Lorentzian Manifolds with no Null Conjugate Points

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Abstract

An integral inequality for a compact Lorentzian manifold which admits a timelike conformal vector field and has no conjugate points along its null geodesics is given. Moreover, equality holds if and only if the manifold has constant sectional curvature. This inequality can be improved if the timelike vector field is assumed to be Killing. As an indirect application of our technique, it is proved that a Lorentzian torus with no conjugate points along its timelike geodesics and admitting a timelike Killing vector field must be flat.

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1. *Introduction*

Conjugate points on geodesics of a Lorentzian manifold have been sistematically studied [**BEE**, Chaps. 10, 11], [**O'N**, Chap. 10]. Of course, their definition in Lorentzian geometry is formally the same as the Riemannian case, but there are here several geometric behaviours which are not possible or have no sense for a Riemannian manifold. In fact, on a Lorentzian manifold the geodesics are divided into three classes, spacelike, null and timelike, according to their causal character, and consequently conjugate points are classified into spacelike, null and timelike points (last two cases, i.e. causal conjugate points, have been extensively considered because their Physical meaning). Let us remark that there are also different approaches and techniques to deal with timelike or null conjugate points. With respect to this class of conjugate points, let us recall that no null geodesic in a $2-$dimensional Lorentzian manifold has conjugate points [**BEE**, Lemma 10.45]. On the other hand, there is a simple but surprising fact which asserts that there are no null conjugate points (i.e conjugate points on null geodesics) in a Lorentzian manifold of constant sectional curvature. The converse obviously is not true; in fact, it is easy to construct a Lorentzian manifold (even geodesically complete) with no null conjugate points and which does not have constant sectional curvature. The following question arises then in a natural way:

Under what assumption a Lorentzian manifold with no conjugate points along its null geodesics has constant sectional curvature?

The main aim of this paper is to find a satisfactory answer to this question. In order to face it, we will introduce an integral inequality on a remarkable family of Lorentzian manifolds such that no null geodesic contains a pair of mutually conjugate points.

Our efforts to deal with this problem have been inspired from several important contributions in Riemannian geometry. Following the philosophy to extend properties of Riemannian manifolds of nonpositive seccional curvature to Riemannian manifolds with no conjugate points, E. Hopf [**Hp**] proved in 1948 that

The total scalar curvature of a closed surface with no conjugate points is nonpositive and vanishes only if the surface is flat.

Therefore, this result combined with the Gauss-Bonnet theorem gives the following relevant fact

A Riemannian torus with no conjugate points must be flat.

Note that the same conclusion trivially holds if the Riemannian metric on the torus is assumed to have nonpositive Gauss curvature. The Hopf theorem was extended by L.W. Green [**Gr**] to any dimension to obtain

If a compact Riemannian manifold (B, g) has no conjugate points then its scalar curvature S satisfies

$$\int_B S \, d\mu_g \leq 0,$$

and equality holds if and only if the metric is flat.

More recently, F. Guimaraes [**Gu**] has generalized Green's theorem to complete Riemannian manifolds under the assumption that the Ricci curvature has an integrable positive or negative part on the unit tangent bundle.

Let us recall now that the no conjugancy assumption on a fixed geodesic in a Riemannian manifold or a causal geodesic in a Lorentzian manifold has been used by Ehrlich and Kim in [**EK**, Th. 3.1] to obtain a generalization of the Hawking-Penrose conjugancy theorem (see [**BEE**, Th. 12.18], for instance) of Singularity theory.

In this paper we will consider two assumptions on the Lorentzian manifolds to give an answer to the previous question: compacteness and the existence of certain symmetry of the metric tensor defined by a timelike conformal vector field. The first requirement is due to technical reasons (we will use integration on the manifold) and that we set our approach into global geometry. Of course, a compact Lorentzian manifold is not always geodesically complete but, as it was proved in [**RS1**] a compact Lorentzian manifold which admits a timelike conformal vector field must be geodesically complete.

Let us remark that the study of conformal vector fields (with certain causal character) on Lorentzian manifolds is a topic of special importance. It has been developed mainly under assumptions of interest in Physics. On the other hand, the Bochner technique on compact Lorentzian manifolds admitting a timelike conformal (and more general) vector field [**RS2**], [**RS3**], [**R**] has shown to be fruitfull to get several obstructions and classifications results. The particular case of timelike Killing vector fields was previously considered as a useful tool to get several kinds of classification results in Lorentzian Geometry [**KR**], [**Ka**], [**Sa1**], [**ARS1**], [**Sa2**] and references therein.

The main results of this paper are Theorem 4·1 and Corollary 4·3 and can be summarized as follows:

If (M,g) *is an* $n(\geq 3)-$*dimensional compact Lorentzian manifold with no null conjugate points and which admits a timelike conformal vector field* K*, then*

$$\int_M [n\mathrm{Ric}(U,U)+S]h^n d\mu_g \leq 0$$

and equality holds if and only if (M,g) *has constant sectional curvature.*

Here Ric and S are, respectively the Ricci tensor and the scalar curvature of (M,g), h is the function $1/\sqrt{-g(K,K)}$ and U is the unit timelike vector field hK.

If the conformal vector field K is assumed to be Killing this result can be improved, Theorem 4·5, to get

If (M,g) *is an* $n(\geq 3)-$*dimensional compact Lorentzian manifold with no null conjugate points and which admits a timelike Killing vector field* K*, then*

$$\int_M S h^n d\mu_g \leq 0$$

and equality holds if and only if (M,g) *is isometric to a flat Lorentzian* $n-$*torus up to a (finite) covering. In particular, in this case* K *is parallel, the first Betti number of* M *is not zero and the Levi-Civita connection of* g *is Riemannian.*

This result widely extends a theorem of Kamishima. In fact, he proved in [**Ka**, Th. A] that if a compact Lorentzian manifold with constant sectional curvature $k \in \mathbb{R}$ admits a timelike Killing vector field, then $k \leq 0$. Moreover, if $k=0$, then the manifold is affinelly diffeomorphic to a Riemannian manifold with non zero first Betti number. It should be noted that Kamishima uses a very different technique in [**Ka**], strongly depending on the Lie group machinery of Lorentzian space forms, than the present one.

Moreover, Theorem 4·5 permits to reprove in Corollary 4·7 the Hopf-Green inequality in Riemannian Geometry [**Gr**] previously mentioned.

Although we deal here with null conjugate points on $n \geq 3-$dimensional Lorentzian manifolds, by the use of a trick we are able to prove Corollary 4·9 that

A compact Lorentzian surface admitting a timelike Killing vector field with no conjugate point along its timelike geodesic must be flat.

This result resembles Hopf theorem [**Hp**], but in our case the no conjugancy hypothesis involves only timelike geodesics. It should be noted that a Lorentzian torus which admits a timelike Killing vector field is conformally flat [**Sa1**]. Theorem 4·9 gives an answer to the natural question to decide when a Lorentzian torus with a timelike Killing vector field must be flat. Finally, in the last section we illustrate our results on remarkable examples of compact Lorentzian manifolds.

2. *Preliminaries*

Let (M, g) be an $n(\geq 2)-$dimensional Lorentzian manifold; that is a (connected) smooth manifold M endowed with a non-degenerate metric g with signature $(-,+,...,+)$. We shall write ∇ for the Levi-Civita connection of g, R for its Riemannian curvature tensor (our convention on the curvature tensor is $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$), Ric for its Ricci tensor, S for its scalar curvature and $d\mu_g$ for the canonical measure associated with g.

As usual, the causal character of a tangent vector $v \in T_pM$ is timelike (resp. null, spacelike) if $g(v,v) < 0$ (resp. $g(v,v) = 0$ and $v \neq 0$, $g(v,v) > 0$ or $v = 0$). If $v \in T_pM$ then, γ_v will denote the unique (maximal) geodesic which satisfies $\gamma_v'(0) = v$. The causal character of $\gamma_v'(t)$ is independent of t and it is called the causal character of the geodesic. A null geodesic γ_v of (M, g) is then a geodesic such that v is a null vector. From now on, we will write P_a^b for the parallel translation along γ_v from $\gamma_v(a)$ to $\gamma_v(b)$.

Unless we indicate otherwise, (M, g) will denote an $n(\geq 3)-$dimensional Lorentzian manifold which admits a timelike vector field K. We put $h = 1/\sqrt{-g(K,K)}$ and write $U = hK$ for the normalized vector field obtained from K.

Recall [**Ha**], [**Ko**] that the *null congruence* on M associated to K is the subset of the tangent bundle TM given by

$$C_KM = \{v \in TM : g(v,v) = 0 \text{ and } g(v, K_{\pi v}) = 1\},$$

where $\pi : TM \longrightarrow M$ is the natural projection. This subset is an orientable embedded submanifold of TM with dimension $2(n-1)$, and can be naturally endowed with the Lorentzian metric $\widehat{g}$ obtained by restriction of the Sasaki metric $\mathbf{g}$ of TM (note that $\mathbf{g}$ is non-degenerate with signature $(-,-,+,...,+)$). Moreover, now $\pi : (C_KM, \widehat{g}) \to (M, g)$ is a semi-Riemannian submersion with spacelike fibers.

Let us suppose that the vector field K is conformal (i.e. $\mathfrak{L}_K g = \rho g$ with $\rho \in C^\infty(M)$). Under that assumption we have that C_KM is invariant by any (local) flow of the geodesic vector field. Furthermore, if M is also assumed to be compact, then it must be geodesically complete [**RS1**], and moreover, in this case, we have

$$\int_{C_KM} (f \circ \Phi_t)\, d\mu_{\widehat{g}} = \int_{C_KM} f \; d\mu_{\widehat{g}}, \tag{2·1}$$

for all $f \in C^0(M)$ and $t \in \mathbb{R}$ where $\Phi_t(v) = \gamma_v'(t)$ is the geodesic flow. (See details on those properties of C_KM in [**GPR**]).

As a key tool to get our main results in this paper we will use [**GPR**, Th. 3.2] which reads as follows: *Let (M, g) be an $n(\geq 3)$-dimensional compact Lorentzian manifold that admits a timelike conformal vector field K. If there is $a > 0$ such that every null geodesic γ_v, with $v \in C_KM$, has no conjugate point of $\gamma_v(0)$ in $[0, a)$, then*

$$\mathrm{Vol}(C_KM) \geq \frac{a^2}{\pi^2(n-2)} \int_{C_KM} \widetilde{\mathrm{Ric}}\; d\mu_{\widehat{g}}, \tag{2·2}$$

where $\widetilde{\mathrm{Ric}}$ denotes the quadratic form associated with the Ricci tensor. Moreover, equality in (2·2) has been characterized by using null sectional curvature.

In order to fix the notation to be used later, we recall the well-known notion of warped product (see [**O'N**, Def. 7.33], for instance). Let (B, g_B) and (F, g_F) be semi-Riemannian manifolds. We write π_B and π_F for the projections of $B \times F$ onto B and F, respectively.

Let $f > 0$ a smooth function on B. The warped product $B \times_f F$ is the product manifold $B \times F$ furnished with the metric tensor $g_f = {\pi_B}^*(g_B) + (f \circ \pi_B)^2 {\pi_F}^*(g_F)$.

3. *Null Geodesics with no Conjugate Points*

It is well-known that if $\gamma_v : [0, a] \longrightarrow M$ is a null geodesic such that there is no conjugate point of $\gamma_v(0)$ in $[0, a)$, then the *Hessian form* $H^{\perp}_{\gamma_v}$ is positive semidefinite, i.e. it satisfies

$$H^{\perp}_{\gamma_v}(V, V) = \int_0^a \Big[g\Big(\frac{\nabla V}{dt}, \frac{\nabla V}{dt}\Big) - g(R(V, \gamma_v')\gamma_v', V)\Big] dt \geq 0, \tag{3·1}$$

for every piecewise smooth vector field V along γ_v such that $V(0) = 0$, $V(a) = 0$ and $g(\gamma_v', V) = 0$, [**O'N**, p. 290-1].

The following construction, similar to the one made in [**BEE**, Chap. 10], emerges in a natural way in our setting. For every null tangent vector $v \in T_pM$, we consider the quotient vector space $\mathcal{N}(v) = \langle v \rangle^{\perp} \big/ \langle v \rangle$, where $\langle v \rangle$ denotes the vector subspace of T_pM spanned by v. If $x \in \langle v \rangle^{\perp}$, then $[x]$ will denote the element of $\mathcal{N}(v)$ defined by its equivalence class. Next, we will project on $\mathcal{N}(v)$ several geometric objects to be used later. Given $[x]$, $[y] \in \mathcal{N}(v)$, if we put $\overline{g}([x], [y]) = g(x, y)$, then $\overline{g}$ is a (well-defined) positive definite scalar product on $\mathcal{N}(v)$. The curvature operator may also be projected on $\mathcal{N}(v)$ as $\overline{R}([x], v)v = [R(x, v)v]$. Finally, since $P_0^t(v) = \gamma_v'(t)$ we can define $\overline{P}_0^t[x] = [P_0^t(x)] \in \mathcal{N}(\gamma_v'(t))$.

To face our end, $\gamma_v(t)$ is considered to be defined for all $t \in \mathbb{R}$. Let $\mathfrak{X}(\gamma_v)$ be the $C^{\infty}(\mathbb{R})-$module of vector fields along γ_v and $\mathfrak{X}^{\perp}(\gamma_v) = \{V \in \mathfrak{X}(\gamma_v) : g(V, \gamma_v') = 0\}$, which is clearly a submodule of $\mathfrak{X}(\gamma_v)$. If $[V] \in \mathfrak{X}^{\perp}(\gamma_v)/\langle \gamma_v' \rangle$, where $\langle \gamma_v' \rangle$ denotes the submodule of $\mathfrak{X}^{\perp}(\gamma_v)$ spanned by γ_v', then we can define $[V](t)$ to be $[V(t)] \in \mathcal{N}(\gamma_v'(t))$. Now, it is natural to put $\frac{\overline{\nabla}[V]}{dt} = \left[\frac{\nabla V}{dt}\right]$. Finally, if $[V](0) = [0], [V](a) = [0]$, we set $\overline{H}^{\perp}_{\gamma_v}([V], [V]) = H^{\perp}_{\gamma_v}(V, V)$ where $V \in [V]$ with $V(0) = 0$ and $V(a) = 0$.

A class $[V]$ is said to be a *Jacobi class* along γ_v if

$$\frac{\overline{\nabla}^2 [V]}{dt^2} + \overline{R}([V], \gamma_v')\gamma_v' = [0].$$

It is not difficult to show that $\gamma_v(0)$ and $\gamma_v(a)$, with $a \neq 0$, are conjugate along γ_v if and only if there exists a Jacobi class $[V](\neq [0])$ along γ_v which satisfies $[V](0) = [0]$ and $[V](a) = [0]$. In fact, any representant $V \in \mathfrak{X}^{\perp}(\gamma_v)$ of such a Jacobi class must satisfy

$$\frac{\nabla^2 V}{dt^2} + R(V, \gamma_v')\gamma_v' = f\gamma_v',$$

where $V(0) = \tau_1 v$, $V(a) = \tau_2 \gamma_v'(a)$, with $\tau_1, \tau_2 \in \mathbb{R}$, and f is a smooth function. In this case, if we choose a function F such that $\frac{d^2F}{dt^2} = f$, then

$$J(t) = V(t) + \Big[\frac{(a - t)(F(0) - \tau_1) + t(F(a) - \tau_2)}{a} - F(t)\Big]\gamma_v'(t)$$

is a nonzero Jacobi vector field along γ_v, with $J(0) = 0$ and $J(a) = 0$.

Several properties of Jacobi classes below are similar to the well-known ones of the Jacobi vector fields; for instance [**O'N**, Lemmas 10.15, 10.16] give respectively:

Lemma 3·1.

(1) *For every Jacobi classes* $[V], [W]$ *along* γ_v, *with* $[V](c) = [W](c) = [0]$ *for some* $c \in \mathbb{R}$, *we have*

$$\overline{g}\Big(\frac{\overline{\nabla}[V]}{dt}, [W]\Big) = \overline{g}\Big([V], \frac{\overline{\nabla}[W]}{dt}\Big). \tag{3·2}$$

(2) *If* $[J_1], ..., [J_k]$ *are Jacobi classes along* γ_v *such that* $[J_i]$ *and* $[J_j]$ *satisfy previous equation* (3·2) *for all* i, j *and we put* $[X] = \sum_{j=1}^{k} \lambda_j [J_j] \in \mathfrak{X}^{\perp}(\gamma_v)/\langle\gamma_v'\rangle$, *with* $[X](0) = [0]$ *and* $[X](a) = [0]$, *then*

$$H^{\perp}_{\gamma_v|_{[0,a]}}([X], [X]) = \int_0^a \overline{g}\Big(\sum_{i=1}^{k} \frac{d\lambda_i}{dt}[J_i], \sum_{j=1}^{k} \frac{d\lambda_j}{dt}[J_j]\Big) dt. \tag{3·3}$$

■

Let $\mathrm{End}(\mathcal{N}(v))$ be the space of all $\mathbb{R}-$linear operators of $\mathcal{N}(v)$ and $\mathcal{R} : \mathbb{R} \to \mathrm{End}(\mathcal{N}(v))$ the smooth path in $\mathrm{End}(\mathcal{N}(v))$ given by $\mathcal{R}(t) = \overline{P}_t^0 \circ \overline{R}(\ \ , \gamma_v'(t))\gamma_v'(t) \circ \overline{P}_0^t$. From the classical symmetries of the curvature tensor R we get that $\mathcal{R}(t)$ is self-adjoint with respect to $\overline{g}$. One associates to $\mathcal{R}$ the *matrix Jacobi equation*

$$\frac{d^2\mathcal{X}}{dt^2} + \mathcal{R}(t) \circ \mathcal{X} = 0, \tag{3·4}$$

where $\mathcal{X} : \mathbb{R} \to \mathcal{N}(v)$ is smooth. One easily checks that $[J]$ is a Jacobi class along γ_v if and only if $\mathcal{X}(t) = \overline{P}_t^0[J(t)]$ is a solution to (3·4).

In all what follows $\mathcal{A}$ will represent the solution to (3·4) determined from the initial conditions $\mathcal{A}(0) = 0$ and $\frac{d\mathcal{A}}{dt} \mid_0 = Id$, where Id denotes the identity transformation. For every $x \in \langle v\rangle^{\perp}$, let us denote J_x the unique Jacobi vector field along γ_v such that $J_x(0) = 0$ and $\frac{\nabla J_x}{dt} \mid_0 = x$. It is not difficult to show that $\mathcal{A}(t)[x] = \overline{P}_t^0[J_x(t)]$.

LEMMA 3·2. *Let* γ_v *be a null geodesic, then there is no conjugate point of* $\gamma_v(0)$ *along* γ_v *if and only if* $\mathrm{Ker}\,\mathcal{A}(t) = 0$ *for all* $t \neq 0$.

Proof. Let J be a nonzero Jacobi field along γ_v such that $J(0) = 0$ and $J(a) = 0$ for $a \neq 0$. If $\frac{\nabla J}{dt} \mid_0 = x$, then $0 \neq [x] \in \mathrm{Ker}\,\mathcal{A}(a)$. If $0 \neq [x] \in \mathrm{Ker}\,\mathcal{A}(a)$, then $[Z] = [J_x]$ is a Jacobi class and satisfies $[Z](0) = [0]$, $[Z](a) = [0]$ but $[Z] \neq [0]$. ■

If γ_v has no pair of mutually conjugate points, then for any $c \neq 0$, let $\mathcal{B}_c$ be the solution to (3·4) determined from the boundary data $\mathcal{B}_c(0) = Id$, and $\mathcal{B}_c(c) = 0$. Clearly, it satisfies $\mathcal{B}_c(t)[x] = \overline{P}_t^0[I_x^c(t)]$ where I_x^c is the unique Jacobi vector field along γ_v such that $I_x^c(0) = x$ and $I_x^c(c) = 0$. A similar argument as in [**Gr**] or [**Ch**, Chap.5], permits assert that there exists $\lim_{c\to+\infty} \frac{dB_c}{dt} \mid_0 = \frac{dB}{dt} \mid_0$ and taking B as the solution to (3·4) determined from the initial data $B(0) = Id$ and $\frac{dB}{dt} \mid_0$, the continuous dependence of solutions on the initial data gives that $\lim_{c\to+\infty} B_c(t) = B(t)$ and $\lim_{c\to+\infty} \frac{dB_c}{dt} \mid_t = \frac{dB}{dt} \mid_t$, with $B(t)$ non singular for all $t \geq 0$.

COROLLARY 3·3. *If* γ_v *has no a pair of mutually conjugate points, then for every* $[x] \in \mathcal{N}(v)$, $[x] \neq [0]$, *there exists a Jacobi class* $[I_x]$ *such that* $[I_x] = \lim_{c\to+\infty}[I_x^c]$ *and* $\frac{\overline{\nabla}[I_x]}{dt} = \lim_{c\to+\infty} \frac{\overline{\nabla}[I_x^c]}{dt}$. *Moreover* $[I_x](t) \neq 0$ *for all* $t \geq 0$.

Proof. It is enought to take $[I_x](t) = \overline{P}_0^t(\mathcal{B}(t)[x])$. ■

REMARK 3·4. It should be pointed out that the previous construction is analogous to the one made in [**BEE**, Chap. 12]. However, the ingredients of equation (3·4) differs from the used ones in the above mentioned cite.

4. *Integral Inequalities and their Applications*

THEOREM 4·1. *Let (M, g) be an $n(\geq 3)$-dimensional compact Lorentzian manifold admitting a timelike conformal vector field K. If (M, g) has no conjugate points along its null geodesics, then*

$$\int_{C_K M} \widetilde{\mathrm{Ric}}\; d\mu_{\widehat{g}} \leq 0. \tag{4·1}$$

Moreover, equality holds if and only if (M, g) has constant sectional curvature $k \leq 0$.

Proof. The inequality (4·1) follows taking into account that if (M, g) has no conjugate points along its null geodesics, then previous inequality (2·2) is valid for any positive real number a.

Now, we will show, in the equality case, that $\widetilde{\mathrm{Ric}} = 0$ on null tangent vectors. For every $v \in C_K M$, let $\{E_1, ..., E_{n-2}\}$ be a parallel frame along γ_v such that $\{E_1(0) = e_1, ..., E_{n-2}(0) = e_{n-2}\}$ give us an orthonormal basis for $\langle v\rangle^{\perp} \cap K^{\perp}_{\pi v}$ with $\widetilde{\mathrm{Ric}}(\gamma_v') = \sum_{j=1}^{n-2} g(R(E_j, \gamma_v')\gamma_v', E_j)$. It is easily seem that $\{[E_1](t), ..., [E_{n-2}](t)\}$ is a basis of $\mathcal{N}(\gamma_v'(t))$ for all t. We put $X_i^c(t) = \cos\left(\frac{\pi t}{2c}\right) E_i(t)$ for every integer $c \geq 1$, and take I_i^c the unique Jacobi vector field along γ_v such that $I_i^c(0) = e_i$ and $I_i^c(c) = 0$.

It follows that $\{[I_1^c](t), ..., [I_{n-2}^c](t)\}$ is a basis of $\mathcal{N}(\gamma_v'(t))$, for each $t \neq c$, since there are no conjugate points along γ_v. Therefore, there are smooth functions λ_{ij}^c such that

$$[X_i^c] = \sum_{j=1}^{n-2} \lambda_{ij}^c [I_j^c], \tag{4·2}$$

on $\mathbb{R} - \{c\}$, but taking into account that $X_i^c(c) = I_i^c(c) = 0$ it is not hard to show that every function λ_{ij}^c has a smooth extension to $\mathbb{R}$. From Corollary 3·3, there exist Jacobi classes $[I_j] = \lim_{c\to+\infty}[I_j^c]$, and it is clear that $\lim_{c\to\infty}[X_i^c] = [E_i]$. Hence, there are smooth functions $\lambda_{ij} = \lim_{c\to\infty} \lambda_{ij}^c$ such that

$$[E_i] = \sum_{j=1}^{n-2} \lambda_{ij} [I_j]. \tag{4·3}$$

Taking into account that $\lim_{c\to+\infty} \frac{\overline{\nabla}[X_i^c]}{dt} = 0$ and $\lim_{c\to+\infty} \frac{\overline{\nabla}[I_j^c]}{dt} = \frac{\overline{\nabla}[I_j]}{dt}$, a straightforward computation from (4·2) and (4·3) gives $\lim_{c\to+\infty} \frac{d\lambda_{ij}^c}{dt} = \frac{d\lambda_{ij}}{dt}$.

On the other hand, we get

$$\sum_{i=1}^{n-2} \overline{H}^{\perp}_{\gamma_v|_{[-c,c]}}([X_i^c], [X_i^c]) =$$

$$\int_{-c}^{c} \Big[\frac{(n-2)\pi^2}{4c^2} \sin^2\left(\frac{\pi t}{2c}\right) - \cos^2\left(\frac{\pi t}{2c}\right) \widetilde{\mathrm{Ric}}(\gamma_v')\Big] dt \geq 0.$$

Next, let $\{f_c\}$ be the following sequence of nonnegative continuous functions

$$f_c : C_K M \to \mathbb{R}, \qquad v \mapsto \sum_{i=1}^{n-2} \overline{H}^{\perp}_{\gamma_v|_{[-c,c]}}([X_i^c], [X_i^c]).$$

If equality holds in (4·1) and we take into account (2·1) and $\int_{-c}^{c}\sin^2(\frac{\pi t}{2c})dt=c$, then we have

$$\int_{C_KM} f_c\; d\mu_{\widehat{g}} = \frac{(n-2)\pi^2}{4c}\mathrm{Vol}(C_KM,\widehat{g}).$$

Threrefore, the Fatou lemma implies that $\int_{C_KM}\liminf f_c\; d\mu_{\widehat{g}}=0$, and so $\liminf f_c(v)=0$ for almost all $v\in C_KM$. Assume now we are considering such a $v\in C_KM$. Since $0\leq \overline{H}^{\perp}_{\gamma_v|_{[-c,c]}}([X_i^c],[X_i^c])\leq f_c(v)$ holds, we have

$$\liminf \overline{H}^{\perp}_{\gamma_v|_{[-c,c]}}([X_i^c],[X_i^c])=0, \tag{4·4}$$

for every $i=1,...,n-2$.

From Lemma 3·1 we obtain

$$\overline{H}^{\perp}_{\gamma_v|_{[-c,c]}}([X_i^c],[X_i^c])=\int_{-c}^{c}\overline{g}\Big(\sum_{j=1}^{n-2}\frac{d\lambda_{i,j}^c}{dt}[I_j^c],\sum_{j=1}^{n-2}\frac{d\lambda_{i,j}^c}{dt}[I_j^c]\Big)dt, \tag{4·5}$$

and so, from (4·4), by using the Fatou lemma again, we get

$$\overline{g}\Big(\sum_{j=1}^{n-2}\frac{d\lambda_{ij}}{dt}[I_j],\sum_{j=1}^{n-2}\frac{d\lambda_{ij}}{dt}[I_j]\Big)=0,\quad \text{for almost all } t\geq 0. \tag{4·6}$$

But $\overline{g}$ is positive definite, and therefore previous (4·6) implies $\sum_{j=1}^{n-2}\frac{d\lambda_{ij}}{dt}[I_j]=[0]$ on $[0,+\infty)$. Since $[I_j](t)\neq[0]$ for all $j=1,...,n-2$ and $t\geq 0$, we infer from (4·3) that $\{[I_1](t),...,[I_{n-2}](t)\}$ is a basis of $\mathcal{N}(\gamma_v'(t))$ which implies that all λ_{ij} are constant on $[0,+\infty)$. Therefore, every $[E_i]$ is a Jacobi class on $[0,+\infty)$. We achieve that $\widetilde{\mathrm{Ric}}(v)=0$ for almost every $v\in C_KM$. An standard continuity argument gives $\widetilde{\mathrm{Ric}}=0$ on null tangent vectors of (M,g).

By using [**EK**], we deduce that $\overline{R}(\ ,\gamma_v')\gamma_v'=[0]$ for every null vector v and so [**O'N**, Prop. 8.28] combined with the Schur theorem gives that (M,g) has constant sectional curvature k.

Since (M,g) must be geodesic complete from a result of completeness by Klingler [**Kl**] (this fact can be also deduced from [**RS1**]) it should be globally isometric to a quotient of the De Sitter space. Nevertheless, it is classical [**CM**] that no discrete subgroup of the isometry group of the De Sitter space which acts properly and discontinouslly produces a compact quotient of the De Sitter space. In brief, there exists no compact Lorentzian manifold of positive constant curvature. Therefore, we conclude $k\leq 0$ as required. ■

Remark 4·2. Compare Theorem 4·1 with [**EK**] where it is assumed that there is no conjugate points along a fixed null geodesic, and the conclusion only concerns to that geodesic.

Taking into account [**GPR**, Prop. 2.3 and Lemma 3.4], Theorem 4·1 can be rewriten as follows

Corollary 4·3. *Let (M,g) be an $n(\geq 3)$-dimensional compact Lorentzian manifold admitting a timelike conformal vector field K. If (M,g) has no conjugate points along its null geodesics, then*

$$\int_M [n\,\widetilde{\mathrm{Ric}}(U)+S]\,h^n d\mu_g\leq 0. \tag{4·7}$$

Moreover, equality holds if and only if (M, g) *has constant sectional curvature* $k \leq 0$.

Counter-example 4·4. Corollary 4·3 (or Theorem 4·1) does not remain true for compact Lorentzian surfaces. In fact, let (M, g) be a compact Lorentzian surface admitting a timelike conformal vector field. Now, we recall that there are no null conjugate points on two dimensional Lorentzian manifolds, and that, clearly, $\widetilde{\mathrm{Ric}} = 0$ holds on null tangent vectors in this case. Thus, if Corollary 4·3 were applicable, then (M, g) should have constant Gauss curvature. But now the Gauss-Bonnet theorem for Lorentzian surfaces (see [**BN**], for instance) can be claimed to get that (M, g) must be flat. However, it is known that there exist nonflat Lorentzian tori which admit a timelike Killing vector field [**Sa1**].

If the vector field K is assumed, more restrictibly, to be Killing, then [**GPR**, Cor. 3.10] can be improved as follows

Theorem 4·5. *Let* (M, g) *be an* $n \geq 3$*-dimensional compact Lorentzian manifold with no conjugate points along its null geodesics and admitting a timelike Killing vector field* K*, then*

$$\int_M S h^n d\mu_g \leq 0. \tag{4·8}$$

Equality holds if and only if (M, g) *is isometric to a flat Lorentzian* $n-$*torus up to a (finite) covering. In particular, in this case* U *is parallel, the first Betti number of* M *is not zero and the Levi-Civita connection of* g *is Riemannian.*

Proof. The inequality is clear from Corollary 4·3 taking into account [**GPR**, Lemma 3.7]. If equality holds in (4·8), then $\int_M h^n \widetilde{\mathrm{Ric}}(U)\, d\mu_g = 0$. On the other hand, Corollary 4·3 says that (M, g) has constant sectional curvature $k \leq 0$; in fact, $k = 0$ from our assumption. Let us consider the Riemannian metric g_R defined by $g_R(X, Y) = g(X, Y) + 2g(X, U)g(Y, U)$, for every $X, Y \in \mathfrak{X}(M)$. Making use of [**GPR**, Lemma 3.7], the vector field U must be parallel. Hence, the Levi-Civita connection associated to g_R on M agrees with the Levi-Civita connection of g (in particular, the $1-$form ω, given by $\omega(X) = g(X, U)$, is closed and clearly is not exact). Therefore, (M, g_R) is a compact flat Riemannian manifold and then, from [**W**, Cor. 3.46], there is a (finite) Riemannian covering $\rho : (\mathbb{T}^n, g_R^0) \to (M, g_R)$ by a flat Riemannian torus. It is easily seen that $\rho : (\mathbb{T}^n, g^0) \to (M, g)$ is a Lorentzian covering, where g_R^0 and g^0 are related in a similar way as previous g_R and g. ■

Remark 4·6. Kamishima proved in [**Ka**, Th. A] that if a compact Lorentzian manifold with constant sectional curvature $k \in \mathbb{R}$ admits a timelike Killing vector field, then it is geodesically complete and $k \leq 0$. Moreover, if $k = 0$, then the manifold is affinelly diffeomorphic to a Riemannian manifold with non zero first Betti number. So that Corollary 4·5 is a wide extension to this result. It should be noted that Kamishima uses a very different technique in [**Ka**] than the present one; in fact his tools are strongly depending on the Lie group machinery of Lorentzian space forms (see [**RS2**], [**RS3**] for another extensions of the Kamishima result).

Theorem 4·5 contains, as a very particular case, the classical Green result in Riemannian geometry ([**Gr**], [**Ch**, Th. 5.11]) mentioned in the introduction.

COROLLARY 4·7. *Let (B,g) be a compact Riemannian manifold of dimension $n \geq 2$ and scalar curvature S. If (B,g) has no conjugate points then*

$$\int_M S\, d\mu_g \leq 0.$$

Moreover, equality holds if and only if (B,g) is flat.

Proof. Consider the Lorentzian manifold $(M, g_L) = (\mathbb{S}^1 \times B, -g_{can} + g)$, where $\mathbb{S}^1$ is the unit circle and g_{can} denote its canonical metric. We may take the vector field K as the lift to $\mathbb{S}^1 \times B$ of the vector field $z \longmapsto iz$ on $\mathbb{S}^1 \subset \mathbb{C}$. Now, we have only to specialize Theorem 4·5 to this Lorentzian manifold. ■

REMARK 4·8. Theorem 4·5 cannot be extended to the case in which K is assumed to be conformal. In fact, let $(\mathbb{T}^2, g_0)$ be a flat Riemannian torus and let us consider $(\mathbb{S}^1, -g_{can})$ as previously. For every $f \in C^\infty(\mathbb{S}^1)$, $f > 0$, the Lorentzian manifold $\mathbb{S}^1 \times_f \mathbb{T}^2$ has no conjugate points along its null geodesics [**FS**, Th. 5.4]. If U is the vector field on $\mathbb{S}^1 \times F$ given by the lift of the vector field $z \longmapsto iz$ on $\mathbb{S}^1 \subset \mathbb{C}$, then $K = fU$ is a timelike conformal vector field (see [**ARS2**], for instance).

Making use of the well-known formulae for the curvature of a warped product metric [**O'N**, Chap. 7], we can write

$$\int_{\mathbb{S}^1 \times \mathbb{T}^2} Sh^3 d\mu_{g_f} = \text{area}(\mathbb{T}^2) \int_{\mathbb{S}^1} \left(4f\Delta f + 2 \parallel \text{grad} f \parallel^2\right)/f^3\, d\mu_{g_{can}},$$

where Δ, grad and $\parallel\ \parallel$ are, respectively, the Laplacian, the gradient and the norm of $(\mathbb{S}^1, g_{can})$. Now, the classical Green divergence theorem permits us to write

$$0 = \int_{\mathbb{S}^1} \Delta\Big(\frac{1}{f}\Big) d\mu_{g_{can}} = \int_{\mathbb{S}^1} \left(-\frac{\Delta f}{f^2} + \frac{2 \parallel \text{grad} f \parallel^2}{f^3}\right) d\mu_{g_{can}}.$$

Hence, if f is not constant, then $\int_{\mathbb{S}^1 \times \mathbb{T}^2} Sh^3 d\mu_{g_f} > 0$.

After showing Theorem 4·5, it might be natural to ask if the function h^n which appears in the integral inequality (4·8) could be omitted. At this time we don't know the answer, but we think that this is not possible.

As it was noted above, Corollary 4·5 cannot be used to study Lorentain surfaces. On the other hand, it does not give consequences from assumptions on timelike conjugate points. However, by means of a trick, the following result can be obtained.

COROLLARY 4·9. *Let (M,g) be a compact Lorentzian surface admitting a timelike Killing vector field K. If (M,g) has no conjugate points along its timelike geodesics, then (M,g) must be flat.*

Proof. Because M is $2-$dimensional, no of its null geodesics has conjugate points. Using this fact and the form of the geodesics in a semi-Riemannian product, we conclude that the Lorentzian manifold $(M \times \mathbb{S}^1, g + g_{can})$ has no conjugate points along its null geodesics. Moreover, K naturally induces a timelike Killing vector field on this semi-Riemannian product. From Theorem 4·5 we obtain

$$\int_M Gh^3 d\mu_g \leq 0,$$

where $G = (1/2)S$ is the Gauss curvature of M. Moreover equality holds if and only if

(M,g) is flat. On the other hand, the integrand in (4·7) is $[3\widetilde{\mathrm{Ric}}(U)+S]h^3=[-3G+2G]h^3=-Gh^3$. So, Corollary 4·3 can be also claimed to get the opposite inequality to the previous one. ■

5. *Examples*

1. *Compact standard static* Lorentzian manifolds are warped products $B\times_f F$ with (B,g_B) a compact Riemannian manifold, dim $B=n\geq 2$, and $(F,g_F)=(\mathbb{S}^1,-g_{can})$. If K is the timelike vector field on $B\times\mathbb{S}^1$ given by the lift of the vector field $z\longmapsto iz$ on $\mathbb{S}^1\subset\mathbb{C}$, then $h=1/\sqrt{-g_f(K,K)}=1/(f\circ\pi_B)$. Moreover, from [**O'N**, Lemma 12.37] K is Killing.

Assume $B\times_f\mathbb{S}^1$ has no conjugate points along its null geodesics. Using [**O'N**, Chap. 7], previous Corollary 4·3 gives

$$\int_B\Big[\frac{(n-1)\Delta f}{f^{n+1}}+\frac{S^B}{f^n}\Big]d\mu_{g_B}\leq 0,$$

where Δ and S^B are the Laplacian and the scalar curvature of (B,g_B), respectively. Taking now into account $\Delta\Big(\frac{-1}{nf^n}\Big)=\frac{\Delta f}{f^{n+1}}-\frac{n+1}{f^{n+2}}\|\mathrm{grad}f\|^2$, the classical Green divergence theorem permits us to write

$$\int_B\frac{\Delta f}{f^{n+1}}d\mu_{g_B}=(n+1)\int_B\frac{\|\mathrm{grad}f\|^2}{f^{n+2}}d\mu_{g_B}\geq 0.$$

Therefore, if $B\times_f\mathbb{S}^1$ has no conjugate points along its null geodesics, then

$$\int_B\frac{S^B}{f^n}d\mu_{g_B}\leq 0,$$

and equality holds if and only if f is constant and (B,g_B) is flat.

2. *Generalized Robertson-Walker Compact Spacetimes* are warped products $B\times_f F$ with $(B,g_B)=(\mathbb{S}^1,-g_{can})$ and (F,g_F) a compact Riemannian manifold with dim$F=n\geq 2$. If U is the vector field on $\mathbb{S}^1\times F$ given by the lift of the vector field $z\longmapsto iz$ on $\mathbb{S}^1\subset\mathbb{C}$, then $K=fU$ is a timelike conformal vector field, [**ARS2**] and $h=1/(f\circ\pi_{\mathbb{S}^1})$. Assume $\mathbb{S}^1\times_f F$ has no conjugate points along its null geodesics (this happen if the fiber (F,g_F) has no conjugate points [**FS**, Theor. 5.4]). A straightforward computation, by using the curvature formulae of a warped product [**O'N**, Chap. 7], permits to write (4·7) as follows

$$n(n-1)\mathrm{Vol}(F)\int_{\mathbb{S}^1}\Big[\frac{(f')^2}{f^3}-\frac{f''}{f^2}\Big]d\mu_{g_{can}}+\int_F S^F d\mu_{g_F}\int_{\mathbb{S}^1}\frac{1}{f^3}d\mu_{g_{can}}\leq 0, \tag{5·1}$$

where f' denotes $U(f)$. This inequality can be also obtained from an inderect method. In fact, we have $\Delta\Big(\frac{-1}{f}\Big)=-\frac{f''}{f^2}+\frac{2(f')^2}{f^3}$ (Δ denotes here the Laplacian operator for $-g_{can}$; i.e. $\Delta=-\Delta^0$, where $\Delta^0=\frac{d^2}{d\theta^2}$ is the usual Laplacian of $\mathbb{S}^1$). Now, with the help of the classical Green divergence theorem, (5·1) is rewriten as follows

$$\int_F S^F d\mu_{g_F}\int_{\mathbb{S}^1}\frac{1}{f^3}d\mu_{g_{can}}\leq n(n-1)\mathrm{Vol}(F)\int_{\mathbb{S}^1}\frac{(f')^2}{f^3}d\mu_{g_{can}},$$

which is of course true if it is noted that $\int_F S^F d\mu_g\leq 0$ because of the Green theorem. In the equality case, $\mathbb{S}^1\times_f F$ has constant sectional curvature $k\leq 0$. Therefore $\widetilde{\mathrm{Ric}}(U)=-nk=-\frac{nf''}{f}$. On the other hand, $\Delta(\ln f)=-\frac{f''}{f}+\frac{(f')^2}{f^2}$, and so we use again the divergence theorem to get that f is constant (i.e. $\mathbb{S}^1 \times_f F$ is, in fact, a semi-Riemannian product) and $k = 0$. Therefore the fiber (F, g_F) must be flat (alternatively, this can be also achieved observing that equality in (5·1) gives $\int_F S^F d\mu_g = 0$, and now the Green theorem can be claimed again).

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REFERENCES

[**ARS1**] L.J. Alías, A. Romero and M. Sánchez, *Spacelike hypersurfaces of constant mean curvature in certain spacetimes*, Nonlinear Anal. **30** (1997), 655–661.

[**ARS2**] L.J. Alías, A. Romero and M. Sánchez, *Spacelike hypersurfaces of constant mean curvature and Calabi-Bernstein type problems*, Tôhoku Math. J. **49** (1997), 337–345.

[**BEE**] J.K. Beem, P.E. Ehrlich and K.L. Easley, *Global Lorentzian Geometry*, Second ed., Pure and Applied Math. **202**, Marcel Dekker, 1996.

[**BN**] G. Birman and K. Nomizu, *The Gauss-Bonnet theorem for two-dimensional spacetimes*, Mich. Math. J. **31** (1984), 77–81.

[**CM**] E. Calabi and L. Markus, *Relativistic space forms,* Ann. of Math. **75** (1962), 63–76.

[**Ch**] I. Chavel, *Riemannian Geometry: A modern Introduction*, Cambridge Univ. Press, 1993.

[**EK**] P.E. Ehrlich and S-B Kim, *From the Riccati Inequality to the Raychaudhuri Equation*, Contemp. Math. **170** (1994), 65–78.

[**FS**] J.L. Flores and M. Sánchez, *Geodesic connectedness and conjugate points in GRW spacetimes*, J. Geom. Phys. **30** (2000), 285–314.

[**Gr**] L.W. Green, *A theorem of E. Hopf*, Mich. Math. J. **5** (1958), 31–34.

[**Gu**] F. Guimaraes, *The integral of the scalar curvature of complete manifolds without conjugate points.* J. Differential Geom. **36** (1992), 651–662.

[**GPR**] M. Gutiérrez, F.J. Palomo and A. Romero, *A Berger-Green type inequality for Compact Lorentzian Manifolds,* Trans Amer. Math. Soc. **354** (2002), 4505–4523.

[**Ha**] S.G. Harris, *A characterization of Robertson-Walker Spaces by null sectional curvature*, Gen. Relativity Gravitation **17** (1985), 493–498.

[**Hp**] E. Hopf, *Closed surfaces without conjugate points,* Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 47–51.

[**Ka**] Y. Kamishima, *Completeness of Lorentz manifolds of constant curvature admitting Killing vector fields*, J. Differential Geom. **37** (1993), 569–601.

[**Kl**] B. Klinger, *Completude des varietes lorentziennes a courbure constante,* Math. Ann, **306** (1996), 353–370.

[**Ko**] L. Koch-Sen, *Infinitesimal null isotropy and Robertson-Walker metrics*, J. Math. Phys. **26** (1985), 407–410.

[**KR**] R. Kulkarni and F. Raymond, *3-dimensional Lorentz space-forms and Seifert fiber spaces*, J. Differential Geom. **21** (1985), 231–268.

[**O'N**] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.

[**R**] A. Romero, *The introduction of Bochner's technique on Lorentzian manifolds*, Nonlinear Anal. TMA **47** (2001), 3047–3059.

[**RS1**] A. Romero and M. Sánchez, *Completeness of compact Lorentz manifolds admiting a timelike conformal-Killing vector field*, Proc. Amer. Math. Soc. **123** (1995), 2831–2833.

[**RS2**] A. Romero and M. Sánchez, *An integral inequality on compact Lorentz manifolds and its applications*, Bull. London Math. Soc. **28** (1996), 509–513.

[**RS3**] A. Romero and M. Sánchez, *Bochner's technique on Lorentz manifolds and infinitesimal conformal symmetries*, Pacific J. Math. **186** (1998), 141–148.

[**Sa1**] M. Sánchez, *Structure of Lorentzian tori with a Killing vector field*, Trans. Amer. Math. Soc. **349** (1997), 1063–1080.

[**Sa2**] M. Sánchez, *Lorentzian manifolds admitting a Killing vector field*, Nonlinear Anal. TMA **30** (1997), 643–654.

[**Sa3**] M. Sánchez, *On the Geometry of Generalized Robertson-Walker Spacetimes: Geodesics*, Gen. Relativity Gravitation **30** (1998), 915–932.

[**W**] J.L. Wolf, *Spaces of constant curvature*, Fourth ed., Publish or Perich, 1979.