

Application of Lorentz geometry to nonimaging optics: new three-dimensional ideal concentrators

Manuel Gutiérrez

*Departamento de Álgebra, Geometría y Topología, Facultad de Ciencias,
Campus de Teatinos, Universidad de Málaga, 29071 Málaga, Spain*

Juan C. Miñano

*Instituto de Energía Solar, Escuela Técnica Superior de Ingenieros Telecomunicación,
Universidad Politécnica, Ciudad Universitaria, 28040 Madrid, Spain*

Carlos Vega

*Departamento de Matemática Aplicada a las Tecnologías de la Información, Escuela Técnica Superior
de Ingenieros Telecomunicación, Universidad Politécnica, Ciudad Universitaria, 28040 Madrid, Spain*

Pablo Benítez

*Instituto de Energía Solar, Escuela Técnica Superior de Ingenieros Telecomunicación,
Universidad Politécnica, Ciudad Universitaria, 28040 Madrid, Spain*

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A new family of three-dimensional ideal nonimaging concentrators with rotational symmetry is presented. The flow-line concentrator and the cone concentrator are particular cases of this family. First, we looked for elliptic bundles of rays, i.e., bundles such that the subset of rays passing through any point of the space forms a cone with an elliptic base (this search was done with the Lorentz geometry formalism.) Second, the concentrators, defined by their reflectors and receiver shapes, were derived from these elliptic bundles with the flow-line design method.

Key words: Nonimaging concentrators, Lorentz geometry, optics design. © 1996 Optical Society of America

1. INTRODUCTION

Optimum transfer of incoherent radiation from a source to a receiver is the general task of a nonimaging concentrator. The description of the concentrator's input and output bundles (\mathcal{M}_i and \mathcal{M}_o , respectively) characterizes the transference of radiation through the concentrator. Rays of \mathcal{M}_i entering the concentrator exit it as rays of \mathcal{M}_o , and vice versa. In general, the interest of the nonimaging concentrator is only in the radiation transfer, and so no additional information is required about the ray-to-ray correspondence between the rays of \mathcal{M}_i and \mathcal{M}_o . The bundles \mathcal{M}_i and \mathcal{M}_o can be described as regions of a four-dimensional phase space: any ray entering (or exiting) the concentrator may be uniquely determined with four parameters. For instance, assume that the concentrator's entry and exit apertures are flat and that x, y are coordinates on these apertures. Giving the point of incidence of the ray on the aperture (x, y) and the direction cosines of the ray at this point fully determines the ray.

The problem of designing a nonimaging concentrator may be stated as the problem of finding an optical system coupling a given \mathcal{M}_i with a given \mathcal{M}_o . Not all nonimaging devices are concentrators, and so not all nonimaging problems can be stated in this way. For instance, in the problem of a nonimaging reflector the bundle \mathcal{M}_o is

not prescribed; only the irradiance that it produces on a given surface is prescribed.¹⁻⁴ In general, the problem of designing a nonimaging concentrator has no perfect (ideal) solution. Only a few cases of three-dimensional (3D) ideal nonimaging concentrators are known. Some of them use graded refractive-index media,⁵ some others are formed by thin layers of transparent material that channel the light by total internal reflection,^{6,7} and only two of them are formed only by reflectors: the cone concentrator (CC) and the flow-line concentrator⁸ (FLC). The interest of the last two concentrators is obvious: they are much easier to manufacture. In addition to the CC and the FLC there are trivial cases, such as the case of a reflective cylinder.

The CC and the FLC were designed with the flow-line design method.^{8,9} The key point of this method is to calculate the flow lines of the bundle of rays issuing from a given surface. Let us explain this method in short with an example: consider a disk D of radius 1 on the plane $x_1 = 0$ (see Fig. 1). For an arbitrary point X , trace the rays linking the disk and the point X . The trajectories of these rays will form a cone with vertex at X . The rays whose trajectories form the cone surface are called edge rays. It can be proved that the base of this cone is elliptic. Therefore the cone has two planes of symmetry $P_1(X)$ and $P_2(X)$. One of the planes (P_1) contains the x_1

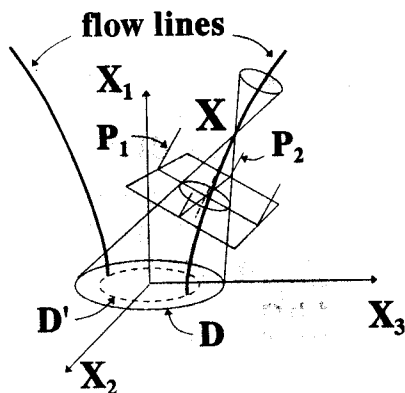


Fig. 1. Cone of edge rays at the point X and two flow lines of the FLC.

axis. Now construct a field of directions on the space x_1, x_2, x_3 by associating with each point the direction given by the intersection of the two planes of symmetry. The flow lines are the integral curves of this field of directions. The integral surfaces tangent to the planes $P_2(X)$ are one-sheet hyperboloids of revolution. The cross sections of these hyperboloids are confocal hyperbolas with foci at points of the circumference bounding D . Consider that one of these hyperboloids is a mirror. It can be proved^{8,9} that the bundle of rays issuing from the disk D is not perturbed by the introduction of this mirror. This means that any ray issuing from the disk D' (defined by the hyperbola's apex) has a trajectory, after any number of reflections, that corresponds to the trajectory that would have another ray issuing from D if there were no reflector. Or, which is the same thing, if a ray is sent from a point within the hyperboloid toward a point of the disk D , it will reach disk D' independently of the reflections that it may undergo. In this way radiation is concentrated on D' , which is called the receiver. The bundle \mathcal{M}_i is formed by the rays emitted within the hyperboloid toward D . The bundle \mathcal{M}_o is formed by all the rays impinging on D' .

In this paper we look for new elliptic bundles with rotational symmetry. We apply the term elliptic bundle to a four-parameter set of rays such that, at any point X of the space, the rays of the bundle passing through X form an elliptic cone. The bundle is said to have rotational symmetry around the x_1 axis if the cone at an arbitrary point P , when rotated around the x_1 axis to a point Q , coincides with the cone at Q .

We have found new families of these bundles. An ideal 3D nonimaging concentrator (defined by its reflector and its receiver shape) corresponds to each of these bundles. The FLC and the CC are particular cases of these concentrators.

The analysis has been formulated with the Lorentz geometry formalism. Because this formulation is highly geometric, the main ideas of the theory have nice geometric counterparts. For example, the cone of edge rays is the light cone of the Lorentz metric, the equation $|\mathbf{G}^{-1}| = 0$ defines the surface of the receiver (\mathbf{G} is the matrix that defines the Lorentz geometry), and the vector flux field is proportional to the eigenvector of the matrix \mathbf{G} associated with the unique negative eigenvalue of \mathbf{G} .

2. GENERALITIES ABOUT LORENTZ GEOMETRY

We shall give some notions about Lorentz geometry. The interested reader may consult Ref. 10 for background on differential geometry. The rapid development of Lorentz geometry is due to Einstein's general relativity theory of gravitation. We shall use Lorentz geometry in a quite different interpretation. The main interest of a Lorentz metric in general and also for our applications in non-imaging optics is that it provides a cone structure on the manifold, that is, a cone in the tangent space of each point of the manifold. We shall interpret this cone as formed by the directions of the edge rays passing through that point. In our study the manifold is an open subset M of \mathbb{R}^3 , and the tangent space $T_x M$ at a point X of M is isomorphic to \mathbb{R}^3 .

Let $B = \{\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3\}$ be the canonical global basis vector field on M . For each point X of M , B_x is the canonical basis of the tangent space $T_x M$. A Lorentz metric g on M is a map that assigns to every point X of M a bilinear map $g_x: T_x M \times T_x M \rightarrow \mathbb{R}$ with some additional properties. These properties may be expressed through the matrix \mathbf{G} associated with the map g in the basis B . Using this matrix, one can formulate the map g_x in the basis B_x as $g_x(Y, Y') = Y^t \mathbf{G} Y'$, where Y and Y' are two vectors of $T_x M$ (the superscript t denotes transposition). The elements of \mathbf{G} are the differentiable functions (of class C^∞) on M defined by $g_{ij}(x) = g_x(\frac{\partial}{\partial x_i}|_x, \frac{\partial}{\partial x_j}|_x)$. The matrix \mathbf{G} must verify the next three properties:

1. \mathbf{G} is symmetric on every point of M ;
2. $|\mathbf{G}| \neq 0$ on every point of M ;
3. \mathbf{G} has one negative and two positive eigenvalues on every point of M ; this is expressed by saying that g has the signature $(-, +, +)$.

The above properties characterize a C^∞ Lorentz metric on M . (Similarly, one can introduce a C^r Lorentz metric, $r \in \mathbb{N}$).

Three different types of vector $Y \in T_x M$ are considered: spacelike if $Y^t \mathbf{G} Y > 0$ or $Y = 0$, timelike if $Y^t \mathbf{G} Y < 0$, and lightlike if $Y^t \mathbf{G} Y = 0$ and $Y \neq 0$.

If $\gamma: \mathbb{R} \rightarrow M$ is a C^1 curve and $\dot{\gamma}: \mathbb{R} \rightarrow TM$ is its derivative, then γ is called spacelike if $\dot{\gamma}(t)$ is spacelike, $\forall t \in \mathbb{R}$. The timelike and the lightlike curves are defined in a similar way.

The set of all lightlike vectors in $T_x M$ forms a cone called the light cone in $T_x M$, whose equation is $Y^t \mathbf{G} Y = 0$. The set of all light cones in $T_x M \forall X \in M$ forms the cone structure on M (see Fig. 2).

Let us analyze now the role of the eigenvalues and eigenvectors of the matrix \mathbf{G} of a Lorentz metric g in the standard chart of \mathbb{R}^3 . This is well known from elementary algebra, but we shall give some details for the Lorentz metric.

Because \mathbf{G} is a symmetric matrix, it can be diagonalized, leading to a matrix $\mathbf{G}' = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. This transformation from \mathbf{G} to \mathbf{G}' is just a change of basis, and thus \mathbf{G}' may be expressed as $\mathbf{G}' = \mathbf{A}^{-1} \mathbf{G} \mathbf{A}$, where \mathbf{A} is the matrix changing between the orthonormal canonical basis B_x and a new one B_x' , which

is also orthonormal (and thus $\mathbf{A}^{-1} = \mathbf{A}^t$). The new basis B_x' is formed by three eigenvectors of \mathbf{G} . The columns of \mathbf{A} are the components of these eigenvectors in the canonical basis B_x , and the elements $\lambda_1, \lambda_2, \lambda_3$ of the diagonal of \mathbf{G}' are the eigenvalues of \mathbf{G} associated with the corresponding eigenvectors. Because g is Lorentzian, the signature is $(-, +, +)$; thus only one eigenvalue is negative, and the other two are positive. There is no loss of generality if we assume that $\lambda_1 < 0$ and $|\mathbf{A}| = 1$ (changing the order of the columns of \mathbf{A} if necessary). Thus the basis B_x' in which \mathbf{G}' is a diagonal matrix has a nice geometric interpretation, since it can be obtained from the canonical basis B_x by a rotation.

Now the study of the cone structure in the basis B_x' is trivial. $B_x' = \{\mathbf{J}, \mathbf{U}, \mathbf{V}\}$, where \mathbf{J}, \mathbf{U} , and \mathbf{V} are the eigenvectors of \mathbf{G} associated with λ_1, λ_2 , and λ_3 , respectively. Let $\mathbf{Y} \neq 0$ be a lightlike vector in $T_x M$. The components of \mathbf{Y} in the basis B_x' are $\mathbf{Y} = (Y_1, Y_2, Y_3)$. The equation of the lightlike vectors in this basis is $\mathbf{Y}'\mathbf{G}'\mathbf{Y} = 0$; i.e.,

$$\lambda_1 Y_1^2 + \lambda_2 Y_2^2 + \lambda_3 Y_3^2 = 0, \quad (1)$$

which is a cone whose axis is parallel to the vector \mathbf{J} (remember that $\lambda_1 < 0$, and $\lambda_2, \lambda_3 > 0$). The intersection of the cone and the plane $Y_1 = 1$ is the ellipse

$$\lambda_2 Y_2^2 + \lambda_3 Y_3^2 = -\lambda_1. \quad (2)$$

The ellipse axes are parallel to the vectors \mathbf{U} and \mathbf{V} (see Fig. 3).

Summarizing,

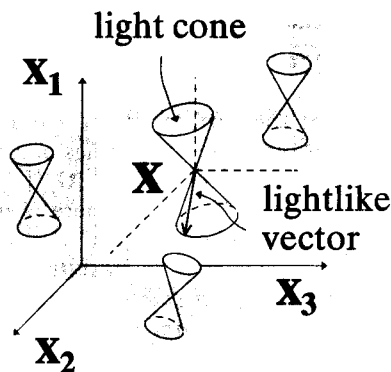


Fig. 2. The light cone at X is formed by the set of lightlike vectors of $T_x M$.

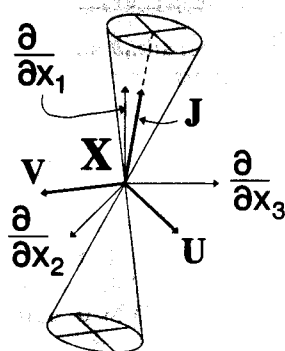


Fig. 3. The basis $B_x' = \{\mathbf{J}, \mathbf{U}, \mathbf{V}\}$ defines the three planes P_1, P_2 , and P_3 of symmetry of the elliptic light cone.

1. The light cone in $T_x M$ is elliptic, $\forall X \in M$.
2. The direction of the elliptic cone axis is the direction of \mathbf{J} , the eigenvector of \mathbf{G} associated with the negative eigenvalue λ_1 of \mathbf{G} .
3. The directions of the principal axis of the elliptic cone are the directions of \mathbf{U} and \mathbf{V} , the eigenvectors of \mathbf{G} associated with the positive eigenvalues of \mathbf{G} .

3. OPTICAL CONDITION

We want to describe an elliptic bundle by means of a Lorentz metric on a suitable open set M of \mathbb{R}^3 . We shall require that the edge rays be the lightlike geodesics of the Lorentz metric. We shall consider media with constant refractive index, and thus the ray trajectories will be straight lines. This imposes the main restriction in our formulation, which will be called the optical condition: *every lightlike geodesic for the Lorentz metric must be a straight line.*

Nevertheless, we shall use another more restrictive statement: *every lightlike geodesic of the Lorentz metric must be a geodesic of the Euclidean metric.* This restriction allows us to formulate the optical condition through a system of differential equations. The difference between the optical condition and the restricted optical condition is only that the parameterization of the straight lines in the restricted statement is not arbitrary.¹⁰

We can express the restricted optical condition as follows: If γ is a C^2 curve in M such that $\dot{\gamma}(t) \neq 0, \forall t \in \mathbb{R}$, then, for $k \in \{1, 2, 3\}$,

$$\left. \begin{aligned} \dot{\gamma}^k(t) + [\dot{\gamma}(t)]^l \Gamma^k_l[\gamma(t)] \dot{\gamma}(t) &= 0 \\ [\dot{\gamma}(t)]^l \mathbf{G}[\gamma(t)] \dot{\gamma}(t) &= 0 \end{aligned} \right\} \Rightarrow \dot{\gamma}^k(t) = 0. \quad (3)$$

The upper-left equation establishes that γ is a geodesic curve in Lorentz geometry, the lower-left one implies that γ is a lightlike curve, and the equation on the right-hand side is the condition for γ to be a Euclidean geodesic curve.

Γ^k is a matrix formed by the elements Γ^k_{ij} , which are the Christoffel symbols of the Lorentz metric.¹⁰ The expression of these symbols using Einstein's notation is

$$\Gamma^k_{ij} = 1/2 g^{km} (g_{m,i} + g_{m,j} - g_{i,j,m}), \quad (4)$$

where $\mathbf{G} = (g_{ij})$; $\mathbf{G}^{-1} = (g^{ij})$; and the subscript after the comma denotes a partial derivative with respect to the variable x_1, x_2 , or x_3 (the one with coincident subscript).

The restricted optical condition [Eq. (3)] is fulfilled if

$$\Gamma^k = f^k \mathbf{G}, \quad k \in \{1, 2, 3\}, \quad (5)$$

in the standard chart of M , where f^k are arbitrary functions on M . The converse is also true, because relation (3) leads to four coincident quadrics, and thus the coefficient matrices must be proportional. Equations (5) will be called restricted optical equations.

Using the expressions for the Christoffel symbols Γ^k_{ij} , we can write the restricted optical equations, in the standard chart of M , as

$$g_{im,j} = g_{mk} f^k g_{ij} + g_{ik} f^k g_{jm}, \quad i, j, m \in \{1, 2, 3\}. \quad (6)$$

Solving this system of equations (see Appendix A), we can show that the inverse matrix \mathbf{G}^{-1} is

$$\begin{bmatrix} -ax_1^2 - 2b_1x_1 + d & -ax_1x_2 - b_2x_1 - b_1x_2 + k & -ax_1x_3 - b_3x_1 - b_1x_3 + l \\ -ax_1x_2 - b_2x_1 - b_1x_2 + k & -ax_2^2 - 2b_2x_2 + m & -ax_2x_3 - b_3x_2 - b_2x_3 + n \\ -ax_1x_3 - b_3x_1 - b_1x_3 + l & -ax_2x_3 - b_3x_2 - b_2x_3 + n & -ax_3^2 - 2b_3x_3 + p \end{bmatrix}, \quad (7)$$

where $a, b_1, b_2, b_3, d, k, l, m, n, p \in \mathbb{R}$.

4. REMAINING CONDITIONS

There are other conditions that we have to impose on the matrix \mathbf{G} in order to describe the elliptic bundles with rotational symmetry.

A. Symmetry about the x_1 Axis

Let (M, x_1, x_2, x_3) be the standard chart such that the x_1 axis is the vertical axis of \mathbb{R}^3 . We want the bundle to have a rotational symmetry about this axis. In Appendix B it is proved that this condition is fulfilled if and only if $b_2 = b_3 = k = l = n = 0$ and $p = m$. If we apply this result it is easy to get

$$\mathbf{G} = \frac{1}{|\mathbf{G}^{-1}|} \begin{bmatrix} -am(x_2^2 + x_3^2) + m^2 & (ax_1 + b)mx_2 & (ax_1 + b)mx_3 \\ (ax_1 + b)mx_2 & -(ax_1^2 + 2bx_1 - d)m - (ad + b^2)x_3^2 & (ad + b^2)x_2x_3 \\ (ax_1 + b)mx_3 & (ad + b^2)x_2x_3 & -(ax_1^2 + 2bx_1 - d)m - (ad + b^2)x_2^2 \end{bmatrix}, \quad (8)$$

where b_1 has been replaced by b for simplicity and

$$|\mathbf{G}^{-1}| = -(ax_1^2 + 2bx_1 - d)m^2 - (ad + b^2)m(x_2^2 + x_3^2). \quad (9)$$

Now consider the matrix \mathbf{G}^{-1} restricted to the plane $x_3 = 0$:

$$\mathbf{G}^{-1}|_{x_3=0} = \begin{bmatrix} -ax_1^2 - 2bx_1 + d & -x_2(ax_1 + b) & -x_3(ax_1 + b) \\ -x_2(ax_1 + b) & -ax_2^2 + m & -ax_2x_3 \\ -x_3(ax_1 + b) & -ax_2x_3 & -ax_3^2 + m \end{bmatrix}_{x_3=0} = \begin{bmatrix} -ax_1^2 - 2bx_1 + d & -x_2(ax_1 + b) & 0 \\ -x_2(ax_1 + b) & -ax_2^2 + m & 0 \\ 0 & 0 & m \end{bmatrix}. \quad (10)$$

Note that if $\lambda_1, \lambda_2,$ and λ_3 are the eigenvalues of \mathbf{G} and $\mathbf{J}, \mathbf{U},$ and \mathbf{V} are their corresponding eigenvectors, then $\mathbf{J}, \mathbf{U},$ and \mathbf{V} are also eigenvectors of \mathbf{G}^{-1} with eigenvalues $\lambda_1^{-1}, \lambda_2^{-1},$ and λ_3^{-1} , respectively.

Observe that m is an eigenvalue of \mathbf{G}^{-1} (i.e., m satisfies $|\mathbf{G}^{-1} - m\mathbf{I}| = 0$, where \mathbf{I} is the unit matrix), and therefore $m \neq 0$ because of the Lorentz-metric signature. It is easy to verify that the vector $W = (0, 0, 1)$ is an eigenvector of \mathbf{G}^{-1} associated with the eigenvalue m (i.e., $\mathbf{G}^{-1}W = mW$). This implies that the other two eigenvectors of basis B_x' are contained in the plane $x_3 = 0$, and thus the light cone is symmetric with respect to the plane $x_3 = 0$. Because of the rotational symmetry, it is concluded that all the preceding results of this paragraph are also valid at any meridian plane (any plane containing the symmetry axis x_1). In particular, we can affirm that any light cone is symmetric with respect to the meridian plane containing the cone vertex. This result will be necessary for the concentrator design (Section 7), which is fortunate

because it is not true that any bundle with revolution symmetry fulfills it.

B. Signature

Observe that during the process of solving the restricted optical equations, we use the regular and the C^2 features, but not the signature, of the matrix \mathbf{G} . The ten-real-parameter family of matrices [Eq. (7)] includes examples of all types of signature. In the solutions obtained in Section 5 we shall impose the condition that \mathbf{G} have signature $(-, +, +)$ in order to be a Lorentz metric.

5. RESULTS

It can be proved (see Appendix C) that all the edge rays of

the bundle are tangent to the surface $|\mathbf{G}^{-1}| = 0$ and that all the tangents to this surface passing through a point of M are edge rays of the bundle. Therefore the equation of this surface represents the whole elliptic bundle. From Eq. (9) we get

$$|\mathbf{G}^{-1}| = -(ax_1^2 + 2bx_1 - d)m^2 - (ad + b^2)m(x_2^2 + x_3^2) = 0, \quad (11)$$

which implies that the surfaces $|\mathbf{G}^{-1}| = 0$ are quadrics. We have classified the different bundles according to the kind of quadric of its corresponding $|\mathbf{G}^{-1}| = 0$ surface. Parameter combinations for which there is no region of the space where the signature of \mathbf{G} is $(-, +, +)$ have been rejected. Table 1 summarizes the results. In case 4 the expression $b^2 + ad - am$ is positive, and in case 5 it is negative. When $b^2 + ad - am = 0$ the surface $|\mathbf{G}^{-1}| =$

Table 1. Classification of the Elliptic Bundles

Case	Parameter Values			Surface $ \mathbf{G}^{-1} = 0$
	a	m	$ad + b^2$	
1	> 0	> 0	0	Disk
2	0	> 0	> 0	Paraboloid
3	< 0	> 0	> 0	Two-sheet hyperboloid
4	> 0	> 0	> 0	Ellipsoid (minor axis on x_1)
5	> 0	> 0	> 0	Ellipsoid (major axis on x_1)
6	> 0	> 0	< 0	One-sheet hyperboloid
7	< 0	< 0	< 0	One-sheet hyperboloid

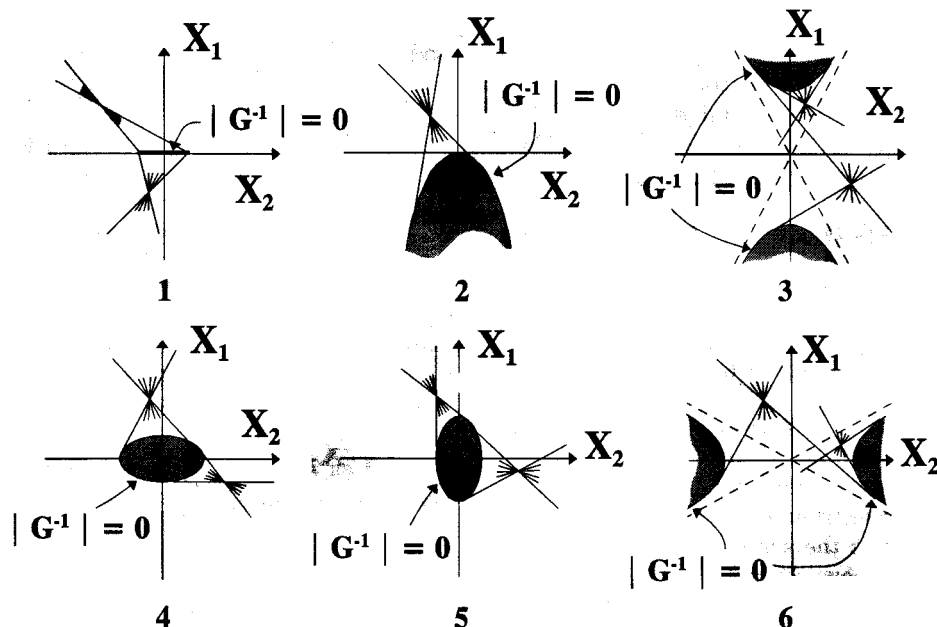


Fig. 4. Elliptic bundles 1–6. The points of the shaded regions do not belong to M .

0 is a sphere. Figure 4 shows the cross section of the surfaces $|\mathbf{G}^{-1}| = 0$ in cases 1–6. The surface $|\mathbf{G}^{-1}| = 0$ in case 1 has a special treatment (see Appendix C). The shaded regions in this figure are the regions $|\mathbf{G}^{-1}| > 0$; thus regions M are the complementary ones. Figure 5 shows the surface $|\mathbf{G}^{-1}| = 0$ for case 7.

In cases 1–5 the rays of the elliptic bundle at an arbitrary point P of M are those that connect P and any point of the surface $|\mathbf{G}^{-1}| = 0$. In cases 6 and 7 the rays of the elliptic bundle are those that pass through P and do not cross any point of the surface $|\mathbf{G}^{-1}| = 0$.

The surface $|\mathbf{G}^{-1}| = 0$ is the same for cases 6 and 7, although the combinations of parameters of Table 1 are not the same. This surface is a one-sheet hyperboloid of revolution that divides the space into two sets: one containing the x_1 axis (M_6) and another one not containing it (M_7). The region M ($|\mathbf{G}^{-1}| < 0$) is different for each of these two cases. In case 6 the region M is M_6 , and in case 7, M is region M_7 . The elliptic bundle is obviously different in these two cases. In all other cases except case 7, the vector \mathbf{J} is contained in a meridian plane. In case 7, \mathbf{J} is perpendicular to the meridian plane.

6. CONCENTRATOR DESIGN

As mentioned in Section 1, the reflector surface with rotational symmetry that contains the flow lines of the bundle does not disturb the bundle; i.e., the reflector exchanges ray trajectories among rays of the bundle, but the cones formed by the rays of the bundle remain the same. The reflector is assumed to be mirrored on both faces. If only one face is reflective, then the bundle will remain the same in one region of M .

The first step in calculating the flow lines is to get the negative eigenvalue λ_1 of \mathbf{G} . The vector field \mathbf{J} is then calculated from λ_1 , giving the equation $\mathbf{G}\mathbf{J} = \lambda_1\mathbf{J}$. The flow lines are the integral curves of \mathbf{J} . Let us explain the procedure with a simple case:

Set $b = 0$, $a = d = m = 1$ (a particular case of 4 and 5 of Fig. 4). The matrix \mathbf{G} is

$$\mathbf{G} = \frac{1}{|\mathbf{G}^{-1}|} \times \begin{bmatrix} 1 - (x_2^2 + x_3^2) & x_1x_2 & x_1x_3 \\ x_1x_2 & 1 - (x_1^2 + x_3^2) & x_2x_3 \\ x_1x_3 & x_2x_3 & 1 - (x_1^2 + x_2^2) \end{bmatrix}, \quad (12)$$

where $|\mathbf{G}^{-1}| = 1 - (x_1^2 + x_2^2 + x_3^2)$ and M is determined by the condition

$$|\mathbf{G}^{-1}| = 1 - (x_1^2 + x_2^2 + x_3^2) < 0. \quad (13)$$

The eigenvalues of \mathbf{G} are calculated from the equation $|\mathbf{G} - \lambda\mathbf{I}| = 0$ (\mathbf{I} is the unit matrix). Their values are

$$\lambda_1 = [1 - (x_1^2 + x_2^2 + x_3^2)]^{-1}, \quad \lambda_2 = \lambda_3 = 1, \quad (14)$$

which shows that this concentrator has circular cones [see Eq. (2)].

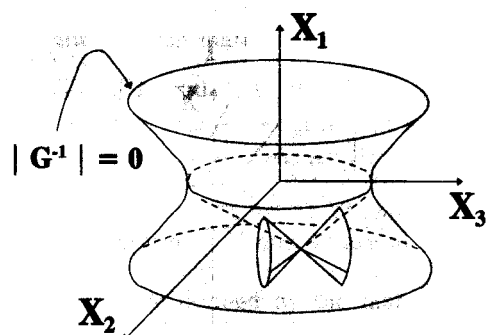


Fig. 5. Elliptic bundle 7. The 3D representation is needed because there are no meridian rays in this bundle. M is the region bounded by the hyperboloid not containing the x_1 axis.

The eigenvector \mathbf{J} is obtained from the equation $\mathbf{G}\mathbf{J} = \lambda_1\mathbf{J}$. The modulus of \mathbf{J} is not fixed by this equation, but this is irrelevant for the calculation of the integral curves, which is our objective. We take \mathbf{J} as $\mathbf{J} = (x_1, x_2, x_3)$ (it is easy to verify that this \mathbf{J} satisfies $\mathbf{G}\mathbf{J} = \lambda_1\mathbf{J}$). The integral curve γ of \mathbf{J} is trivial:

$$\gamma(t) = (c_1 e^t, c_2 e^t, c_3 e^t). \quad (15)$$

Therefore the flow lines are straight lines from the origin. The surface $|\mathbf{G}^{-1}| = 0$ is the sphere

$$1 - (x_1^2 + x_2^2 + x_3^2) = 0. \quad (16)$$

The reflector with rotational symmetry that contains the flow lines has a cone shape. This reflector together with the sphere $|\mathbf{G}^{-1}| = 0$ defines the well-known cone concentrator (CC).

The design procedure is the same for all other cases, although it is more complex. Figure 6 shows two symmetric flow lines for each bundle 1–6 (see Fig. 4), and Fig. 7 shows some flow lines for bundle 7. Table 2 describes the flow lines for all the cases. In cases 1–6 the flow lines are contained in meridian planes and are confocal with the corresponding surfaces $|\mathbf{G}^{-1}| = 0$. It is interesting to note that the flow lines of the bundle formed by the rays intercepting a given ellipsoid are coincident with the flow lines of the bundle formed by the rays intercepting any other ellipsoid that is confocal with the first one (the flow lines in these cases are confocal hyperbolas).

Let us consider one case from Fig. 6, for instance, case 2. The reflector surface (i.e., the surface of revolution whose cross section coincides with the two flow lines appearing in Fig. 6) divides the paraboloid $|\mathbf{G}^{-1}| = 0$ into two regions. The upper region is called the receiver, and the whole paraboloid $|\mathbf{G}^{-1}| = 0$ is the virtual entry aperture. Any ray departing from a point of M inside the reflector and directed toward an arbitrary point of

the virtual entry aperture is a ray of the elliptic bundle, and thus it will reach the receiver directly or after reflections. Also note that any ray reaching any point of

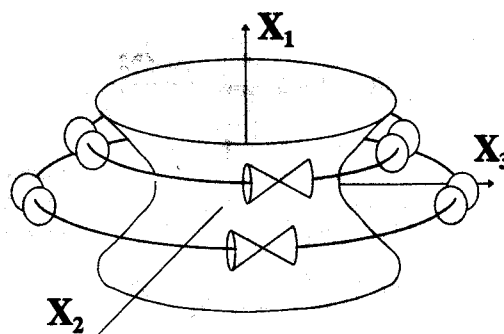


Fig. 7. Flow lines of elliptic bundle 7. These are circumferences contained in planes parallel to the plane $x_1 = 0$ and centered on the x_1 axis.

Table 2. Flow Lines of the Elliptic Bundles

Case	Surface $ \mathbf{G}^{-1} = 0$	Flow Lines
1	Disk	Hyperbolas (axis on plane $x_1 = 0$)
2	Paraboloid	Parabolas (axis on x_1)
3	Two-sheet hyperboloid	Ellipses (major axis on x_1)
4	Ellipsoid (minor axis on x_1)	Hyperbolas (axis on plane $x_1 = 0$)
5	Ellipsoid (major axis on x_1)	Hyperbolas (axis on x_1)
6	One-sheet hyperboloid	Hyperbolas (axis on plane $x_1 = 0$)
7	One-sheet hyperboloid	Circumferences

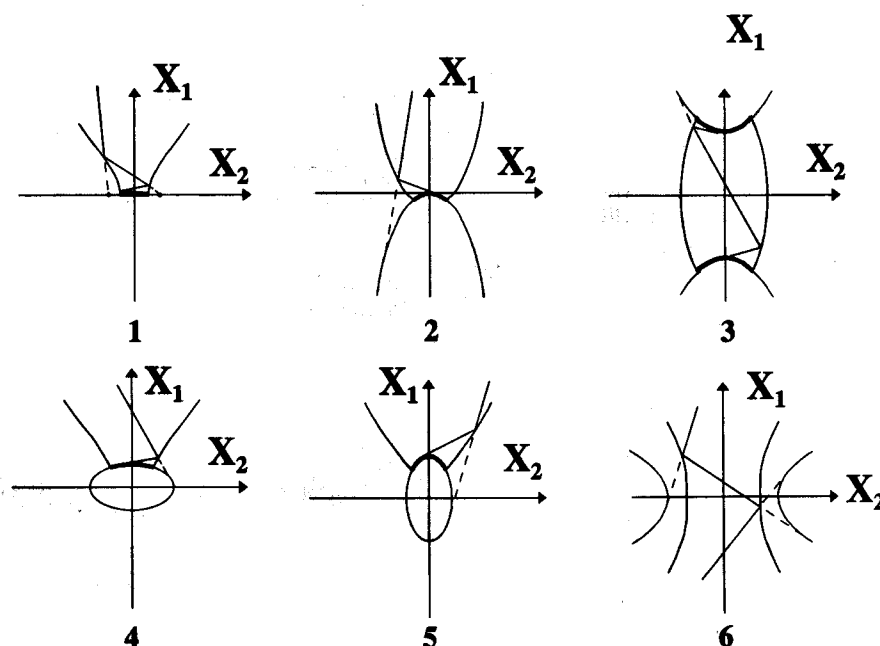


Fig. 6. Cross section of the concentrators derived from elliptic bundles 1–6. Case 1 corresponds to the FLC, and the intermediate case between cases 4 and 5 (in which $|\mathbf{G}^{-1}| = 0$ is a sphere and the flow lines are straight lines) corresponds to the CC.

the receiver is a ray of the elliptic bundle (the receiver is illuminated isotropically by the elliptic bundle). The preceding considerations apply for cases 1–5. Observe that in case 3 the receiver and the virtual entry aperture have two branches. Case 1 is easily recognized as the one corresponding to the FLC, and the CC is the intermediate case between cases 4 and 5.

As we have said in Section 6, the elliptic bundle in cases 6 and 7 does not intercept the hyperboloid $|\mathbf{G}^{-1}| = 0$. Nevertheless, we can also talk of receiver and virtual entry aperture if we consider the complementary bundle, i.e., the bundle of rays that do not belong to the elliptic bundle. These rays' trajectories are spacelike (if we use Lorentz geometry terminology), whereas the rays of the elliptic bundle are timelike. The complementary bundle obviously has the same symmetry properties as the elliptic one. This complementary bundle exists in all the cases, and, in the case of the FLC, it has been used for the design of light-confining cavities in photovoltaic applications.¹¹

7. CONCLUSION

We have reformulated the problem of 3D ideal concentrator design in terms of Lorentz geometry. With this approach we have obtained a new family of ideal 3D concentrators. This family includes the flow-line concentrator and the cone concentrator, two of the few 3D ideal concentrators known before this work.

The application of Lorentz geometry in the formulation of this problem seems quite natural, and many suggestions and new questions arise, of which we want to point out one: are there Lorentz metrics that satisfy the general optical condition and do not fulfill the restricted optical condition that we have used in our analysis?

The new ideal concentrators seem to be easy to manufacture; they use a single reflector. Nevertheless, their practical application is still to be shown. From the theoretical point of view, their existence is itself important.

APPENDIX A: SOLUTION OF THE RESTRICTED OPTICAL EQUATIONS

Let us define the matrices

$$\mathbf{G} = (g_{ij}), \quad \mathbf{f} = \begin{bmatrix} f^1 \\ f^2 \\ f^3 \end{bmatrix},$$

$$d\mathbf{x} = \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}, \quad \mathbf{H} = d\mathbf{x}\mathbf{f}^t + \mathbf{f}d\mathbf{x}^t. \quad (\text{A1})$$

Matrix \mathbf{G} may be considered a map on M with values on $\mathbf{GL}(3, \mathbb{R})$, and \mathbf{H} as a one-form on M with values on the vector space of the 3×3 real matrices $\mathbf{M}(3, \mathbb{R})$. Then it is straightforward to see that system (6) is equivalent to the matrix equation

$$d\mathbf{G} = \mathbf{G}\mathbf{H}\mathbf{G}, \quad (\text{A2})$$

which is a Pfaff system. A standard application of the Frobenius theorem gives the integrability conditions of system (A2).¹² These conditions can be obtained as follows: assume that the metric is C^2 and that the func-

tional coefficients f^k are C^1 . Applying the exterior differential in both sides of the system gives

$$0 = d^2\mathbf{G} = d\mathbf{G} \wedge \mathbf{H}\mathbf{G} + \mathbf{G}d\mathbf{H}\mathbf{G} - \mathbf{G}\mathbf{H} \wedge d\mathbf{G}. \quad (\text{A3})$$

Now if we use system (A2),

$$0 = \mathbf{G}\mathbf{H}\mathbf{G} \wedge \mathbf{H}\mathbf{G} + \mathbf{G}d\mathbf{H}\mathbf{G} - \mathbf{G}\mathbf{H} \wedge \mathbf{G}\mathbf{H}\mathbf{G} = \mathbf{G}d\mathbf{H}\mathbf{G}, \quad (\text{A4})$$

and \mathbf{G} being regular, Eq. (A4) is equivalent to $d\mathbf{H} = 0$; that is,

$$d\mathbf{H} = -d\mathbf{x} \wedge d\mathbf{f}^t + d\mathbf{f} \wedge d\mathbf{x}^t = 0, \quad (\text{A5})$$

and by elemental properties of exterior algebra this is equivalent to the next conditions on the functions f^k :

$$\begin{aligned} f^1 &= ax_1 + b_1, \\ f^2 &= ax_2 + b_2, \\ f^3 &= ax_3 + b_3, \end{aligned} \quad (\text{A6})$$

where a, b_1, b_2 , and $b_3 \in \mathbb{R}$. These are the integrability conditions of the system. To get the solutions notice that if \mathbf{I} is the identity matrix, then

$$\mathbf{I} = \mathbf{G}^{-1}\mathbf{G} \Rightarrow 0 = d\mathbf{I} = d\mathbf{G}^{-1}\mathbf{G} + \mathbf{G}^{-1}d\mathbf{G}. \quad (\text{A7})$$

Thus we obtain $\mathbf{G}^{-1}d\mathbf{G} = -d\mathbf{G}^{-1}\mathbf{G}$, and using this in system (A2) we get

$$d\mathbf{G}^{-1} = -\mathbf{H}. \quad (\text{A8})$$

Finally, if we use expressions (A6) for f^k , identify the coefficient in Eq. (A8), and integrate, it is straightforward to get expression (7) for \mathbf{G}^{-1} .

The inverse of this matrix is the general solution of the restricted optical equations on a suitable open set M of \mathbb{R}^3 , with the assumption that the metric is C^2 and that the functional coefficients f^k are C^1 . By inspection of \mathbf{G}^{-1} and Eqs. (A6), we see that with those assumptions the metric, as well as the functions f^k , is C^∞ .

APPENDIX B: ROTATIONAL SYMMETRY CONDITION ON \mathbf{G}

Let X be a point of M and Y a vector of T_xM . If we rotate X an angle α around the x_1 axis, we get the point $X' = \mathbf{F}X$ where \mathbf{F} is the rotation matrix

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix}. \quad (\text{B1})$$

The vector Y becomes the vector Y' (which belongs to $T_{x'}M$) after the rotation. Y and Y' are also related by $Y' = \mathbf{F}Y$.

The elliptic bundle defined by the matrix \mathbf{G} has rotational symmetry when the following condition is fulfilled for any $X \in M$, $Y \in T_xM$ and for any angle α : If $Y^t\mathbf{G}(X)Y = 0$, then $Y'^t\mathbf{G}(X')Y' = 0$. Using the rotation relationships, we get

$$Y'G(X)Y = 0 \Rightarrow Y'F'G(FX)FY = 0. \quad (B2)$$

For given values of X and α , condition (B2) means that the light cone [left-hand side of condition (B2)] must be contained in the surface defined by the right-hand side of condition (B2), which is also a quadric surface. Therefore this surface must coincide with the light cone; i.e., their corresponding matrices must be proportional:

$$G(X) = h(X)F'G(FX)F, \quad (B3)$$

where $h: M \rightarrow \mathbb{R}$ is an arbitrary function.

Taking the inverse matrix in Eq. (B3) and noting that $F^t = F^{-1}$, we obtain

$$h(X)G^{-1}(X) = F'G^{-1}(FX)F. \quad (B4)$$

Expanding this equation and using expression (7) for G^{-1} , we get $h(X) = 1$ and the following conditions for the parameters in the matrix G^{-1} :

$$b_2 = b_3 = k = l = n = 0, \quad p = m. \quad (B5)$$

Conditions (B5) are fulfilled if and only if Eq. (B4) is accomplished for any point X and for any value of α .

APPENDIX C: SURFACE $|G^{-1}| = 0$

Owing to the rotational symmetry, there is no loss of generality if we take a point P of M contained in the plane $x_3 = 0$. Let $\gamma(t)$ be the trajectory of a ray passing through the point $P = (p_1, p_2, 0)$, with the direction $Y = (Y_1, Y_2, Y_3)$:

$$\gamma(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ 0 \end{bmatrix} + t \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}. \quad (C1)$$

The intersection of this ray trajectory with the quadric $|G^{-1}| = 0$ [given by Eq. (9)] leads to the second-order equation in t :

$$\begin{aligned} & [am^2Y_1^2 + (ad + b^2)m(Y_2^2 + Y_3^2)]t^2 \\ & + 2[am^2p_1Y_1 + bm^2Y_1 + (ad + b^2)mp_2Y_2]t \\ & + [m^2(ap_1^2 + 2bp_1 - d) + (ad + b^2)mp_2^2] = 0. \end{aligned} \quad (C2)$$

The discriminant Δ of Eq. (C2) may be written, after some calculations, as

$$\Delta = m^2(ad + b^2)|G^{-1}(P)|[Y'G(P)Y]. \quad (C3)$$

Note that

$$\begin{aligned} Y'G(P)Y & = [(-amp_2^2m^2)Y_1^2 \\ & - (ap_1^2 + 2bp_1 - d)m(Y_2^2 + Y_3^2) \\ & + 2(ap_1 + b)mp_2Y_1Y_2 \\ & - (ad + b^2)p_2^2Y_3^2]/|G^{-1}(P)|. \end{aligned} \quad (C4)$$

Remember that $|G^{-1}| < 0$ in M and that $m \neq 0$ because m is an eigenvalue of G^{-1} . If we exclude the case $ad + b^2 = 0$, then, because of Eq. (C3), $\Delta = 0$ if and only if $Y'G(P)Y = 0$. Note that the ray is tangent to the surface $|G^{-1}| = 0$ if and only if $\Delta = 0$. The definition of a lightlike

vector Y is $Y'G(P)Y = 0$, and a ray with this direction is an edge ray. Therefore we conclude that the edge rays of the elliptic bundle are the tangents to the surface $|G^{-1}| = 0$ and that only these tangents are edge rays.

Consider now the case $ad + b^2 = 0$. If $a = 0$, then $b = 0$ and $|G^{-1}| = dm^2$ [see Eq. (9)]. G is in that case independent of the point of M , $G = \text{diag}(d^{-1}, m^{-1}, m^{-1})$. G represents a Lorentz metric only if $d < 0$ and $m > 0$. The elliptic bundle in this case forms the same cone (a circular cone) at any point of the space. It is thus a trivial case. If $a \neq 0$, then $|G^{-1}| = -m^2(ax_1 + b)^2a^{-1}$, and so G may represent a Lorentz metric if $a > 0$. The surface $|G^{-1}| = 0$ is the plane $x_1 = -ba^{-1}$.

Let $\gamma(t)$ be the straight line of Eq. (C1). This line cuts the plane $x_1 = -ba^{-1}$ when $t = -(ap_1 + b)a^{-1}Y_1^{-1}$. The coordinates x_2 and x_3 of the point of intersection of this ray with the plane $x_1 = -ba^{-1}$ fulfill [see Eq. (C1)]

$$x_2^2 + x_3^2 = \frac{m}{a} + \frac{Y'G(P)Y}{maY_1^2}. \quad (C5)$$

Note that $ma \neq 0$. The directions Y of the edge rays fulfill $Y'G(P)Y = 0$. It is easy to see that $Y_1 \neq 0$ for an edge ray [$Y \neq 0$ and $Y'G(P)Y = 0$ implies that $Y_1 \neq 0$ for a point of M]. Therefore we can conclude that when $ad + b^2 = 0$ and $a > 0$, the edge rays are the ones that pass through the circumference given by $x_2^2 + x_3^2 = m/a$ and $x_1 = -ba^{-1}$.

The preceding result can be included in the case $ad + b^2 \neq 0$ if we consider that the surface $|G^{-1}| = 0$ is the disk D bounded by the circumference mentioned in the above paragraph, when $ad + b^2 = 0$ and $a > 0$. With this definition we can say for all the cases that the edge rays are the tangents to the surface $|G^{-1}| = 0$ and only these tangents.

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