Construction of examples of b-completion

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Abstract

We survey some techniques for constructing b-completions. First, we give general results relating the b-completion of a product to the b-completion of its factors, the b-completion of a quotient, and the dependence of the b-completion on the bundle chosen to form it. Second, we relate (when possible) some b-completions to those of Riemannian metrics, and study invariant connections on homogenous spaces. Some particular examples are also considered.

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1 Introduction

The classical singularity theorems proved in the sixties by Hawking and Penrose, being as they were timelike incompleteness theorems of Lorentz manifolds, brought to the forefront the need to define those "ideal points" forming the "edge" of the space-time, which were the "limits" of the incomplete geodesics. These ideal points are the singularities of the space-time and its existence indicates the existence of particles with a begining or end according to its proper time. A reasonable choice was to define singularities as equivalence classes of incomplete timelike geodesics. Nevertheless, Geroch [7] showed the shortcomings of this definition, giving an example of a geodesically complete

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space-time admitting endless timelike curves with finite length and bounded acceleration. This example was singularity free with the mathematical definition, but did not fit the intended physical condition that all particles were "eternal".

A much more successful definition of singularity was given by Schmidt [14]. He defined the set of singularities and *endowed it with a topology.* This construction is very general in the sense that it depends only on the existence of a connection on a manifold and not on a metric. The example of Geroch has singularities with this definition. The Schmidt definition has interesting properties, but some drawbacks too. It is very hard to determine the singularity space in many examples, and when it has been obtained, as in the case of closed Friedmann spaces, its properties seem to contradict physical intuition (the Big Bang and the Big Crunch are the same singularity). See [10], [4], and [3]. At any rate, the b-completion stands out as a reference for future definitions, since endless finite length curves (measured via a parallel basis along the curve) are very important for *any* definition of singularity. In this paper we study different examples of b-completion (with the Schmidt definition) along with the tools required to construct them. Some of them are well known; others not so much; therefore we put more emphasis on them. A more complete version of this paper containing proofs are available through the autors.

2 The b-completion of some products and quotients

Let M be a connected m-dimensional manifold, $\pi : P \to M$ a connected subbundle of the frame bundle with structural group $G \subset GL(m, \mathbb{R})$. Let Φ be the fundamental form and λ a connection (form) on P. We define on P the Riemann metric

$$g^\lambda(X, Y) = \langle \Phi(X), \Phi(Y) \rangle + \langle \lambda(X), \lambda(Y) \rangle \,, \tag{1}$$

where $\langle \bullet, \bullet \rangle$ is the standard Euclidean product in $\mathbb{R}^m$ and $\langle A, B \rangle = tr(AB^t)$ ($\bullet^t$ denoting transposition) the standard Euclidean product in $\mathfrak{gl}(m, \mathbb{R})$. All is clear since $\mathfrak{g}$, the Lie algebra of G, is a subspace of $\mathfrak{gl}(m, \mathbb{R})$. Let $d = d^\lambda$ be the distance in P associated to g^λ and $(\hat{P}, \hat{d}) = (P, d)_c$ the Cauchy completion of the metric space (P, d). The right action of G on P can be extended to $\hat{P}$. The quotient space $\widehat{M} = \hat{P}/G$ is now the **b-completion** of (M, λ). The **singularity space** or **b-boundary** is $\partial M = \widehat{M} - M$. Since the distance $\hat{d}$ and the projection $\hat{\pi}$ are extensions of d and π, we simply write d and π.

If (M, g) is a space-time, P is usually the bundle of all positive(ly oriented) frames $(u_1, u_2, \ldots, u_m)$ such that u_1 is timelike future, G is the connected component of the Lorentz group and λ the connection form of the Levi-Civita connection of g. If (M,h) is a Riemann manifold and similar choices are made, $\widehat{M}$ is homeomorphic to the Cauchy completion of (M,δ) , where δ is the distance in M attached to g [14]. In any case we refer to the b-completion of g or h. Although we will not use it, we cite an equivalent definition of the b-completion due to Sachs [13], *only for space-times* (M,g) . In the unit bundle $UM = \{X \in TM \mid X \text{ unit timelike future}\}$ a Riemann metric g^D, depending on g, is given and the g^D-distance d is used to get the Cauchy completion of (UM,d) . Then an equivalence relation R in $(UM,d)_c$ is defined and the quotient space is shown to be homeomorphic to $\widehat{M}$.

It is not hard to prove that $\pi : \hat{P} \to \widehat{M}$ is open, $\widehat{M}$ is arc connected, M is dense in $\widehat{M}$, and the manifold topology of M is the one inherited as a subspace of $\widehat{M}$. Nevertheless $\widehat{M}$ fails to be locally compact or Hausdorff. This happens in important examples of space-times and adds to the difficulties of mathematical handling those of physical interpretation.

Quite often we require M to be inextendible. If not, there are simple examples of b-completion as we may see.

Proposition 2.1 *Let M be a connected manifold with $m > 1$ with a connection. Given $x_1, \ldots, x_n \in M$, let $N = M - \{x_1, \ldots, x_n\}$ with the induced connection. Then $\widehat{N}$ is homeomorphic to $\widehat{M}$ and $\partial N = \partial M \cup \{x_1, \ldots, x_n\}$.* $\square$

Singularities are closely related to b-incomplete curves [9]. Let $c : [a,b) \to M$ be a smooth curve and $C : [a,b) \to P$ a (non unique) horizontal lift of c. It happens that the finiteness of the length of C, measured with g^λ, depends only on c. Accordingly we say that c has **finite or infinite fibre length**.

Definition 2.2 *A smooth curve $c : [a,b) \to M$ is **extendible** if* $\lim_{t\to b} c(t)$ *exists in M, and is **inextendible** otherwise. We say that c is **b-incomplete** if it is inextendible and has finite fibre length.*

Incomplete geodesics are b-incomplete if $b < \infty$, since it is well known that the existence of $\lim_{t\to b} c(t) \in M$ implies that c is extendible as a geodesic to $[a, b+\varepsilon)$ for some $\varepsilon > 0$.

Theorem 2.3 *For any Cauchy sequence (u_n) in P with limit $\widehat{u} \in \widehat{P}$, there is a finite length horizontal curve $C : [0,1) \to P$ such that* $\lim_{t\to 1} C(t) = \widehat{u}$. $\square$

If $c : [0,1) \to M$ is b-incomplete, any horizontal lift C has finite length. This implies the existence of $\lim_{t\to 1} C(t) = \widehat{u}$, necessarily unique and in $\hat{P} - P$. Therefore, $\pi(\widehat{u})$ is a singularity and c converges to it. Nevertheless, since $\widehat{M}$ may be non-Haussdorf, c may converge to other points in $\widehat{M}$. To sum up, b-incomplete curves converge to singularities, but not uniquely (there is a limit set rather than a limit point). In any case, theorem 2.3 shows that any singularity is in the limit set of a curve: If $\widehat{x} = \pi(\widehat{u})$, and $c = \pi \circ C$, then c converges to $\widehat{x}$. However, we must stress that we have no information about a possible limit of c in M (recall $\widehat{M}$ is not Haussdorf). If all limits of c were in ∂M, c would be b-incomplete. Schmidt proved (see [15], [9]) that in space-times all singularities are limits of b-incomplete curves.

Definition 2.4 *A b-incomplete curve c* ***induces a singularity*** *$p \in \partial M$ if there is a horizontal lift C of c with limit in $\pi^{-1}(p)$. (Then all horizontal lifts have limit in $\pi^{-1}(p)$.)*

Lemma 2.5 *Let $C, D : [a, b) \to P$ be curves such that C is horizontal and $D(t) = C(t)s(t)$ for some $s : [a, b) \to G$. Then*

(1) Length$(s) \leq$ Length(D), where lengths in G are measured with the left invariant metric induced by the Euclidean product $\langle \bullet, \bullet \rangle$ already chosen in $\mathfrak{g}$.

(2) For some $k \in \mathbb{R}$ we have Length$(C) \leq k$ Length(D). $\square$

This lemma is used to prove the following theorem.

Theorem 2.6 *Let C, $D : [a, b) \to P$ be finite length curves having limit in $\hat{P}$ as $t \to b$. Then the following statements are equivalent*

(1) The limits of C and D belong to the same fibre $\pi^{-1}(p) \subset \hat{P}$, $p \in \widehat{M}$.

(2) There is an increasing sequence $t_n \to b$ and a family of horizontal curves $\Delta_n : [0, 1] \to P$, joining $C(t_n)$ with the fibre $\pi^{-1}(d(t_n))$, such that Length$(\Delta_n) \to 0$ and the sequence (s_n) in G defined by $\Delta_n(1)s_n = D(t_n)$ converges to some $s \in G$.

Conditions ***1*** *and* ***2*** *are a consequence of*

(3) There is an increasing sequence $t_n \to b$, a family of curves $\Gamma_n : [0, 1] \to P$ joining $C(t_n)$ to $D(t_n)$ and a constant $k \in \mathbb{R}$ such that

$$\int_0^1 \left|\lambda(\dot{\Gamma}_n(t))\right| dt \leq k \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad \int_0^1 \left|\Phi(\dot{\Gamma}_n(t))\right| dt \to 0. \quad (2)$$

If G is connected, conditions ***1,2*** *and* ***3*** *are equivalent.* $\square$

We remark that if C and D are horizontal lifts of b-incomplete curves c and d, condition **1** can be substituted by the condition that c and d induce the same singularity. As an application we give necessary conditions to hold if two b-incomplete curves induce the same singularity.

Theorem 2.7 *Let $\alpha_i : [0, 1) \to M$, $i = 1, 2$ be b-incomplete curves inducing the same singularity. If $J_i = \; Im\,(\alpha_i)$, suppose the closures in M of J_1 and J_2 are disjoint. Then, there is a b-incomplete curve $\beta : [0, 1) \to M$ inducing the same singularity as α_1 and α_2 and an increasing sequence (t_n) in $[0, 1)$ with* *limit* 1 *such that* $\beta(t_{2n}) \in J_1$ *and* $\beta(t_{2n+1}) \in J_2$.

Theorem 2.8 *Let* $\pi : P \to M$ *and* $\chi : Q \to M$ *be connected subbundles of the frame bundle* $L(M)$ *with structural groups* G *and* H *and connections* λ *and* σ. *Let* $F : P \to Q$ *be a morphism of principal bundles such that*

(1) F *is an injective immersion, inducing a difeomorphism* f *on* M *and the inclusion of* G *in* H.
(2) G *is a closed subgroup of* H.
(3) $F^*\sigma = \lambda$.

Then the b-completions $\widehat{M}_P$ *and* $\widehat{M}_Q$ *are homeomorphic.*

Corollary 2.9 *Let* (M, g) *be a semi-riemannian manifold,* P *a connected component of the orthonormal bundle* OM *and* Q *a connected component of the frame bundle* LM. *If they are endowed with the Levi-Civita connections, the completions of* M *are homeomorphic.* $\square$

Corollary 2.10 *Let* M *be a manifold with a connection,* P *a connected component in the holonomy bundle and* Q *a connected component in the frame bundle* $L(M)$. *If the holonomy group is closed, both b-completions of* M *are homeomorphic.* $\square$

Let (M, λ), (N, σ) be manifolds of dimension m, n with connections λ, σ. Let $L'(M \times N)$, $L'M$, and $L'N$ be connected components of the frame bundles and φ, π, χ the projections. Depending on manifold orientability, the structural groups will be the full $GL(k, \mathbb{R})$, $k = m, n, mn$ or connected components of them. They will be denoted by G, H, K. Let $\pi \times \chi : L'M \times L'N \to M$ be the product bundle, which is a principal bundle with structural group $G \times H$. In a natural way, $G \times H$ is a subgroup of K and $L'M \times L'N$ a subbundle of $L'(M \times N)$. If $p : L'M \times L'N \to L'M$, $q : L'M \times L'N \to L'N$ are the projections there is on $L'M \times L'N$ a connection $\lambda \times \sigma$ defined by $\lambda \times \sigma = p^*\lambda + q^*\sigma$. (Since G and H are subgroups of K, its Lie algebras are subalgebras of $\mathfrak{k}$, and $p^*\lambda, q^*\sigma$ are seen as $\mathfrak{k}$-valued forms). Finally $\lambda \times \sigma$ induces a connection ω in $L'(M \times N)$ with formula $\Psi^*\omega = \lambda \times \sigma$.

Theorem 2.11 *With the above notation the completion of* $(M \times N, \omega)$ *is homeomorphic to the product of the b-completions of* (M, λ) *and* (N, σ).

Corollary 2.12 *Let* (M, g) *and* (N, h) *be semi-riemannian manifolds, then the b-completion of* $(M \times N, g \times h)$ *is the product of the b-completions of* (M, g) *and* (N, h). $\square$

Some particular cases of the results derived from theorem 2.8 are in [6], and others appear in [2].

Let $\psi : M \to M$ be an affine map for a connection λ and Ψ its lift to LM; it is an isometry of (LM, d^λ) which can be extended to an isometry of $(LM, d^\lambda)_c$. In the quotient we get a homeomorphism $\widehat{\psi} : \widehat{M} \to \widehat{M}$; hence a group A of affine transformations of M acts also on $\widehat{M}$ and the orbit space $\widehat{M}/A$ can be considered.

If A acts discretely on M, then $N = M/A$ is a manifold and the projection $p : M \to N$ is smooth. Let $L'N$ be a connected component of the frame bundle LN. Then p induces a fibre space morphism $q : LM \to LN$ and there is a unique connection λ_0 in LN determined by $q^*\lambda_0 = \lambda$. We have [1]

Theorem 2.13 *If the action of A on $(LM)_c$ has closed orbits, the b-completion of (N, λ_0) is homeomorphic to $\widehat{M}/A$ with the quotient topology.* $\square$

3 Other examples

In the preceding section we developed theorems which allow us to get new examples from old ones. We give now concrete examples.

3.1 Riemann metrics and related cases

We said at the begining that if (M, h) is a Riemann manifold, the b-completion of the Levi-Civita connection is just the Cauchy completion of (M, δ), where δ is the distance associated to the metric. These cases are considered trivial in the sense that there is a general simple answer (although $(M, \delta)_c$ may be hard to determine). If a Lorentz manifold (M, g) had the same Levi-Civita connection ∇ as that of the Riemann metric h, we would know that the b-completion of g is the b-completion of h. This is clear, since the b-completion can be computed in the full frame bundle (not necessarily in the specific orthonormal bundle of g or h, see theorem 2.8) and the b-completion depends only on the connection.

Let g and h be a Lorentz and Riemann metric on M and $g^\flat, h^\flat : TM \to T^*M$ the induced isomorphisms. We have a field of automorphisms J defined by $h(X, Y) = g(JX, Y)$ for $X, Y \in \mathfrak{X}(M)$. The symmetry of g and h imply that J is self adjoint for both metrics.

Proposition 3.1 *If J is parallel for either metric, the b-completions of them coincide, and $\widehat{M}_g$ is homemorphic to the Cauchy completion of (M, δ). This is the case if the metrics are related by $h(Y, Z) = g(Y, Z) + 2g(X, Y)g(X, Z)$, where X is a unit timelike parallel field.*

We may then show the equivalence of another definition of singularity set (called the *Cauchy boundary* [8]) and the b-completion. Using the fact that X is parallel if and only if the connection form of g is reducible to the $O(m-1)$-structure of frames having X as the first vector, it can be proved that the Riemann metric associated to the space time given in [8] is just the metric h given above.

On the other hand, the b-completion of the example in the last section of [13], can be computed via a Riemann metric h constructed with the help of a parallel field X. It can also be computed with the corollary of theorem 2.11.

3.2 Reductive homogeneous spaces

We fix a connected Lie group P and a closed Lie subgroup G. The quotient space P/G is denoted by M, and the projection of P onto M by π. We have the corresponding Lie algebras $\mathfrak{p}$ and $\mathfrak{g}$. It is well known that P is a principal fibre bundle with base M, group G, and such that the right action of G onto P is the multiplication. Therefore, if $A \in \mathfrak{g} \subset \mathfrak{p}$, the fundamental field $A_{\#}$ in $\mathfrak{X}(P)$ associated to A is just A. This bundle has the important property that P acts on the left on P and (more important) on M by left multiplication: For $v \in P$ we have $\tau_v : M \to M$, $\tau_v(uG) = vuG$. We assume our homogeneus space to be **reductive.** This means there is a supplementary $\mathfrak{m}$ of $\mathfrak{g}$ in $\mathfrak{p}$ such that $Ad_s(\mathfrak{m}) \subset \mathfrak{m}$ for all $s \in G$. We have a homomorphism $\mathcal{A} : G \to GL(\mathfrak{m})$ by defining $\mathcal{A}(s) = \mathcal{A}_s = Ad(s)|_{\mathfrak{m}}$. The associated Lie algebra homomorphism is $X \to \mathbf{a}_X$, obtained by restriction of $ad(X) \in \mathfrak{gl}(\mathfrak{p})$ to $\mathfrak{m}$.

It is convenient to consider frames in M as isomorphisms $r : \mathfrak{m} \to T_xM$. Define $j : P \to LM \quad , \quad j(u)(Z) = \pi_{*u}(Z_u) \quad u \in P,\ Z \in \mathfrak{m}$.

Theorem 3.2 *The map $j : P \to LM$ is a principal fibre bundle homomorphism over the identity of M, with $\mathcal{A} : G \to GL(\mathfrak{m})$ as associated group homomorphism; i.e., $j(us) = j(u) \circ \mathcal{A}_s$ for $u \in P$, $s \in G$. For $v \in P$, $j \circ \tau_v = T_v \circ j$, where T_v is the lift of τ_v to LM.* $\Box$

Let $L'M$ be a connected component of LM. In a reductive homogeneous space $M = P/G$ an **invariant connection** is considered. This means by definition that the translations $\tau_v : M \to M$ are affine maps. Due to high symmetry, it is easier to get information about the b-completion. The study of these examples was started by the authors [2]. We outline the main results.

For each connection λ in LM define $\mu = j^*\lambda \in \Lambda^1(P, \mathfrak{gl}(\mathfrak{m}))$. If $X \in \mathfrak{g} \subset \mathfrak{p} \subset \mathfrak{X}(P)$, then $\mu(X)$ is the *constant* map $\mathbf{a}_X$. This shows that the information that μ may carry about λ is in the maps $\mu(Z)$, $Z \in \mathfrak{m}$. We also denote by $\mu : \mathfrak{m} \to C^\infty(P, \mathfrak{gl}(\mathfrak{m}))$ the induced linear map.

Theorem 3.3 *The connections λ on LM are in bijective correspondence with the linear maps $\mu : \mathfrak{m} \to C^\infty(P, \mathfrak{gl}(\mathfrak{m}))$ such that*

$$\mu(\mathcal{A}_s(Z))(us^{-1}) = \mathcal{A}_s \circ (\mu(Z)(u)) \circ \mathcal{A}_s^{-1} \quad , \quad \textit{for all}\, u \in P,\; s \in G,\; Z \in \mathfrak{m}.$$

The relation between λ and μ is $\mu(Z)(u) = j^*\lambda(Z(u))$. □

If λ is invariant, $\mu(Z) \in C^\infty(P, \mathfrak{gl}(\mathfrak{m}))$ is constant. We identify $\mu(Z)$ with its constant value in $\mathfrak{gl}(\mathfrak{m})$.

Corollary 3.4 *The invariant connections λ on LM are in bijective correspondence with the linear maps $\mu : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ such that*

$$\mu(\mathcal{A}_s(Z)) = \mathcal{A}_s \circ \mu(Z) \circ \mathcal{A}_s^{-1} \quad \textit{for all}\, s \in G,\; Z \in \mathfrak{m}$$

or the bilinear maps $\mu : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ such that

$$\mu(\mathcal{A}_s(X), \mathcal{A}_s(Y)) = \mathcal{A}_s\left(\mu(X, Y)\right) \quad \textit{for all}\, s \in G,\; X, Y \in \mathfrak{m}.$$

□

Theorem 3.5 *Let λ be an invariant connection λ on LM such that there is a connection ω in the G-principal bundle $\pi : P \to M$ with the property that*

$$\mu(X) = j^*\lambda(X) = \mathbf{a}_{\omega(X)} \quad \textit{for all}\, X \in \mathfrak{X}(P).$$

Then λ has no singularities. □

However, many invariant connections have singularities. Let $C_0 : \mathbb{R} \to P$ be given by $C_0(t) = u \exp(\varepsilon t Z)$, for some $Z \in \mathfrak{m}$, $u \in P$ and $\varepsilon = \pm 1$, and let $c : \mathbb{R} \to M$ be its projection on M; i.e., $c = \pi \circ C_0$. The horizontal lift C of c to $L'M$ such that $C(0) = j(u)$ is

$$C(t) = j(C_0(t)) EXP(-\varepsilon t \mu(Z)) \tag{3}$$

and its length measured by g^λ for $a \leq t \leq b$ is

$$\int_a^b |EXP(\varepsilon t \mu(Z))(Z)|\, dt, \tag{4}$$

where EXP is the exponential of an endomorphism.

Theorem 3.6 *Consider an invariant connection λ on LM given by $\mu : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$, and $Z \in \mathfrak{m}$ such that $\mu(Z)(Z) = kZ$ with $k \neq 0$. If $\varepsilon = \pm 1$ has opposite sign to k, the curve $c(t) = \pi(u \exp(\varepsilon t Z))$, $t \geq 0$, is b-incomplete and thus converges to a singularity.*

Invariant connections on Lie groups and homogeneous spaces are classified in [11] and [12].

3.3 An example of b-completion for a homogeneous space.

Let $P = (\mathbb{R}^m, +)$ and $G = 0$. We identify $P = M = \mathbb{R}^m$, $\mathfrak{p} = \mathfrak{m} = \mathbb{R}^m$, $\pi = id$, $u \exp(tZ) = u + tZ$, and $L'M = \mathbb{R}^m \times GL^+(m;\mathbb{R})$. The connection λ is associated to $\mu : \mathfrak{m} = \mathbb{R}^m \to \mathfrak{gl}(\mathfrak{m})$, $\mu(Z) = f(Z)I$, where $f : \mathbb{R}^m \to \mathbb{R}$ is $f(x^1, ..., x^m) = x^m$.

Let $I = [a,b]$, $c : I \to M$ a curve and $C : I \to LM = \mathbb{R}^m \times GL^+(m;\mathbb{R})$ its horizontal lift with initial condition $C(0) = (x, 1)$. Then

$$C(t) = \left(c(t), e^{-(f(c(t))-f(x))}\right) \in \mathbb{R}^m \times GL^+(m;\mathbb{R}), \tag{5}$$

$$Length(C) = \int_a^b e^{(f(c(t))-f(x))} \left|\dot{c}(t)\right| dt \geq \left| \left[e^{(f(c(t))-f(x))} \right]_a^b \right|,$$

since $\left|f(\dot{c}(t))\right| \leq \left|\dot{c}(t)\right|$.

We show that *every singularity is induced by a b-incomplete curve*. Let $p \in \partial M$. By theorem 2.3, there is a horizontal $C : I \to LM$ of finite length such that $\lim_{t\to b} C(t) \in \pi^{-1}(p) \subset \pi^{-1}(\partial M)$. If $c = \pi \circ C$, then $C(t) = \left(c(t), e^{-(f(c(t))-f(x))}\right)$. If c converged to $y \in M$, $\lim_{t\to b} C(t) = \left(y, e^{-(f(y)-f(x))}\right) \in LM$. Therefore c is b-incomplete and induces p.

Define $\gamma : [0, \infty) \to M = \mathbb{R}^m$, $\gamma(t) = (0, ..., 0, -t)$. Then $\Gamma(t) = (\gamma(t), e^t)$ is the horizontal lift of γ such that $\Gamma(0) = (0, 1)$. Clearly, $Length(\Gamma) = \int_0^\infty e^{-t} dt < \infty$; therefore γ is b-incomplete and induces $p^* \in \partial M$. We say that *there is a unique singularity*; i.e., $\partial M = \{p^*\}$. We should prove that any b-incomplete curve c and γ induce the same singularity. This is shown with theorem 2.6 using as curves Δ_n the horizontal lifts of curves joining $c(t)$ and $(0, ..., 0, c^m(t)) = \gamma(c^m(t))$.

Let $K \subset \mathbb{R}^m$ be a closed subset such that $\inf \{f(x) \mid x \in K\} > -\infty$. It can be proved that $\pi^{-1}(K)$ is closed in *the completion* $(L\mathbb{R}^m)_c$. Therefore $\widehat{M} - K = \widehat{\mathbb{R}^m} - K$ is an open neighbourhood of p^* and $\widehat{M}$ is Hausdorff. Moreover, M is open in $\widehat{M}$, but this information is not enough to know exactly the topology of $\widehat{M}$ except in the case $m = 1$.

Consider a *discrete* subgroup A of $\mathbb{R}^m$. Every $a \in A$ can be seen as a translation of $\mathbb{R}^m = M$. Translations are affine maps for λ; therefore theorem 2.13

is applicable to get the b-completion of the manifold $N = M/A$ with the connection λ_0 induced by λ. We get $\widehat{N} = (\mathbb{R}^m \cup \{p^*\})/A$ under the action of A on $\widehat{M} = \mathbb{R}^m \cup \{p^*\}$ given by

$$a \cdot x = a + x \text{ if } x \in \mathbb{R}^m \quad \text{and} \quad a \cdot p^* = p^*.$$

The topology of $\widehat{N}$ is the quotient topology. As we said, the topology of $\widehat{M}$ is not fully determined, but the sets $\{p^*\} \cup f^{-1}((-\infty, r))$ are open in $\widehat{M}$. It follows that if some $a \in A$ is not orthogonal to $(0, ..., 0, 1)$, then $\widehat{N}$ has a unique singularity q^*, and that q^* has a unique neighbourhood $\widehat{N}$. On the other hand, if $A \subset Ker(f)$ and $Ker(f)/A$ is compact, any neighbourhood V of the unique singularity q^* of $\hat{N} = \widehat{\mathbb{R}^m}/A$ contains one of the form

$$\left((\mathbb{S}^1)^{m-1} \times (-\infty, r)\right) \cup \{q^*\}.$$

The homogeneous spaces in this section generalize those given in [5] where $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z} = \mathbb{S}^1$ with a "constant" connection were considered. The b-completions obtained were either $\mathbb{R} \cup \{-\infty\}$ with intervals $[-\infty, r)$ as neighbourhoods of $p^* = -\infty$, or $\mathbb{S}^1 \cup \{q^*\}$ having q^* only one neighbourhood. The techniques of proof were different.

3.4 Other invariant connection in $\mathbb{R}^m$.

For each $x \in \mathbb{R}^m$, $m \geq 2$, let $\mu(x) \in \mathfrak{gl}(\mathbb{R}^m)$ denote the diagonal matrix A such that $A_i^i = x^i$. We consider the Lie group $P = (\mathbb{R}^m, +)$. The invariant connection on P is given by $\mu : \mathbb{R}^m \longrightarrow \mathfrak{gl}(\mathbb{R}^m)$ just defined. The curve $c : [0, a) \to \mathbb{R}^m$ has

$$C(t) = (c(t), \exp(-\mu(c(t) - c(0)))\, S)$$

as the horizontal lift such that $C(0) = (c(0), S)$ for $S \in GL(m, \mathbb{R})$. The length of C is

$$Length(C) = \int_0^a \left| S^{-1} \cdot \exp(\mu(c(t) - c(0)))\,(\dot{c}(t)) \right| dt.$$

Suppose $S = 1$. Then

$$Length(C) \geq \int_0^a \left| \dot{c}^i(t) \exp\left(c^i(t) - c^i(0)\right) \right| dt \geq \int_0^a \left| \frac{d}{dt} \exp\left(c^i(t) - c^i(0)\right) \right| dt.$$

This implies the existence of the limits $\lim_{t \to a} \exp(c^i(t))$, if $Length(C) < \infty$. Consider the curve $\alpha : [0, \infty) \to \mathbb{R}^m$, $\alpha(t) = tu$, where u is a *unit* vector with all $u^i \leq 0$. The length of the horizontal lift A such that $A(0) = (0, 1)$ is

$$Length(A) = \int_0^\infty \left| \left(e^{tu^1}, ..., e^{tu^m}\right) \left(u^1, ..., u^m\right) \right| dt \leq \sum_{u^i \neq 0} \int_0^\infty \left| u^i \right| e^{tu^i} dt < \infty.$$

The curve α is b-incomplete and induces a singularity $z = z(u)$.

Theorem 3.7 *The singularities z are different.*

3.5 The Friedmann spaces.

It is proved in [4] and [10] that the closed Friedmann space in dimension two has exactly one essential past singularity. This singularity has a single neighbourhood (the whole b-completion). Using this as stepping stone, these authors show that in dimension four there is at least one essential singularity with a single neighbourhood and at least one essential past and one essential future singularity are in fact the same singularity. It was conjectured in [10] that there was a unique essential past singularity. Other results on Schwarzschild spaces are also obtained. The definition given in [10] of **essential singularity** is specific for the cases considered and of a technical nature. Its purpose is to rule out singularities which are induced by "physically meaningless" curves.

The present authors [3] prove that 4-dimensional Friedmann closed space has only one singularity (not restricted to being essential) and this singularity has a unique neighbourhood. Briefly $\partial M = \{p\}$ and $\widehat{M}$ is the only neighbourhood of p. They also prove that the non-closed Friedmann four dimensional space has a unique essential past singularity. As for future essential singularities, it is shown that maximal non-spacelike future geodesics are not b-incomplete; hence, if these singularities existed, they would be of little physical importance. To sum up in a very simplified way: There is a Big Bang, only one Big Bang, and if there is a Big Crunch it will be the same as the Big Bang.

4 Final comments

The techniques shown to get the b-completion of a space, related to some other whose b-completion is known gives an important collection of examples, at least from the mathematical viewpoint. Unfortunately, the collection is very short if we look for physically interesting cases. The closed Friedmann space is the only case, and there is interesting information for non-closed Friedmann spaces, but Schwarzschild space has not been tackled. At present, everything seems to indicate that to compute the b-completion, a specific way has to be found for every case. The knowledge of the b-completion of the most important space-times and its physical interpretation (if any) are the more important steps to be taken.

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