned}$$

Therefore (b_n) is the sequence required to prove that C and Γ have the same limit. Now, it follows from Ref. 5, theorem 4.2 that any singularity is the limit of a b-incomplete curve c. With the notation above C and Γ have the same limit, so the limit of c is the limit of γ which is p^*. ∎

Lemma 5. For each $r \in \mathbb{R}$, $\pi^{-1}(f^{-1}(r)) \subseteq L$ is closed in the completion $\bar{L}$.

Proof. Let (u_n) be a sequence in the set with limit $v \in \bar{L}$ not in $\pi^{-1}(f^{-1}(r))$. A finite length curve $C : [0,\infty) \to L$ can be constructed such that $C(n) = u_n$ and converging to v. Clearly its projection c is b-incomplete by lemma 3b and [5], theorem 4.2. Then, by lemma 4, we have $f(c(n)) \to -\infty$. But this is a contradiction because $f(c(n)) = r$ for all n ∎

Theorem 13. For each $r \in \mathbb{R}$ the set $E_r = \{p^*\} \cup \{x \in M / f(x) < r\}$ is a neighbourhood of p^* in the completion of $M = \mathbb{R}^m$, so $\bar{M}$ is Hausdorff.

Proof. Let $\bar{\pi} : \bar{L} \to \bar{M}$ be the projection. We must show that $A = \bar{\pi}^{-1}(E_r)$ is open. The only difficulty arises when showing that a point $v \in \bar{\pi}^{-1}(p^*)$ is an interior point. Let $2\delta = d(v, \pi^{-1}(f^{-1}(r)))$. By lemma 5, $\delta > 0$. We say that if B is the ball centered at v with radius δ, then $B \subseteq A$. If this were false there would be a point $u \in B$ such that $f(\pi(u)) \geq r$. We join u and v with a curve C with length$(C) < \delta$. By continuity, $c = \pi \circ C$ cuts the hyperplane $f(x) = r$, say $f(c(t)) = r$. But since $d(v, C(t)) < \delta$, it cannot be $d(v, \pi^{-1}(f^{-1}(r))) = 2\delta$. Contradiction. The second statement is trivial. ∎

Theorem 14. If N is a neighbourhood of p^*, for any curve $c : [0,\infty) \to M = \mathbb{R}^m$ such that $e^{f(c(t))}|c(t)| \to 0$, and $f(c(t)) \to -\infty$, there is $a \in \mathbb{R}$ such that $c[a,\infty) \subseteq N$.

Proof. We prove the theorem checking the proof of theorem 12. With the notations there we have that the length of the curve

$$\Sigma(s) = \Big((1-s)\gamma(-f(c(t))) + sc(t)\,,\ e^{-f(c(t))} \Big)\,, \quad s \in [0,1]$$

joining $C(t)$ and $\Gamma(-f(c(t)))$, is less than $e^{f(c(t))}|c(t)|$. With our hypothesis, C and Γ have the same limit. Hence s and γ also have the same limit which is p^*, and the conclusion is obvious. ∎

Consider an arbitrary manifold M and L,π,λ etc. with the same meaning as in Section 2. Let A be a group of affine transformations acting discretely, i.e. properly and discontinuously, on M. Let $M_0 = M/A$ be the orbit space, $p : M \to M_0$ the projection, and $q : L \to L_0$ the induced map between certain connected components of the frame bundles. There is a unique connection λ_0 on L_0 such that $q^*\lambda_0 = \lambda$. The action of A on M can be naturally extended to both $\bar{M}$ and $\bar{L}$, the b-completion of M and the Cauchy completion of L respectively. The b-completions of M and M_0 are related as follows.

Theorem 15. If the action of A on $\bar{L}$ has closed orbits, then $\bar{M}_0$ is homeomorphic to $\bar{M}/A$ with the quotient topology.

Proof. See [1].

We shall apply this theorem to the situation described at the beginning of the section. Therefore $\mathbb{R}^m = P$ is identified with the translation group of $\mathbb{R}^m = M = P/\{0\}$. Since $\mathfrak{g} = 0$, we have $\mathfrak{p} = \mathfrak{m} = \mathbb{R}^m$ and a connection λ is given by a linear map $f : \mathbb{R}^m \to \mathbb{R}$ as described above. Any discrete subgroup of translations A of $\mathbb{R}^m$ is a group of affine maps for the connection λ. The manifold $M_0 = M/A$ has a connection λ_0 described by the same linear map f. Therefore taking A as the subgroup G of P we have an example of reductive homogeneous space. Its b-completion is described by the following theorem.

Theorem 16. The b-completion of the homogeneous space $P/G = \mathbb{R}^m/A$ with the connection λ_0 given by the linear map $f : \mathbb{R}^m \to \mathbb{R}$ is the orbit space of $\mathbb{R}^m \cup \{p^*\}$ under the action of $A = G$ given by

$$a \cdot x = a + x \quad \text{for } x \in \mathbb{R}^m \quad \text{and} \quad a \cdot p^* = p^*.$$

The topology of this orbit space is the quotient topology of the topology of the b-completion $\mathbb{R}^m \cup \{p^*\}$ under the action of $A = G$.

Proof. We only need apply theorem 15. Since the action of A on $\mathbb{R}^m \cup \{p^*\}$ is the extension of the action on $\mathbb{R}^m$ the only possibility is $a \cdot p^* = p^*$ for all $a \in A$. We show that for each $u \in \bar{L}$ the orbit $A \cdot u$ is closed. First, if $u = (x, h) \in L$ we have $a \cdot (x, h) = (a + x, h)$ and $A \cdot u$ is closed. If $u \in \bar{L} - L$, it will be the limit of a Cauchy sequence in L. As we have shown in the proof of theorem 12, we may assume that this sequence is $((-nU, e^n)) = \Gamma(n)$. But $a \cdot \Gamma(t) = (a - tU, e^t)$, which is also a horizontal curve. The "transversal curves" $\Sigma_n(s) = (-nU + sa, e^n)$, $s \in [0, 1]$ join $\Gamma(n)$ and $a \cdot \Gamma(n)$ and have length $\leq e^{-n}|a| \to 0$. Therefore the sequences $(\Gamma(n))$ and $(a \cdot \Gamma(n))$ have the same limit and we have got that all points in $\bar{\pi}^{-1}(p^*)$ are left fixed by A so these orbits are clearly closed. ∎

Corollary A. If $G = A \subseteq \mathbb{R}^m$ is such that for some $a \in A$ is $f(a) \neq 0$, then the only singularity of P/G has only one neighbourhood, the whole space.

Proof. $N \subseteq \bar{\mathbb{R}}^m/A$ is a neighbourhood of the singularity if and only if $\bar{p}^{-1}(N)$ is a neighbourhood of p^* which is A-invariant. ($\bar{p}$ is the natural extension of p to the completions.) This only happens if $\bar{p}^{-1}(N)$ is the whole space, hence N is the whole space. ∎

Corollary B. If $A \subseteq \ker f$ and $\ker f/A$ is compact, any neighbourhood N of the unique singularity q^* of $\bar{\mathbb{R}}^m/A$ contains one of the form $(\mathbb{S}^1)^{m-1} \times (-\infty, a) \cup \{q^*\}$.

Proof. Let $\bar{\sigma} : \bar{\mathbb{R}}^m \to \bar{\mathbb{R}}^m/A$ be the projection. If for all $a \in \mathbb{R}$ the inclusion

$$\mathbb{R}^{m-1} \times (-\infty, a) \cup \{q^*\} \subseteq \bar{\sigma}^{-1}(N) \tag{6}$$

were false, using theorem 16, a sequence (x_n) not in $\bar{\sigma}^{-1}(N)$ could be constructed in a strip $K \times \mathbb{R}$ with K compact and $f(x_n) \to -\infty$. Then a curve $c : [0, \infty) \to K \times \mathbb{R}$ can be given with $c(n) = x_n$. This curve verifies the hypothesis of theorem 14 but not the conclusion. The contradiction shows that (6) holds for some a and the corollary follows. ∎

The theorems in this section generalize those given in [3] where (with our notation) the cases $P = \mathbb{R}$ and $G = \{0\}$ or $\mathbb{Z}$ are considered, obtaining the quotient spaces $M = \mathbb{R}$ or $\mathbb{S}^1$. In both cases the linear map $f : \mathbb{R} \to \mathbb{R}$ gives the connection and can be identified with a scalar λ. In [3] the b-completions obtained are either $\mathbb{R} \cup \{-\infty\}$ with intervals $[-\infty, r)$ as neighbourhoods of $p^* = -\infty$, or $\mathbb{S}^1 \cup \{p^*\}$ having p^* only one neighbourhood.

REFERENCES

1. Amores, A. M. (1988). *Lett. Math. Phys.* **15**, 341.
2. Dodson, C. T. J. (1978). *Int. J. Theor. Phys.* **17**, 389.
3. Dodson, C. T. J., and Sulley, L. J. (1977). *Lett. Math. Phys.* **1**, 301.
4. Hawking, S. W., and Ellis, G. F. R. (1973). *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge).
5. Schmidt, B. G. (1971). *Gen. Rel. Grav.* **1**, 269.