



Global decomposition of a Lorentzian manifold as a Generalized Robertson–Walker space [☆]

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ABSTRACT

Generalized Robertson–Walker (GRW) spaces constitute a quite important family in Lorentzian geometry, and it is an interesting question to know whether a Lorentzian manifold can be decomposed in such a way. It is well known that the existence of a suitable vector field guarantees the local decomposition of the manifold. In this paper, we give conditions on the curvature which ensure a global decomposition and apply them to several situations where local decomposition appears naturally. We also study the uniqueness question, obtaining that the de Sitter spaces are the only nontrivial complete Lorentzian manifolds with more than one GRW decomposition. Moreover, we show that the Friedmann Cosmological Models admit an unique GRW decomposition, even locally.

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1. Introduction

If $I \subset \mathbb{R}$ is an open interval, (L, g_0) a Riemannian manifold and $f \in C^\infty(I)$, the warped product $I \times_f L$ is the product manifold $I \times L$ with metric $-dt^2 + f(t)^2 g_0$ and is called a Generalized Robertson–Walker space, GRW space in short. These spaces were introduced in [1] and they have been widely studied since then. It is well known that if a Lorentzian manifold admits a timelike, closed and conformal vector field, then it is locally a GRW space, [9,15], but the presence of such a vector field does not allow us to decide if the decomposition is global. The best we can say, under suitable completeness conditions, is that the manifold is a quotient of a GRW space, which is not always a global product, see Examples 2.5 and 2.6.

In this paper we study purely geometric conditions to achieve a global decomposition as a GRW space and apply them to obtain global GRW from local ones. For example, any Robertson–Walker space is a perfect fluid, so it is natural to find out under which conditions a perfect fluid is a Robertson–Walker space. Other situation is suggested by photon surfaces in General Relativity, which are timelike totally umbilic submanifolds, [3]. In a local GRW space, photon surfaces inherit the local GRW structure of the ambient space. We study how far they are in fact global GRW spaces.

Finally, we investigate the GRW decomposition uniqueness obtaining that the de Sitter spaces are the unique complete Lorentzian manifolds with more than one (with nonconstant warping function) GRW decomposition. On the other hand, in a nonnecessarily complete manifold, if the lightlike sectional curvature is not zero for any degenerate planes at a point p , then there is at most one local GRW structure in a neighborhood of p .

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The noncomplete case is interesting in Cosmology. In fact, Friedmann spaces are incomplete GRW spaces with a distinguished family of comoving observers which physically represents the average galaxy evolution in the spacetime. From a mathematical point of view, comoving observers are integral curves of the unitary of a timelike, closed and conformal vector field which gives the global decomposition as a GRW space. Since the lighthlike sectional curvature of the Friedmann spaces is always positive, this decomposition is unique, even locally, which means that the Friedmann mathematical representation of the galaxy evolution is unambiguous.

2. Preliminaries

As we said, we are going to study when a space splits as a GRW space $M = I \times_f L$. In this space there are two distinguished vector fields. The first one is $U = \frac{\partial}{\partial t}$ which is a *reference frame*, i.e., a timelike and unit vector field. Moreover, it is closed (its metrically equivalent one form is closed) and orthogonally conformal ($L_U g(X, Y) = 2\alpha g(X, Y)$ for all $X, Y \in U^\perp$ for some function $\alpha \in C^\infty(M)$). The other one is $V = f \frac{\partial}{\partial t}$ which is timelike, closed and conformal. Our interest is to study the implications that the existence of a vector field with this kind of properties has on the global structure of a Lorentzian manifold, when we have additional geometric information.

Let (M, g) be a connected Lorentzian manifold with dimension $n > 1$ and U a closed reference frame. We will call Φ the flow of U , ω the metrically equivalent one-form and L_p the orthogonal leaf through $p \in M$. If there is not confusion we drop the point p and write simply L .

Proposition 2.1. *If U is a complete and closed reference frame, then for all $p \in M$ the map $\Phi : (\mathbb{R} \times L_p, -dt^2 + g_t) \rightarrow (M, g)$ is a normal Lorentzian covering map, where g_t is a metric tensor on L_p for each $t \in \mathbb{R}$ and $g_0 = g|_{L_p}$.*

Proof. We have $L_U \omega = d \circ i_U \omega + i_U \circ d \omega = 0$, and therefore Φ is foliated, i.e. $\Phi_t(L_p) = L_{\Phi_t(p)}$ for all $t \in \mathbb{R}$ and $p \in M$. Thus, the map $\Phi : \mathbb{R} \times L_p \rightarrow M$ is a local diffeomorphism which is onto and $\Phi^* g = -dt^2 + g_t$, where g_t is a metric on L_p .

Now, we show that it is a covering map. Let $\sigma : [0, 1] \rightarrow M$ be a geodesic and $(t_0, x_0) \in \mathbb{R} \times L_p$ a point such that $\Phi(t_0, x_0) = \sigma(0)$. We must show that there exists a lift $\alpha : [0, 1] \rightarrow \mathbb{R} \times L_p$ of σ through Φ starting at (t_0, x_0) , [12]. There is a geodesic $\alpha : [0, s_0] \rightarrow \mathbb{R} \times L_p$, $\alpha(s) = (t(s), x(s))$, such that $\Phi \circ \alpha = \sigma$ and $\alpha(0) = (t_0, x_0)$ because Φ is a local isometry. If we suppose $s_0 < 1$, there is a geodesic $(t_1(s), x_1(s))$ such that $\Phi(t_1(s), x_1(s)) = \sigma(s)$ with $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$, then in the open interval $(s_0 - \varepsilon, s_0)$ it holds $\Phi(t(s), x(s)) = \Phi(t_1(s), x_1(s))$. Differentiating and using that Φ is foliated, it is easy to see that $t_1(s) - t(s) = c \in \mathbb{R}$. Therefore it exists $\lim_{s \rightarrow s_0} \alpha(s)$ and the geodesic α is extendible.

It remains to show that the group of deck transformations acts transitively on the fibres. Take $(t_0, x_0) \in \mathbb{R} \times L_p$ such that $\Phi(t_0, x_0) = \Phi(0, p) = p$. Since Φ is foliated, it follows that $\Phi_{-t_0}(L_p) = L_p$ and thus the map $\mathbb{R} \times L_p \rightarrow \mathbb{R} \times L_p$ given by $(t, x) \rightarrow (t + t_0, \Phi_{-t_0}(x))$ is a deck transformation and takes $(0, p)$ to (t_0, x_0) . \square

The following proposition, for the case that the flow is conformal when restricted on the leaves, is proved in a general form in [13], but it does not include an explicit expression for the warping function. We include here a sketch of the proof to make the paper self-contained.

Proposition 2.2. *Let $I \subset \mathbb{R}$ be an open interval, L a manifold and g a Lorentzian metric on $I \times L$ such that the canonical foliations are orthogonal. If $U = \frac{\partial}{\partial t}$ is an orthogonally conformal reference frame such that $\nabla \operatorname{div} U$ is proportional to U , then g is the warped product $-dt^2 + f(t)^2 g_0$ where $g_0 = g|_L$ and $f(t) = \exp(\int_0^t \frac{\operatorname{div} U(s, q)}{n-1} ds)$, being q a fixed point of L .*

Proof. It is clear that U is also orthogonally integrable, that is, the distribution U^\perp is integrable, thus

$$L_U g(v, w) = 2 \frac{\operatorname{div} U}{n-1} g(v, w) \quad \text{for all } v, w \in U^\perp.$$

Therefore $\Phi_t : L_{(0,p)} \rightarrow L_{\Phi_t(0,p)}$ is a conformal diffeomorphism with factor $\exp(2 \int_0^t \frac{\operatorname{div} U(\Phi_s(0,p))}{n-1} ds)$. Since $L_{(t,p)} = \{t\} \times L$ and $\Phi_t(s, p) = (t + s, p)$, we have

$$g((0_t, v_p), (0_t, w_p)) = \exp\left(2 \int_0^t \frac{\operatorname{div} U(s, p)}{n-1} ds\right) g_0(v, w)$$

for all $v, w \in T_p L$. But $\operatorname{div} U$ is constant on L and the proposition follows. \square

If a reference frame U is closed, orthogonally conformal and $\nabla \operatorname{div} U$ is proportional to U , we say that U is a *warped reference frame*. This name is justified by the following corollary, which is a combination of the above two results.

Corollary 2.3. *Let (M, g) be a Lorentzian manifold and U a complete warped reference frame. Then the map $\Phi : (\mathbb{R} \times L_p, -dt^2 + f(t)^2 g_0) \rightarrow (M, g)$ is a normal Lorentzian covering for all $p \in M$, where $f(t) = \exp(\int_0^t \frac{\operatorname{div} U(\Phi_s(p))}{n-1} ds)$ and $g_0 = g|_{L_p}$.*

In a warped product $-dt^2 + f(t)^2g_0$ the vector field $\frac{\partial}{\partial t}$ is a warped reference frame. Observe that if we drop the completeness hypothesis in the above corollary we do not obtain a covering map, but we do obtain a local splitting around any point as $-dt^2 + f(t)^2g_0$ with $U = \frac{\partial}{\partial t}$. Thus, the existence of a warped reference frame is equivalent to the local splitting as a warped product $-dt^2 + f(t)^2g_0$.

Remark 2.4. If M admits a complete warped reference frame then it is isometric to a quotient $(\mathbb{R} \times_f L_p)/\Gamma$, where Γ is a group of isometries which preserves the canonical foliations. If $\psi \in \Gamma$ then it is of the form $\psi(t, x) = (t + k_\psi, \varphi(x))$, for certain constant k_ψ , where φ is an homothety of coefficient c^2 and $f(t) = cf(t + k_\psi)$. Therefore, in order to show that M is globally a GRW space we must assure that $k_\psi = 0$ for all $\psi \in \Gamma$, or equivalently, integral curves of U meet the orthogonal leaves at only one value of its parameter.

Recall that the quotient $(\mathbb{R} \times_f L_p)/\Gamma$ is not necessarily isometric to a warped product $\mathbb{S}^1 \times_f N$ (called twisted product in [11]) even if f is periodic, as the following example shows.

Example 2.5. Take $\tilde{M} = \mathbb{R} \times \mathbb{R}$ with the metric $g = -dt^2 + (2 + \cos(2\pi t))^2 dx^2$ and Γ the group of isometries generated by $\psi(t, x) = (t + 1, x + 1)$. Consider $M = \tilde{M}/\Gamma$. It has a complete warped reference frame U which lifts to $\frac{\partial}{\partial t}$, since $\frac{\partial}{\partial t}$ is invariant under Γ , but M is neither a global GRW nor a warped product of type $\mathbb{S}^1 \times_f L$, see Remark 2.4.

On the other hand, we can construct a quotient of a GRW that is not a global GRW even if f is nonperiodic.

Example 2.6. Consider $\tilde{M} = \mathbb{R} \times \mathbb{R}^2$, with the metric $g = -dt^2 + e^t(dx^2 + dy^2)$ which is a portion of de Sitter spacetime \mathbb{S}_1^3 , [2]. Let Γ be the isometry group generated by $\psi(t, (x, y)) = (t + 1, e^{-\frac{1}{2}}(x, y))$. It acts proper and discontinuously and the quotient \tilde{M}/Γ is neither a global GRW nor a warped product of type $\mathbb{S}^1 \times_f L$.

A timelike, closed and conformal vector field V is characterized by the equation $\nabla_X V = a X$ for all vector field X , where a is certain function. Call $\lambda = |V|$ and $U = \frac{V}{\lambda}$. It is easy to show that $a = U(\lambda)$ and λ is constant through the orthogonal leaves. Moreover, U is a warped reference frame. We write some useful formulas and facts.

Lemma 2.7. Let V be a timelike, closed and conformal vector field, $\lambda = |V|$, $U = \frac{V}{\lambda}$ and L an orthogonal leaf. Then

1. If h is a constant function through the orthogonal leaves of V , then $U(h)$ is also constant through the orthogonal leaves.
2. $\Delta\lambda = -U(U(\lambda)) - (n - 1)\frac{U(\lambda)^2}{\lambda}$.
3. $\text{div } U = (n - 1)U(\ln \lambda)$.
4. If $v \in TL$ is an unit vector, then

$$\text{Ric}(v + U) = \text{Ric}_L(v) + \frac{n - 2}{\lambda} \left(\frac{U(\lambda)^2}{\lambda} - U(U(\lambda)) \right),$$

$$\text{Ric}(v) = \text{Ric}_L(v) + \frac{1}{\lambda} \left(U(U(\lambda)) + (n - 2)\frac{U(\lambda)^2}{\lambda} \right)$$

where Ric_L is the Ricci tensor of L .

The above formulas can be directly proved from the fact that V is closed and conformal, but observe that if we take U the unitary of a closed and conformal vector field V in Corollary 2.3, then we get that M is locally isometric to $\mathbb{R} \times_f L$, where $f(t) = \frac{\lambda(\Phi_t(p))}{\lambda(p)}$ and we can deduce the above lemma from the standard formulas for a warped product, [12].

We finish this section establishing the relation between warped reference frames and closed and conformal vector fields.

Lemma 2.8. Let M be a complete Lorentzian manifold and U a warped reference frame. Then there exists a closed and conformal vector field V such that $U = \frac{V}{|V|}$.

Proof. Take $\Phi : \mathbb{R} \times_f L \rightarrow M$ the normal covering given in Corollary 2.3. If $\psi(t, x) = (t + k, B(x))$ is a deck transformation, $\frac{f(t+k)}{f(t)} = c$ for all $t \in \mathbb{R}$, being c certain constant, then using that $\mathbb{R} \times_f L$ is complete, [15]

$$\infty = \int_0^\infty f(t) dt = \sum_{n=0}^\infty c^n \int_0^k f(t) dt,$$

$$\infty = \int_{-\infty}^0 f(t) dt = \sum_{n=0}^{\infty} c^{-n-1} \int_0^k f(t) dt.$$

Thus $c = 1$ and $f(t+k) = f(t)$ for all $t \in \mathbb{R}$. The vector field $f \frac{\partial}{\partial t}$ is closed and conformal and it is preserved by the deck transformations. Thus there exists a closed and conformal vector field V on M such that $U = \frac{V}{|V|}$. \square

3. Global GRW decompositions

In [11] it was proved that a complete Lorentzian manifold with nonnegative constant scalar curvature which admits a nonparallel warped reference frame is isometric to a global GRW space. Now we get decomposition theorems on Lorentzian manifolds using Ricci curvature hypothesis. Before, we need the following lemma.

Lemma 3.1. *Let (M, g) be a noncompact Lorentzian manifold and U a closed reference frame with compact orthogonal leaves. Then M is isometric to $(I \times L, -dt^2 + g_t)$, where $I \subset \mathbb{R}$ is an open interval and L a compact Riemannian manifold.*

Proof. Let A be the domain of Φ , L an orthogonal leaf and $(a, b) \subset \mathbb{R}$ the maximal interval of \mathbb{R} such that $(a, b) \times L \subset A$. We claim that it is the maximal definition interval of each integral curve with initial value on L . Indeed, suppose that $\Phi_t(p_0)$ is defined in $(a, b + \delta)$ for some $p_0 \in L$. There is a $\eta \in \mathbb{R}$ such that $(-\eta, \eta) \times L_{\Phi_b(p_0)} \subset A$. Since $L_{\Phi_b(p_0)}$ is compact, $\Phi_{-\frac{\eta}{2}} : L_{\Phi_b(p_0)} \rightarrow L_{\Phi_{b-\frac{\eta}{2}}(p_0)}$ is onto, and therefore a diffeomorphism. Now, for an arbitrary $p \in L$, $\Phi_t(p)$ can be defined in $(a, b + \eta)$ and we obtain a contradiction.

If there were $(t_0, p) \in (a, b) \times L$ such that $\Phi(t_0, p) \in L$, $t_0 \neq 0$, then $M = \bigcup_{t \in [0, t_0]} \Phi_t(L)$, and it would be compact. Therefore $\Phi : (a, b) \times L \rightarrow M$ is an injective map and we obtain the desired result. \square

Theorem 3.2. *Let M be a complete and noncompact Lorentzian manifold with $n \geq 3$ and U a nonparallel warped reference frame. If one of the following conditions is true*

1. $\text{Ric}(U) \leq 0$,
2. $\text{Ric}(v) \geq 0$ for all $v \in U^\perp$,
3. $\text{Ric}(w) \geq 0$ for all lightlike vector w ,

then M is globally a GRW space.

Proof. Take V the closed and conformal vector field provided in Lemma 2.8. Then $\lambda = |V|$ is a nonconstant function because U is nonparallel. Let $\gamma(t)$ be an integral curve of U .

(1) If γ returned to L then $\lambda(\gamma(t))$ would be periodic, since Φ_t is foliated and λ is constant through L . Using $\text{Ric}(U) = -(n-1) \frac{U(U(\lambda))}{\lambda}$, the hypothesis implies that $\frac{d^2 \lambda \gamma(t)}{dt^2} \geq 0$. Contradiction. The result follows applying Corollary 2.3.

(2) First suppose $\Delta \lambda \leq 0$. Take L an orthogonal leaf and suppose the $\gamma(0) \in L$. We know that if γ returned to L then $\lambda(\gamma(t))$ would be periodic. We have $-\Delta \lambda = U(U(\lambda)) + (n-1) \frac{U(\lambda)^2}{\lambda}$, thus if $z(t) = \ln \frac{\lambda(\gamma(t))}{\lambda(\gamma(0))}$ we get $0 \leq z'' + nz'^2$. Since $\lambda(\gamma(t))$ is periodic and nonconstant there exists $t_0 < t_1$ such that $z'(t_0) = z'(t_1) = 0$ and $z'(t) \neq 0$ for all $t \in (t_0, t_1)$. Suppose that $z' > 0$ in (t_0, t_1) (the case $z' < 0$ is similar). Then for all $t \in (t_0 + \varepsilon, t_1)$, where ε is small enough, we get

$$\int_{t_0+\varepsilon}^t \frac{-z''}{z'^2} \leq \int_{t_0+\varepsilon}^t n.$$

Thus $\frac{1}{z'(t)} \leq n(t - t_0 - \varepsilon) + \frac{1}{z'(t_0+\varepsilon)}$ and we get a contradiction taking $t \rightarrow t_1$. Therefore, the covering map $\Phi : \mathbb{R} \times L \rightarrow M$ is injective (see Remark 2.4) and Corollary 2.3 finishes the proof.

Now, suppose that there exists a point $p \in M$ with $\Delta \lambda(p) > 0$ and call L the leaf through p . Take $z \in L$ and $v \in T_z L$ an unit vector. Since

$$\text{Ric}_L(v) = \text{Ric}(v) - \frac{1}{\lambda} \left(U(U(\lambda)) + (n-2) \frac{U(\lambda)^2}{\lambda} \right),$$

we get $\text{Ric}_L(v) \geq -\frac{1}{\lambda} (U(U(\lambda)) + (n-1) \frac{U(\lambda)^2}{\lambda}) = \frac{\Delta \lambda}{\lambda}(z)$. But λ and $\Delta \lambda$ are constant through the orthogonal leaves, thus $\text{Ric}_L(v) \geq \frac{\Delta \lambda}{\lambda}(p) > 0$ for all unit vector $v \in T L$. Using the completeness of M and the theorem of Myers we conclude that L is compact. Lemma 3.1 and Proposition 2.2 show that M is globally a GRW space.

(3) Let $h : M \rightarrow \mathbb{R}$ be given by $h = U(U(\lambda)) - \frac{U(\lambda)^2}{\lambda}$. We consider two possibilities: there exists $p \in M$ with $h(p) > 0$ or $h(q) \leq 0$ for all $q \in M$.

Assume the second one. The function $z(t) = \ln \frac{\lambda(\gamma(t))}{\lambda(\gamma(0))}$ verifies $z'' \leq 0$. Therefore γ cannot return to $L_{\gamma(0)}$ and applying Corollary 2.3, M is globally a GRW space.

Suppose now that there exists $p \in M$ such that $h(p) > 0$ and take L the leaf through p . If $v \in T_z L$ is an unit vector then $v + U_z$ is lightlike and

$$0 \leq \text{Ric}(v + U_z) = \text{Ric}_L(v) + \frac{n-2}{\lambda} \left(\frac{U(\lambda)^2}{\lambda} - U(U(\lambda)) \right).$$

Then we get $\frac{n-2}{\lambda(z)} h(z) \leq \text{Ric}_L(v)$. But λ and h are constant through the orthogonal leaves. Hence $0 < \frac{n-2}{\lambda(p)} h(p) \leq \text{Ric}_L(v)$ for all unit vector $v \in TL$ and we can conclude as in the point 2. \square

We cannot use $\text{Ric}(U) \geq 0$ because in this case U would be parallel. If $\text{Ric}(u) \geq 0$ for all timelike vector u then it is said that the timelike convergence condition holds (TCC) and if $\text{Ric}(u) \geq 0$ for all lightlike vector u then it is said that the null convergence condition holds (NCC). We cannot suppose the more restrictive TCC condition because U would be parallel too, [15].

A condition like $\text{div } U \geq 0$ (resp. $\text{div } U \leq 0$), leads trivially to a GRW decomposition because $\lambda(\gamma(t))$ would be increasing (resp. decreasing).

Corollary 3.3. *Let (L, g_0) be a noncompact and complete Riemannian manifold and $M = \mathbb{S}^1 \times L$ endowed with a warped product metric $-dt^2 + f(t)^2 g_0$. If the NCC holds then f is constant.*

Causality hypotheses are frequently used in Lorentzian geometry, besides curvature hypotheses. Since the injectivity of Φ depends on the behavior of some timelike curves it seems natural to impose a causality condition to reach global decompositions. However, a hard condition like being globally hyperbolic is not sufficient to obtain a global product, as the following example shows.

Example 3.4. (Compare with Proposition 2 in [9].) Take $\tilde{M} = \mathbb{R}^2$ with the metric $-dt^2 + f(t)^2 dx^2$, where $f(t) = 4 + \sin(2\pi t)$. Call Γ the isometry group generated by $\Psi(t, x) = (t + 1, x + 1)$ and $\Pi : \tilde{M} \rightarrow M = \tilde{M}/\Gamma$ the projection. The vector field $V = \Pi_*(f \frac{\partial}{\partial t})$ is timelike, closed and conformal. The manifold M verifies any causality condition. In fact, $\Pi(\{(t, x) : t = x\})$ is a Cauchy hypersurface. But M does not split as a GRW, see Remark 2.4.

Lemma 3.5. *Let M be a complete Lorentzian manifold and V a timelike, closed and conformal vector field with unitary U . If there exists $t_0 > 0$ and $p \in M$ such that $\Phi_{t_0}(p) \in L_p$, being Φ the flow of U , then*

1. $\Phi_{t_0} : M \rightarrow M$ is an isometry.
2. If M is chronological, then the isometry group Ω generated by Φ_{t_0} is isomorphic to \mathbb{Z} and acts on M in a properly discontinuous manner.

Proof. (1) Use that V is conformal, Φ_t is foliated and $\lambda = |V|$ is constant through the orthogonal leaves.

(2) Suppose that the manifold is future oriented by the vector field U . We will construct for each $q \in M$ an open set Θ of q such that $\Phi_{nt_0}(\Theta) \cap \Theta = \emptyset$ for all $n \in \mathbb{Z} - \{0\}$. Recall that $\Phi : \mathbb{R} \times_f L_q \rightarrow M$ is a local isometry with $f(t) = \frac{\lambda(\Phi_q(t))}{\lambda(q)}$, see comments after Lemma 2.7. Take $k = \max\{f(t)^2 : t \in [-t_0, t_0]\}$ and the ball $W = B(q, \frac{\varepsilon}{2\sqrt{k}}) \subset L_q$, where $\varepsilon < t_0$ is small enough to $\Phi : (-\varepsilon, \varepsilon) \times_f W \rightarrow \Theta$ be an isometry and W a normal neighborhood of q . Given $\Phi_s(z) \in \Theta$, with $z = \exp_q(v) \in W$ and $s \in (-\varepsilon, \varepsilon)$, we can construct future timelike curves for $t \in [0, 1]$

$$\alpha(t) = \Phi \left(\frac{\varepsilon}{2}(t-1), \exp_q((1-t)v) \right),$$

$$\beta(t) = \Phi \left(\frac{\varepsilon}{2}t, \exp_q(tv) \right)$$

from $\Phi_{-\frac{\varepsilon}{2}}(z)$ to q and from q to $\Phi_{\frac{\varepsilon}{2}}(z)$ respectively.

Suppose $\Phi_{nt_0}(\Theta) \cap \Theta \neq \emptyset$, $n \in \mathbb{Z} - \{0\}$. Take $x, y \in \Theta$ with $y = \Phi_{nt_0}(x)$. The open set Θ is an union of segments $\Phi_z(-\varepsilon, \varepsilon)$ of integral curves of U , with $z \in W$, thus we have $x \in \Phi_{z_1}(-\varepsilon, \varepsilon)$, $y \in \Phi_{z_2}(-\varepsilon, \varepsilon)$. We can use a timelike curve α from $\Phi_{-(\varepsilon/2)}(z_2)$ to q and β from q to $\Phi_{(\varepsilon/2)}(z_1)$. The curve formed by the segment of Φ_{z_1} from $\Phi_{(\varepsilon/2)}(z_1)$ to $\Phi_{-(\varepsilon/2)}(z_2)$, α and β , is timelike and closed in contradiction with the chronology hypothesis. Thus, $\Phi_{nt_0}(\Theta) \cap \Theta = \emptyset$ for all $n \in \mathbb{Z} - \{0\}$.

Take the Riemannian metric $g_R = g + 2\omega \otimes \omega$. The group Ω is a group of isometries for g_R too. Since g_R is Riemannian, the existence of the above open set Θ for each $q \in M$ is sufficient to show that the action of Ω in M is properly discontinuous. \square

Note that we must use a Riemannian argument in the above proof because the analogous statement in Lorentzian geometry is not true.

Theorem 3.6. *Let M be a chronological complete Lorentzian manifold and U a nonparallel warped reference frame. Then M is a global GRW space or there is a Lorentzian covering map $\Psi : M \rightarrow \mathbb{S}^1 \times N$, where $(\mathbb{S}^1 \times N, -dt^2 + f^2g_N)$ is a Lorentzian warped product.*

Proof. Suppose M is not a GRW space. Then there is $p \in M$ such that the integral curve $\Phi_p(t)$ of U returns to L , the leaf of U^\perp through p (Remark 2.4). Take $t_0 = \inf\{t > 0 : \Phi_p(t) \in L\}$ and call $\lambda(t) = \lambda(\Phi_p(t))$, where $\lambda = |V|$ and V is the closed and conformal vector field with $U = \frac{V}{|V|}$ (Lemma 2.8). We can suppose that $\lambda'(0) > 0$, since $\lambda(t)$ is not constant. Then, $t_0 > 0$ and it is a minimum. Therefore $\Phi_p(t_0) \in L$ and applying the above lemma, Ω acts on M in a properly discontinuous manner. Using Corollary 2.3 it is easy to show that M/Ω is isometric to a Lorentzian warped product $(\mathbb{S}^1 \times N, -dt^2 + f^2g_N)$, where $N = L/\Omega$. \square

4. Perfect fluids

A Lorentzian four-dimensional manifold M is called a perfect fluid if there is a reference frame U and $\rho, \eta \in C^\infty(M)$ (the energy and pressure) such that the stress-energy tensor is $T = (\rho + \eta)\omega \otimes \omega + \eta g$, or equivalently $\text{Ric} = (\rho + \eta)\omega \otimes \omega + \frac{1}{2}(\rho - \eta)g$. A perfect fluid satisfies $U(\rho) = -(\rho + \eta) \text{div} U$ (energy equation) and $(\rho + \eta)\nabla_U U = -\nabla^\perp \eta$ (force equation), where $\nabla^\perp \eta$ is the component of $\nabla \eta$ orthogonal to U .

It is well known that any Robertson–Walker (RW) spacetime is a perfect fluid [12], and it is straightforward to see that a four-dimensional GRW spacetime is a perfect fluid if and only if it is a RW spacetime.

Moreover, in a RW spacetime we have the following basic relation between the warping function f and pressure and energy.

$$\frac{3f''}{f} = -\frac{1}{2}(\rho + 3\eta), \quad \rho' = -3(\rho + \eta)\frac{f'}{f}.$$

Eliminating f , the above equations implies the following equation of state in the open set $\rho + \eta \neq 0$.

$$\left(\frac{\rho'}{\rho + \eta}\right)' = \frac{1}{3}\left(\frac{\rho'}{\rho + \eta}\right)^2 + \frac{1}{2}(\rho + 3\eta). \tag{1}$$

A perfect fluid is called barotropic if it satisfies an equation of state $\eta = \eta(\rho)$. Observe that a RW perfect fluid $\mathbb{R} \times_f L$ is barotropic in the open set $\rho + \eta \neq 0$ and $\frac{d\rho}{d\eta} \neq 0$.

A natural question is which conditions on a perfect fluid make it a global Robertson–Walker space, [6,7]. Using the decomposition theorems of the previous section we can give an answer to this question. In fact, the following theorem shows that under mild conditions, the equation of state (1) give rise to a global decomposition as a RW space.

Theorem 4.1. *Let M be a (four-dimensional) noncompact spacetime with a barotropic perfect fluid (U, ρ, η) such that U is geodesic, $\frac{d\eta}{d\rho} \neq 0$, $\frac{d^2\eta}{d^2\rho} \neq 0$, $\rho + \eta \neq 0$ and $\rho > 0$ is not constant. If it verifies the equation of state (1), then either M is incomplete or a global Robertson–Walker space.*

Proof. We first show that U is closed, that is, $d\omega = 0$. From the force equation $g(X, \nabla \eta) = -(\rho + \eta)g(X, \nabla_U U)$ for all $X \in U^\perp$ we get $d\eta = h\omega$, being h certain function, and

$$-X(h) = X(g(U, \nabla \eta)) = g(U, \nabla_X \nabla \eta) = g(X, \nabla_U \nabla \eta) = 0.$$

Hence $0 = dh \wedge \omega + h d\omega = h d\omega$. Thus, to show $d\omega_p = 0$ it is enough to consider a critical point p of η . Call $\eta(t) = \eta(\gamma(t))$ and $\rho(t) = \rho(\gamma(t))$ where γ is the integral curve of U with $\gamma(0) = p$. Then $\eta' = 0$ if and only if $\rho' = 0$. If $\rho'(t) = 0$ for all $t \in (-\varepsilon, \varepsilon)$ then $\rho + 3\eta = 0$ in contradiction with $\frac{d^2\eta}{d^2\rho} \neq 0$. Thus there is a sequence t_n converging to 0 with $\rho'(t_n) \neq 0$ and therefore $d\omega_p = 0$.

Now, consider the self-adjoint endomorphism $A : U^\perp \rightarrow U^\perp$ given by $A(X) = \nabla_X U$. A straightforward computation give us $\text{Ric}(U) = -U(\text{div} U) - \|A\|^2$ (to prove this take $\{U_1, U_2, \dots, U_n\}$ a local orthonormal basis around an arbitrary point p , such that $\nabla_{U_i} U_j(p) = 0$ and $U_1(p) = U(p)$). From this, the energy equation and the equation of state (1), we get

$$\|A\|^2 = \left(\frac{\rho'}{\rho + \eta}\right)' - \frac{1}{2}(\rho + 3\eta) = \frac{1}{3}\left(\frac{\rho'}{\rho + \eta}\right)^2 = \frac{1}{3}(\text{trace } A)^2.$$

Then $A = \frac{\text{div} U}{3} \text{id}$ and U is orthogonally conformal. We know that η and ρ are constant on the orthogonal leaves. The energy equation implies that $\text{div} U$ is constant through the orthogonal leaves, that is, $\nabla \text{div} U$ is proportional to U and hence U is a warped reference frame. If U were parallel then M would be locally a direct product, which implies that ρ is constant, [12].

If M is complete, we apply Theorem 3.2(2) if $\rho + \eta < 0$ and Theorem 3.2(3) if $\rho + \eta > 0$. (See the expression of the Ricci tensor of a perfect fluid at the beginning of this section.) \square

5. Lightlike sectional curvature and timelike hypersurfaces

Let M be a Lorentzian manifold and U a reference frame. We can define a curvature for degenerate planes as follows, [8]. Take Π a degenerate plane and a basis $\{u, v\}$, where u is the unique lightlike vector in Π with $g(u, U) = 1$. The lightlike sectional curvature of Π is

$$\mathcal{K}_U(\Pi) = \frac{g(R(v, u, u), v)}{g(v, v)}.$$

This sectional curvature depends on the choice of the reference frame U , but its sign does not change if we choose another vector field. In fact, if U' is another reference frame then $\mathcal{K}_U(\Pi) = g(u, U')^2 \mathcal{K}_{U'}(\Pi)$ where $u \in \Pi$ is the unique lightlike vector such that $g(u, U) = 1$. Thus, it makes sense to say positive lightlike sectional curvature or negative lightlike sectional curvature.

If U is geodesic, $n > 3$ and \mathcal{K}_U is a never zero point function then U is a warped reference frame [9,10] and, under completeness hypothesis, it follows from Corollary 2.3 that M is covered by a Robertson–Walker space $\mathbb{R} \times L$. As an application of the decomposition results presented above, we obtain conditions on the lightlike sectional curvature which ensure the global decomposition of M . First, we give some relations between lightlike sectional curvature and Ricci curvature.

Lemma 5.1. *Let M be a Lorentzian manifold and U a timelike and unit vector field. If u is a lightlike vector with $g(u, U) = 1$, then*

$$\text{Ric}(u) = \sum_{i=1}^{n-2} \mathcal{K}_U(\text{span}(u, e_i)),$$

where $\{e_1, \dots, e_{n-2}\}$ is an orthonormal basis of $u^\perp \cap U^\perp$.

Lemma 5.2. *Let $M = I \times_f L$ be a GRW space and u a lightlike vector such that $g(u, \frac{\partial}{\partial t}) = 1$. If $u = -\frac{\partial}{\partial t} + w$, $w \in TL$, and $\Pi = \text{span}(u, v)$ is a degenerate plane, where $v \in TL$, then $\mathcal{K}_{\frac{\partial}{\partial t}}(\Pi) = \frac{K^L(\text{span}(v, w)) + f'^2 - f''f}{f^2}$, where K^L is the sectional curvature of L . If moreover L has constant sectional curvature then $\text{Ric}(u) = (n - 2)\mathcal{K}_{\frac{\partial}{\partial t}}(\Pi)$.*

Proof. It is a straightforward calculation. If L has constant sectional curvature, then $\mathcal{K}_{\frac{\partial}{\partial t}}$ is a point function and the above lemma gives us $\text{Ric}(u) = (n - 2)\mathcal{K}_{\frac{\partial}{\partial t}}(\Pi)$. \square

Lemma 5.3. *Let $M = \mathbb{R} \times_f L$ be a GRW space. If M is lightlike complete and $\text{Ric}(u) > 0$ for all lightlike vector u , then $\text{Ric}_L(v) > 0$ for all $v \in TL$.*

Proof. Suppose there is $v \in TL$ unit vector such that $\text{Ric}_L(v) \leq 0$. Then

$$0 < \text{Ric}\left(\frac{\partial}{\partial t} + v\right) = \text{Ric}_L(v) + \frac{n-2}{f} \left(\frac{f'^2}{f} - f''\right).$$

If we call $y = \ln f$ then $y'' = \frac{f''f - f'^2}{f^2} < 0$. We can suppose $y'(0) > 0$. Now, $\int_{-\infty}^0 e^y \leq \int_{-\infty}^0 e^{y'(0)t+y(0)} < \infty$ and we conclude from [15] that M is lightlike incomplete. \square

Proposition 5.4. *Let M be a noncompact and complete Lorentzian manifold with $n > 3$ and U a geodesic reference frame. If the lightlike sectional curvature \mathcal{K}_U is a positive point function, then M is globally a Robertson–Walker space $\mathbb{R} \times L$ with L of constant positive sectional curvature.*

Proof. As we said at the beginning of this section, M is covered by a RW space $\mathbb{R} \times L$. From Lemmas 5.1, 5.2 and 5.3, L has positive constant curvature and therefore is compact. Now, we apply Lemma 3.1. \square

Theorem 5.5. *Let M be a noncompact and complete Lorentzian manifold with $n > 3$ and U a geodesic reference frame. Suppose that the lightlike sectional curvature \mathcal{K}_U is a never zero point function such that $\frac{1}{n-2} \text{Ric}(U) < \mathcal{K}_U$. Then M is globally a Robertson–Walker space.*

Proof. If U were parallel then $\text{Ric}(U) = 0$ and the above proposition give us the desired result. If U is not parallel, take $v, w \in U^\perp$ unit and orthogonal vectors and the degenerate plane $\Pi = \text{span}(-U + w, v)$. Since M is locally a RW space, we can apply Lemma 5.2 and we get $\text{Ric}(-U + w) = (n - 2)\mathcal{K}_U(\Pi)$. But $\text{Ric}(-U + w) = \text{Ric}(U) + \text{Ric}(w)$ and therefore $\text{Ric}(w) > 0$ for all $w \in U^\perp$. Now, we apply Theorem 3.2(2). \square

Spacelike hypersurfaces are widely studied in General Relativity due to their role as initial data hypersurfaces in the Cauchy problem. On the other hand, a k -dimensional timelike submanifold can be interpreted as the history of a $(k - 1)$ -dimensional spacelike submanifold. Timelike totally umbilic hypersurfaces are called photon surfaces and were studied in [3]. Now, we show that in a local GRW spaces, photon surfaces are global GRW spaces under certain curvature condition.

Lemma 5.6. *Let M be a Lorentzian manifold and U a reference frame. If S is a timelike totally umbilic hypersurface and $u \in TS$ is a lightlike vector with $g(u, U) = 1$, then*

$$\text{Ric}_S(u) = \sum_{i=1}^{n-3} \mathcal{K}_U(\text{span}(u, e_i)),$$

where $\{e_1, \dots, e_{n-3}\}$ is an orthonormal basis of $u^\perp \cap U^\perp \cap TS$.

Proof. Suppose that $II(X, Y) = g(X, Y)N$ for all $X, Y \in TS$. Take $u \in TS$ a lightlike vector such that $g(u, U) = 1$ and $\{v_1, \dots, v_n\}$ an orthonormal basis with v_n the unitary of N . By the Gauss equation and the fact that u is lightlike

$$\text{Ric}(u) = \sum_{i=1}^n \varepsilon_i g(R(v_i, u, u), v_i) = \text{Ric}_S(u) + \mathcal{K}_U(\text{span}(u, v_n)).$$

Now, take $\{e_1, \dots, e_{n-2}\}$ an orthonormal basis of $u^\perp \cap U^\perp$ such that e_{n-2} is the unitary of the projection of N on $u^\perp \cap U^\perp$. Then $\text{span}(u, v_n) = \text{span}(u, e_{n-2})$ and using Lemma 5.1, $\text{Ric}(u) = \sum_{i=1}^{n-2} \mathcal{K}_U(\text{span}(u, e_i))$. Therefore $\text{Ric}_S(u) = \sum_{i=1}^{n-3} \mathcal{K}_U(\text{span}(u, e_i))$. \square

Theorem 5.7. *Let M be a complete Lorentzian manifold with $n \geq 4$ and U a warped reference frame. If S is a timelike, noncompact, complete and totally umbilic hypersurface of M with never zero mean curvature such that $\mathcal{K}_U(\Pi) \geq 0$ for all degenerate plane Π tangent to S , then S is globally a GRW space.*

Proof. Take V closed and conformal with $U = \frac{V}{|V|}$ (Lemma 2.8). Suppose $\nabla V = a \cdot \text{id}$ and $II(X, Y) = g(X, Y)N$ for all $X, Y \in TS$, where N is the mean curvature vector field of S . If $V = \alpha \frac{N}{|N|} + W$, where $W \in TS$, then it follows that $\nabla_X^S W = (a + g(N, V))X$ for all $X \in TS$, i.e. W is closed and conformal in S .

Suppose first that W is parallel in S , that is, $a = -g(N, V)$. Call $c = g(W, W)$ and take the function on S given by $h(p) = g(V, V)$ which is constant on the orthogonal leaves of W . On the other hand, $W(h) = 2ac$ and $W(a) = -W(g(V, N)) = cg(N, N) + a \frac{W(g(N, N))}{2g(N, N)}$. If γ is an integral curve of W and $h(t) = h(\gamma(t))$, then $h''(t_0) > 0$ at each critical point t_0 of h , showing that h cannot be periodic. Therefore Remark 2.4 and Corollary 2.3 gives us that S is globally a direct product.

Suppose now that W is not parallel. The above lemma shows that S satisfies the NCC condition, and Theorem 3.2(3) finishes the proof. \square

Corollary 5.8. *Let M be a complete Lorentzian manifold with $n \geq 4$ and U a parallel reference frame. If $K(\Pi) \geq 0$ for all plane $\Pi \in U^\perp$, then any timelike, noncompact, complete and totally umbilic hypersurface of M with never zero mean curvature is globally a GRW space.*

Proof. If $\Pi = \text{span}(-U + v, w)$ is a degenerate plane to the hypersurface, from Lemma 5.2 we have $\mathcal{K}_U(\Pi) = K(\text{span}(v, w)) \geq 0$. \square

In general, it is a difficult question to check the completeness of a timelike hypersurface S of a complete Lorentzian manifold M . In the case of totally umbilic hypersurfaces of a GRW space we can give a simple criterium.

If U is a reference frame on M , we can define the hyperbolic angle $\theta \in [0, \infty)$ between S and U as the hyperbolic angle between U and the projection of U in S . More explicitly, if N_0 is the normal unit vector field to S and $U = \alpha N_0 + W$, $W \in N_0^\perp$, then θ is characterized by

$$\cosh \theta = \frac{-g(U, W)}{\sqrt{-g(W, W)}}.$$

Proposition 5.9. *Let $\mathbb{R} \times_f L$ be a complete GRW space. If S is a timelike, closed (as a subset of $\mathbb{R} \times L$) and totally umbilic hypersurface of $\mathbb{R} \times_f L$ such that the hyperbolic angle between $\frac{\partial}{\partial t}$ and S is bounded, then S is complete.*

Proof. Let N_0 be the normal unit vector field to S and $V = f \frac{\partial}{\partial t}$. Put $II(X, Y) = g(X, Y)N$ and $V = \alpha N_0 + W$, $W \in N_0^\perp$. We already know that W is closed and conformal in S and thus $U = \frac{W}{|W|}$ is a warped reference frame. The Riemannian metric $g_R = g + 2dt^2 = dt^2 + f^2 g_0$ is complete, [12,15], and we have $g_R(U, U) = -1 + 2g(\frac{\partial}{\partial t}, U)^2 = -1 + 2 \cosh^2 \theta$, where θ is the

hyperbolic angle between $\frac{\partial}{\partial t}$ and S . Therefore $g_R(U, U)$ is bounded and thus it is a complete vector field. Using Corollary 2.3 there is a Lorentzian covering $\mathbb{R} \times_h Q \rightarrow S$, where $h(s) = \frac{|W|_{\gamma(s)}}{|W|_{\gamma(0)}}$ and γ is an integral curve of U . We can suppose without loss of generality $|W|_{\gamma(0)} = 1$. Since S is totally umbilic, its lightlike geodesics are geodesics of $\mathbb{R} \times L$, thus S and $\mathbb{R} \times_h Q$ are lightlike complete and hence Q is complete. If we show that

$$\int_0^\infty \frac{h(s)}{\sqrt{1+h(s)^2}} ds = \int_{-\infty}^0 \frac{h(s)}{\sqrt{1+h(s)^2}} ds = \infty,$$

then $\mathbb{R} \times_h Q$ is complete and so is S , [15]. Let $T : \mathbb{R} \times L \rightarrow \mathbb{R}$ be the projection and consider the diffeomorphism $\rho = T \circ \gamma : \mathbb{R} \rightarrow \mathbb{R}$. Then $\frac{d}{ds}\rho = -g(\frac{\partial}{\partial t}, U) = \cosh \theta \leq c$, where c is a certain constant. Since $g(W, W)_{\gamma(s)} \leq g(V, V)_{\gamma(s)} = -f^2(T(\gamma(s)))$ it follows that

$$\int_0^\infty \frac{f \circ \rho}{\sqrt{1+(f \circ \rho)^2}} ds \leq \int_0^\infty \frac{h}{\sqrt{1+h^2}} ds,$$

but

$$\infty = \int_0^\infty \frac{f(t)}{\sqrt{1+f(t)^2}} dt \leq c \int_0^\infty \frac{f \circ \rho(s)}{\sqrt{1+(f \circ \rho(s))^2}} ds. \quad \square$$

Example 5.10. Take $M = \mathbb{S}^1 \times \mathbb{R} \times \mathbb{S}^2$ with metric $g = -dt^2 + f(t)^2 g_0$ where $f(t) = 2 + \cos t$, $g_0 = dx^2 + \frac{1}{2} g_S$ and g_S is the canonical metric in \mathbb{S}^2 . Take the universal covering $\varrho : \tilde{M} \rightarrow M$, $F : \tilde{M} \rightarrow \mathbb{R}$ given by $F(t, x, p) = x - \int_0^t \frac{1}{2f(s)} ds$ and $\tilde{S} = F^{-1}(0)$. It follows that $\nabla F = \frac{1}{2f} \frac{\partial}{\partial t} + \frac{1}{f^2} \frac{\partial}{\partial x}$ and $(\nabla F)^\perp = T\mathbb{S}^2 \oplus \text{span}(\frac{\partial}{\partial t} + \frac{1}{2f} \frac{\partial}{\partial x})$. Now, a direct computation gives us that $\text{Hess}_F(X, Y) = \frac{f'}{2f^2} g(X, Y)$ for all $X, Y \in \nabla F^\perp$. Hence \tilde{S} is a timelike, closed, noncompact and totally umbilic hypersurface of \tilde{M} . Since the hyperbolic angle between \tilde{S} and $\frac{\partial}{\partial t}$ is constant, it follows that \tilde{S} is complete.

Take now Π a degenerate plane tangent to \tilde{S} . Suppose $\Pi = \text{span}(u, v)$ where $g(u, u) = g(u, v) = g(\frac{\partial}{\partial t}, v) = 0$ and $g(v, v) = g(u, \frac{\partial}{\partial t}) = 1$. Then $u = -\frac{\partial}{\partial t} + w$, where $g(w, w) = 1$ and $g(w, \frac{\partial}{\partial t}) = g(v, w) = 0$.

Since Π is tangent to \tilde{S} , the structure of $(\nabla F)^\perp$ implies that $v \in T\mathbb{S}^2$ and the relation $g(u, \nabla F) = 0$ that $w = -\frac{1}{2f} \frac{\partial}{\partial x} + X$, where $X \in T\mathbb{S}^2$. Now, if we denote K^0 the sectional curvature of $(\mathbb{R} \times \mathbb{S}^2, g_0)$, then $K^0(\text{span}(v, w)) = \frac{3}{2}$, therefore Lemma 5.2 give us

$$\mathcal{K}_{\frac{\partial}{\partial t}}(\Pi) = \frac{K^0(\text{span}(v, w)) + f'^2 - f''f}{f^2} = \frac{5/2 + 2 \cos(t)}{f^2} > 0.$$

The hypersurface $S = \varrho(\tilde{S})$ is not compact. Applying Theorem 5.7 it is a global GRW space.

Example 5.11. From Corollary 5.8 we can deduce the well-known fact that the anti de Sitter space $\mathbb{H}_1^n(r) = \{p \in \mathbb{R}_2^{n+1} : \langle p, p \rangle = -r^2\}$ cannot be immersed completely in Minkowski space \mathbb{R}_1^{n+1} . In fact, if $S \subset \mathbb{R}_1^{n+1}$ is a complete Lorentzian manifold with constant negative curvature, then S is totally umbilic, [12]. On the other hand, the Gauss equation implies that it has never zero mean curvature vector. Using the corollary, S is globally a GRW space. But a GRW space of constant negative curvature is incomplete, [15].

6. Uniqueness of GRW decompositions

In [4], the uniqueness of direct product decompositions of a Riemannian manifold is studied, where only products into indecomposable factors are considered and the uniqueness is understood in the following sense: a direct product decomposition into indecomposable factors is unique if the corresponding foliations are uniquely determined. Euclidean space admits more than one direct product decomposition, but this is essentially the only Riemannian manifold with this property.

We know that in a GRW decomposition $\mathbb{R} \times_f L$ the vector field $f \frac{\partial}{\partial t}$ is timelike, closed and conformal. Conversely, a timelike, closed and conformal vector field gives rise to a warped reference frame and therefore to a local GRW decomposition. Thus, to deal with the uniqueness of GRW decompositions, it is sufficient to study how many timelike, closed and conformal vector fields can exist on a Lorentzian manifold, up to scalar multiplication.

The uniqueness question has been recently analyzed in [16] for static spacetimes. A possible interest of this kind of results for GRW spacetimes comes from the recently introduced “big rip” models which try to explain the accelerated expansion of the Universe. The qualitative properties of the models depend on the behavior of the warping function f , [5]. So we must ensure that the qualitative behavior of this function (or the function itself) is univocally determined.

Example 6.1. The de Sitter space $\mathbb{S}_1^n(r) = \{p \in \mathbb{R}_1^{n+1} : \langle p, p \rangle = r^2\}$ admits different timelike, closed and conformal vector fields. It is a straightforward consequence of the fact that it is a GRW, homogeneous and isotropic. This vector fields are obtained fixing $p_0 \in \mathbb{R}_1^{n+1}$ with $\langle p_0, p_0 \rangle = -1$ and taking $V_p = p_0 - \frac{\langle p_0, p \rangle}{r^2} p$. The different decompositions of the de Sitter space are $\mathbb{R} \times_f \mathbb{S}^{n-1}(\mu)$ where $f(t) = \frac{r}{\mu} \cosh(\frac{1}{r}t + b)$ with $r, \mu \in \mathbb{R}^+$ and $b \in \mathbb{R}$.

Lemma 6.2. Let M be a complete Riemannian manifold with $n \geq 2$ and V a closed and conformal vector field. Call U its unitary and $A = \{p \in M : \lambda(p) \neq 0\}$, where $\lambda = |V|$. If the equation $U(U(\lambda)) = -c^2\lambda$ holds in A , then M is isometric to a sphere of curvature c^2 , $\mathbb{S}^n(\frac{1}{c})$.

Proof. It is known that V has at most two zeroes, [14]. Suppose $\nabla V = a \cdot \text{id}$. Then $\nabla a = U(a)U = U(U(\lambda))U = -c^2\lambda U = -c^2V$ and $\text{Hess}_a = -c^2ag$ in M . Using [14] (Theorem 2, III), we conclude that M is a sphere of curvature c^2 . \square

The following theorem shows that the sign of the lightlike sectional curvature is an obstruction for the existence of more than one GRW decomposition. In addition, we show that the de Sitter space is the unique complete Lorentzian manifold with more than one nontrivial GRW decomposition.

Theorem 6.3. Let M be a Lorentzian manifold with $n \geq 3$ and V a timelike, closed and conformal vector field. If there exists another closed and conformal vector field W , without zeros and linearly independent to V at some point $p \in M$, then

1. There are degenerate planes Π of T_pM such that $\mathcal{K}_U(\Pi) = 0$, being U the unitary of V .
2. If V is not parallel and M is complete then it is isometric to a de Sitter space.

Proof. (1) Suppose $\nabla W = b \cdot \text{id}$ and $W = \alpha U + X$, where $U = \frac{V}{|V|}$ and $X \in U^\perp$. Call $-\lambda^2 = g(V, V)$, $\sigma^2 = g(X, X)$, $A = \{p \in M : \sigma(p) \neq 0\}$ and define in A the vector field $F = \frac{X}{\sigma}$.

Since $\nabla_U X = (b - U(\alpha))U$ and $0 = Ug(X, U) = g(\nabla_U X, U)$ we have $b = U(\alpha)$ and $\nabla_U X = 0$. Now, $\nabla_X X = \nabla_X(W - \alpha U) = (b - \alpha U(\ln \lambda))X - X(\alpha)U$, and taking derivative along X in $\sigma^2 = g(X, X)$ we have $F(\sigma) = b - \alpha U(\ln \lambda)$. Thus we get the equation $U(\alpha) = \alpha U(\ln \lambda) + F(\sigma)$. On the other hand, $F(\alpha) = -\sigma U(\ln \lambda)$.

Taking derivative along U in $\sigma^2 = g(X, X)$ we get $U(\sigma) = 0$ and we can easily check that $[U, \lambda F] = 0$. From the expression $U(\lambda F(\alpha)) = \lambda F(U(\alpha))$ we obtain the equation

$$F(F(\sigma)) = -\sigma U(U(\ln \lambda)). \tag{2}$$

Take $p \in A$ and call $\lambda(t) = \lambda(\gamma(t))$ and $\sigma(s) = \sigma(\zeta(s))$, where γ is the integral curve of U and ζ of F through p . Then M is locally isometric to $((-\varepsilon, \varepsilon) \times L_p, -dt^2 + (\frac{\lambda(t)}{\lambda(0)})^2 g_{L_p})$ being $\frac{\partial}{\partial t}$ identified with U . Since X is closed and conformal in U^\perp , (L_p, g_{L_p}) is locally isometric to $((-\delta, \delta) \times S, ds^2 + (\frac{\sigma(s)}{\sigma(0)})^2 g_S)$, where $\frac{\partial}{\partial s}$ is identified with F , see the comments after Lemma 2.7.

Consider the degenerate plane $\Pi = \text{span}(-U_p + F_p, v)$, where v is a unit vector orthogonal to U_p and F_p . From Lemma 5.2 we obtain $\mathcal{K}_U(\Pi) = K^{L_p}(\text{span}(v, X)) - (\ln \lambda)''(0)$. But the curvature formulas for a warped product gives us $K^{L_p}(\text{span}(v, X)) = -\frac{\sigma''(0)}{\sigma(0)}$ and from Eq. (2) we get $(\ln \lambda)''(0) = -\frac{\sigma''(0)}{\sigma(0)}$. Therefore $\mathcal{K}_U(\Pi) = 0$.

(2) Suppose now that M is complete and take $\gamma : \mathbb{R} \rightarrow M$ an integral curve of U with $\gamma(0) = p \in A$. Since $U(\sigma) = 0$ we have that $\gamma(t) \in A$ for all $t \in \mathbb{R}$. Then, using $[U, \lambda F] = 0$, there is a constant k such that $F(F(\sigma))(\gamma(t)) = -\frac{k}{\lambda^2}\sigma$ for all t , and therefore $\lambda^2(\ln \lambda)'' = k$.

If $k < 0$ the solutions of this differential equation are not positive on the whole \mathbb{R} and if $k = 0$ we obtain an incomplete warped product. If $k > 0$ the solutions are $\lambda(t) = \frac{\cosh(\sqrt{k}ct+b)}{c}$ where $c > 0$. Since $\lambda(t)$ is not periodic, γ cannot return to L_p and then by Corollary 2.3, M is isometric to the GRW space $\mathbb{R} \times_{\frac{\lambda(t)}{\lambda(p)}} L_p$.

Now, for any $q \in L_p$ we have $F(F(\sigma))(q) = -\sigma(p)U(U(\ln \lambda))(p) = -\frac{k}{\lambda^2}\sigma$ and the above lemma says that L_p is an euclidean sphere of curvature $\frac{k}{\lambda(p)^2}$. Then, comparing the warping function with that of Example 6.1, M is isometric to the de Sitter space of constant curvature c^2k . \square

In the case $\dim M = 2$, M complete and V nonparallel, the orthogonal leaves of V are trivially isometric to (\mathbb{R}, dx^2) or (\mathbb{S}^1, dx^2) . We can obtain as in the above proof that $\lambda(t) = \frac{\cosh(\sqrt{k}at+b)}{b}$ and therefore M is isometric to the 2-de Sitter space or its universal covering.

The completeness hypothesis in Theorem 6.3(2) is necessary as is shown by the following example.

Example 6.4. Take (L, g_0) any Riemannian manifold and consider $M = (0, \infty) \times \mathbb{R} \times L$ with metric $g = -dt^2 + t^2(ds^2 + e^{2s}g_0)$. The vector fields $t \frac{\partial}{\partial t}$ and $(t - e^s) \frac{\partial}{\partial t} + \frac{e^s}{t} \frac{\partial}{\partial s}$ are closed and conformal. Observe that the second vector field is not timelike on all the spacetime.

Corollary 6.5. *The Friedmann models admit an unique GRW decomposition, even locally.*

Proof. From Friedmann equation and [Lemma 5.2](#) we get $\mathcal{K}_{\frac{\partial}{\partial t}}(\Pi) > 0$ for any degenerate plane Π . \square

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