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LORENTZ GEOMETRY TECHNIQUE IN NONIMAGING OPTICS.

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Abstract. Nonimaging optics is a field that study optimal concentration of light from a source distribution to a receiver. The relevant information is codified by a field of cones at each point of the concentrator, formed by those rays that we want to reach the receiver (perhaps after some reflections on the wall of the concentrator). This suggests that we can use Lorentz geometry to analyze the problem. We will establish a technique to design three dimensional ideal concentrators with arbitrary media which generalize a previous one for the homogeneous case.

1. **Introduction.** A nonimaging concentrator is a device that transfers a light radiation from an entry aperture to a receiver of less area such that the intensity of the radiation is higher at the receiver. Mathematically it can be determined by giving the set of the admissible light rays at each point of the aperture surface (usually a plane disc), the equation of the wall of the concentrator, formed by a mirror, and the equation of the receiver where the concentration of light achieves a maximum [7], [8]. At each point inside the concentrator the set of admissible light rays is determined by a cone in the tangent space at that point, and it is possible to determine the concentrator using Lorentz geometry. In this paper we consider the general case in which we have an arbitrary refractive index media and we will establish a technique to design nonimaging concentrator which generalize a previous one developed for the homogeneous case. The main technical difficulty is to establish and analyze a system of first order partial differential equations which models the problem.

A Lorentz metric on a manifold provides to each tangent space with a scalar product of signature $(-,+,\dots,+)$. This mean that the maximum dimension of the subspaces where the metric is negative definite is 1. Formally it is close related to Riemannian Geometry but they have different properties. Lorentz manifolds are used as the mathematical background for General Relativity and, from a mathematical point of view, there is an increasing research interest in recent years. One of the main properties of Lorentz Geometry is that it provides a field of cones on the manifold, and this is the feature that we want to use in nonimaging optics.

There is a limit to the concentration of light and a concentrator which achieves this limit is called ideal. They are characterized by the property that each point of the receiver has isotropic illumination. At each point inside an ideal concentrator the cone of rays that will reach the receiver is wider and wider as the points approximate the receiver, and it degenerates to a semispace just at the points of the receiver. The problem of design ideal concentrators has a simpler version in dimension 2. Once we have a solution in dimension 2 that has a symmetry with respect to an axis, we can have a 3-dimensional concentrator making a rotation around the axis of symmetry, but in general it will not be ideal. This is because we know the behavior of all light rays inside any plane that contains the symmetry axis, but there are other "non vertical" rays. Thus it is interesting to develop proper three dimensional methods such as the Lorentz geometry technique discussed in this paper.

It is enough to consider the trajectories of the edge rays of the cones at each point because if they reach the receiver then the inside rays of the cone at that point will also reach the receiver

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[8]. If we fix our attention to the trajectory of a light ray of the edge of the cone at a point it is clear that it is a broken geodesic of the euclidean space $\mathbb{R}^3$ (in the homogeneous case), where there is a broken point each time the ray reflects on the mirror. At each point of a smooth segment of the trajectory, the ray remains in the edge of the cone in that point. In Lorentz geometry this means that it is a lightlike curve for an appropriate Lorentz metric. After the first reflection, it is not guaranteed that the ray remains on the edge of the cone at the reflection point and this would force us to consider a different cone at this reflection point. In the non-homogeneous case the same argument applies with a suitable metric in $\mathbb{R}^3$ conformal to the euclidean metric.

To avoid this difficulty we only consider concentrators with the property that the field of cones is invariant by the (two sided mirror) wall of the concentrator, or equivalently, the reflected ray of any lightlike ray remains being a lightlike ray of the same Lorentz metric. This is called the detailed balance property [9].

2. **Lorentz geometry approach.** Let (M, g) be a Lorentz manifold where M is an open set in $\mathbb{R}^3$. Recall that this means that g is a symmetric twice covariant tensor field on M such that at each point $p \in M$, the bilinear form $g_p : T_pM \times T_pM \to \mathbb{R}$ is not degenerate and has signature $(-, +, +)$. As usual, a vector $v \in T_pM$ is called timelike if $g(v, v) < 0$, lightlike if $g(v, v) = 0$ and $v \neq 0$, and spacelike if $g(v, v) > 0$ or $v = 0$, and it is know as the causal character of the vector v. If $\gamma : \mathbb{R} \to M$ is a C^1 curve, then it is called timelike, lightlike or spacelike if all its tangent vectors has the corresponding causal character. The set of lightlike vectors in T_pM forms the light cone at p [6].

Let (x_1, x_2, x_3) be the standard chart in $\mathbb{R}^3$, and $G(p)$ the matrix which represent g_p in the coordinate basis $B_p = (\frac{\partial}{\partial x_1}_{|p}, \frac{\partial}{\partial x_2}_{|p}, \frac{\partial}{\partial x_3}_{|p})$. Let δ be the euclidean metric on $\mathbb{R}^3$. The matrix G can be diagonalized, leading to a matrix $G' = diag(\lambda_1, \lambda_2, \lambda_3)$ where one of the eigenvalues is negative, say λ_1, and the other two positives. Let $B' = (J, U, V)$ be the δ-orthonormal basis of eigenvectors of G. Changing the order of U and V if it were necessary, we can assume that B and B' are related by a rotation at each point.

Let $v \in T_pM$ be a lightlike vector, $v = \sum_{i=1}^{3} y_i \frac{\partial}{\partial x_i}_{|p}$. The condition $g(v, v) = 0$ can be expressed in the basis B' by the equation

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 0$$

which is the equation of a cone with axis in the direction of J. The intersection of this cone with the hyperplane $y_1 = 1$ gives the ellipse

$$\lambda_2 y_2^2 + \lambda_3 y_3^2 = -\lambda_1$$

with semiaxis $\sqrt{\frac{-\lambda_1}{\lambda_2}}$ in the U direction and $\sqrt{\frac{-\lambda_1}{\lambda_3}}$ in the V direction. Thus the Lorentz metric g provides a field of elliptic cones on M whose principal elements are codified by the eigensystem of G.

A key fact in this approach is that the edge rays of the cone at each point must be lightlike curves of the Lorentz metric, but these edge rays are true light rays, and then, by the Fermat principle, their trajectories are geodesic of the metric $r^2\delta$ where r is the refractive index function on M. This imposes the following condition to the Lorentz metric g called the restricted optical condition: *Every lightlike geodesic of the Lorentz metric must be a geodesic of the metric* $r^2\delta$. Here the word "restricted" is used to distinguish with the optical condition that assert that *every lightlike geodesic of the Lorentz metric must be a pregeodesic of the metric* $r^2\delta$.

Let ∇ be the Levi-Civita connection of g, and Γ_{ij}^k the Christoffel symbols of ∇ in the canonical basis B, and $\overline{\nabla}$, $\overline{\Gamma}_{ij}^k$ the Levi-Civita and the Chirstoffel symbols of the metric $r^2\delta$ in the same basis. Let Γ^k and $\overline{\Gamma}^k$ be the matrices formed with the corresponding Christoffel symbols. Let $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ be a differentiable curve in M. Then the restricted optical condition may be rewritten as the following matrix condition

$$\left.\begin{array}{r} \ddot{\gamma}^k + \dot{\gamma}\, \Gamma^k\, \dot{\gamma} = 0 \\ \dot{\gamma}\, G\, \dot{\gamma} = 0 \end{array}\right\} \Rightarrow \ddot{\gamma}^k + \dot{\gamma}\, \overline{\Gamma}^k\, \dot{\gamma} = 0. \tag{1}$$

Metrics satisfying the above condition form a wide class including all types of signatures, thus we will impose that the metric be Lorentz in a suitable open set M of $\mathbb{R}^3$.

From the above analysis of the field of cones of a Lorentz metric, it is clear that the concentrator is ideal if for each point p of the receiver, $\lim_{x \to p} \frac{-\lambda_1}{\lambda_i} = +\infty$, $i = 1, 2$. This gives us a tool in order to determine the surface of the receiver. Another consequence of the analysis is that for a concentrator with rotational symmetry around the x_1-axis whose generator curve is an integral curve of J, the detailed balance holds.

3. **The restricted optical equation.** There are some solutions to the three dimensional ideal concentrator design in the non-homogeneous case using other techniques, see [3] and [4]. As far as we know, the most important case for applications is when M has constant refractive index (the homogeneous case). This case has well known solutions, some of then has been obtained using the Lorentz geometry technique. Here we generalize this technique establishing the optical equations for arbitrary (positive) refractive index function and their integrability conditions. We will give a solution to the integrability condition in the important case in which the refractive index function is symmetric around the x_1-axis. The optical equation forms a system of eighteen first order partial differential equation whose general solution for the homogeneous case was found in [2].

The restricted optical condition (1) can be equivalently expressed with the following system called the restricted optical equation

$$\Gamma^k = \overline{\Gamma}^k + f^k G, \quad k \in \{1,2,3\}, \tag{2}$$

where $\overline{\Gamma}^k$ is the matrix formed by the Christoffel symbols of the metric $r^2\delta$ in the standard chart,

$$\overline{\Gamma}^k_{ij} = \frac{1}{2n}\left(-\frac{\partial n}{\partial x_i}\delta_{jk} - \frac{\partial n}{\partial x_j}\delta_{ki} + \frac{\partial n}{\partial x_k}\delta_{ij}\right), \quad i,j,k \in \{1,2,3\}, \tag{3}$$

being $n = r^{-2}$ the square of the speed of light in the media, and f^k are coefficient functions on M to be determined. Let $H \in \Lambda^1(M, \mathfrak{gl}(3,\mathbb{R}))$ be the one-form given by

$$H = d\mathbf{x}\mathbf{f}^t + \mathbf{f}d\mathbf{x}^t, \tag{4}$$

where $d\mathbf{x} = \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$ and $\mathbf{f} = \begin{pmatrix} f^1 \\ f^2 \\ f^3 \end{pmatrix}$, and let $A \in \Lambda^1(M, \mathfrak{gl}(3,\mathbb{R}))$ be the connection form of $r^2\delta$ in the coordinate basis B, $A = \sum_{i=1}^3 dx_i\overline{\Gamma}_i$, being $\overline{\Gamma}_i = \left(\overline{\Gamma}^k_{ij}\right)$ the matrix whose element in the j-row and k-column is given by $\overline{\Gamma}^k_{ij}$. Then the restricted optical equation is equivalent to the following Pfaff system,

$$\varpi = dG - GHG - AG - GA^t = 0. \tag{5}$$

Note that the homogeneous case corresponds to $A = 0$.

The 2-form $dA - A\wedge A$ is skewsymmetric, that is, $dA - A\wedge A \in \Lambda^1(M, \mathfrak{o}(3,\mathbb{R}))$. In fact, write $A = \sum_{i=1}^3 dx_i\overline{\Gamma}_i$ as $A = \alpha_0 I + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3$ for suitable 1-forms α_i, where I is the identity matrix, and $E_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ is the standard basis of the Lie algebra $\mathfrak{o}(3,\mathbb{R})$. A direct computation gives

$$dA - A\wedge A = (d\alpha_1 - \alpha_2\wedge\alpha_3)E_1 + (d\alpha_2 - \alpha_3\wedge\alpha_1)E_2 + (d\alpha_3 - \alpha_1\wedge\alpha_2)E_3.$$

Using the fact that if $A \in \Lambda^r(\mathbb{R}^n, \mathfrak{gl}(n,\mathbb{R}))$, $B \in \Lambda^s(\mathbb{R}^n, \mathfrak{gl}(n,\mathbb{R}))$ are $\mathfrak{gl}(n,\mathbb{R})$-valued r and s forms, then $(A\wedge B)^t = (-1)^{rs}B^t\wedge A^t$, and taking the exterior derivative in both sides of equation (5), we get

$$\begin{aligned} d\varpi \; &= \; (GH + A)\wedge\varpi - \varpi\wedge(GH + A) - \\ &\quad -G(dH + A^t\wedge H - H\wedge A)G - (dA - A\wedge A)G + G(dA - A\wedge A) \end{aligned}$$

thus, using the Frobenius theorem [1], the integrability conditions may be expressed imposing that the coefficients of the six second degree polynomials in the six unknown g_{ij}, given by

$$G(dH + A^t\wedge H - H\wedge A)G + (dA - A\wedge A)G - G(dA - A\wedge A)$$

must be identically null. The coefficients of the second order terms are clear. For the other terms it is immediate that the matrix $dA - A\wedge A$ commutes with the unknown symmetric matrix G if and only if $dA - A\wedge A$ is proportional to the identity, but being skewsymmetric, it must be zero. This leads us to the integrability conditions

$$\left.\begin{array}{c} dH + A^t\wedge H - H\wedge A = 0 \\ dA - A\wedge A = 0. \end{array}\right\} \tag{6}$$

Note that the second equation involves only the refractive index function r whereas the first one involves r and f^i.

An important family of solutions of (6) can be obtained assuming rotational symmetry for the refractive index function. This is equivalent to

$$n(x_1, x_2, x_3) = t(x_1, x_2^2 + x_3^2) \tag{7}$$

for some differentiable function $t : \mathbb{R}^2 \to \mathbb{R}$. Writing the second equation of (6) in the standard chart and using (7), after a straightforward computation we obtain the following solution

$$n(x_1, x_2, x_3) = e^b \left(4(x_1^2 + x_2^2 + x_3^2) + (4x_1 + c)c\right)^2 \tag{8}$$

being $b, c \in \mathbb{R}$, such that $(-\frac{c}{2}, 0, 0) \notin M$. Introducing this function in the first equation of (6) we get

$$\begin{aligned} f^1(x_1, x_2, x_3) &= a(x_1 + \frac{c}{2}) \\ f^2(x_1, x_2, x_3) &= ax_2 \\ f^3(x_1, x_2, x_3) &= ax_3 \end{aligned} \tag{9}$$

where $a \in \mathbb{R}$.

Because we are looking for a solution G on M where it is not singular, taking into account that $G^{-1}dG = -dG^{-1}G$, the equation (5) is equivalent to the following linear equation in the component of G^{-1} as unknown

$$dG^{-1} + H + G^{-1}A + A^t G^{-1} = 0 \tag{10}$$

Once we have a solution of this system, the technique to construct the concentrator follows the same steps than in the homogeneous case below and we will not describe it here. We point out that such a solution must satisfy some additional properties to determine an ideal concentrator.

4. **The homogeneous case.** Without loss of generality we can suppose $n = 1$. The euclidean metric has null Christoffel symbols in the coordinate basis B, thus, the restricted optical equation in this case is

$$\Gamma^k = f^k G, \quad k \in \{1, 2, 3\} \tag{11}$$

and the Pfaff system becomes equivalent to

$$\varpi = dG + GHG = 0. \tag{12}$$

Using the Frobenius theorem we have the integrability conditions $dH = 0$, that is,

$$\begin{aligned} f^1 &= ax_1 + b_1 \\ f^2 &= ax_2 + b_2 \\ f^3 &= ax_3 + b_3 \end{aligned} \tag{13}$$

where $a, b_i \in \mathbb{R}^3$.

The system (12) is equivalent to that in (10) making $A = 0$, that it to say, $dG^{-1} = -H$, and it can be solved by quadratures taking into account (13). In order to simplify the solution we impose rotational symmetry around the x_1-axis, obtaining the following solution

$$G = \frac{1}{|G^{-1}|} \begin{pmatrix} -am(x_2^2 + x_3^2) + m^2 & (ax_1 + b)mx_2 & (ax_1 + b)mx_3 \\ (ax_1 + b)mx_2 & h(x_1, x_3) & (ad + b^2)x_2x_3 \\ (ax_1 + b)mx_3 & (ad + b^2)x_2x_3 & h(x_1, x_2) \end{pmatrix}$$

where $h(y, z) = -(ay^2 + 2by - d)m - (ad + b^2)z^2$, $\left|G^{-1}\right| = -(ax_1^2 + 2bx_1 - d)m^2 - (ad + b^2)m(x_2^2 + x_3^2)$, and a, b, d, m are real parameters to be determined such that G is a Lorentz metric.

The matrix G has a constant eigenvalue $\frac{1}{m}$, thus if we take $m > 0$, then g cannot be negative definite and the open set where g is Lorentz is $M = \left|G^{-1}\right| < 0$. After the discussion about the role of the eigenvalues of G in the determination of the receiver in order to be an ideal concentrator, it is clear that the natural candidate to be this surface must be contained in the quadric $\left|G^{-1}\right| = -(ax_1^2 + 2bx_1 - d)m^2 - (ad + b^2)m(x_2^2 + x_3^2) = 0$. If we call μ_1, μ_2, μ_3 the eigenvalues of G^{-1} taking the first negative and the other two positives on M, it is easy to see that at each point of the above quadric we have

$$\begin{aligned} \mu_1 &= 0 \\ \mu_2 &= \frac{1}{m}\left((ad + b^2 - am)(x_2^2 + x_3^2) + m^2\right) \\ \mu_3 &= m \end{aligned}$$

this mean that the surface of the receiver must be contained in the region, which is called virtual entry aperture, determined by the conditions

$$\begin{aligned} \left|G^{-1}\right| &= 0 \\ (ad+b^2-am)(x_2^2+x_3^2)+m^2 &> 0 \end{aligned} \tag{14}$$

to guarantee that the concentrator, if it exist, be ideal.

Using the axial symmetry of g around the x_1-axis, it is easy to see that at each point $p \in M$ the eigenvector V associated with $\lambda_3 = \frac{1}{m}$ is tangent to the horizontal circle with center in the x_1-axis which contains p. Thus taking an integral curve of J as the generator of the revolution surface that forms the wall of the concentrator, it will satisfy the detailed balance property.

The following table summarizes the parameter combinations that makes g to be Lorentz on $\left|G^{-1}\right| < 0$.

Case	a	m	$ad+b^2$	$ad+b^2-am$	virtual entry aperture
1	+	+	0		Disk $r=\sqrt{\frac{m}{a}}$
2	0	+	+		Paraboloid
3	-	+	+		Two-sheet hyperboloid
4	+	+	+	+	Ellipsoid (major axis on x_1)
4.5	+	+	+	0	Sphere
5	+	+	+	-	Ellipsoid (minor axis on x_1)
6	+	+	-		Empty
7	-	-	-		Empty

Classification of Lorentz metrics that satisfy the restricted optical condition and has rotational symmetry around the x_1-axis.

The case 4.5 is the intermediate case between 4 and 5. The cases 6 and 7 have the same quadric $\left|G^{-1}\right| = 0$ which is a one-sheet hyperboloid, but in both cases they have empty virtual entry aperture. In 6 the open set M is the region that contains the x_1-axis. The case 7 has the parameter value $m < 0$, and the metric g is Lorentz in the region which does not contain the x_1-axis. Neither of the two define properly a concentrator because the rays inside the cone at each point of M do not reach the quadric $\left|G^{-1}\right| = 0$. A similar situation has been used to design light-confining cavities in photovoltaic applications [5].

All other cases define a concentrator. To obtain the equation of the wall we use the flowline technique [8]. Let p be a point in M that does not belong to the x_1-axis and α an integral curve of the eigenvector J through p such that it has a limit point in the receiver. The wall of the concentrator can be obtained as the surface of revolution around the x_1-axis that has the curve α as a generator. Then any ray at a point p inside the concentrator and directed toward an arbitrary point of the virtual entry aperture is a ray inside the cone at p, and thus it will reach the receiver directly or after reflections. Reciprocally, any ray reaching any point of the receiver is a ray that comes from inside the cone at each point of its trajectory, because the receiver is illuminated isotropically by the family of rays forming the field of cones of the Lorentz metric. In cases 2 - 5, the quadric $\left|G^{-1}\right| = 0$ and the wall intersects a meridian plane (a plane containing the x_1-axis) forming two confocal conics. See [2] for more details. Case 1 is the flow line concentrator FLC, and Case 4.5 is the cone concentrator, CC. Both are well known and they was previously obtained with a different technique [8].

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