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Equivalence of Cauchy singular boundary and b-boundary in $O(3)$-reducible space–times ☆

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ABSTRACT

The singular Cauchy boundary construction for a space–time depends on an arbitrary choice of a timelike vector field. Nevertheless, if the Levi-Civita connection is reducible to an $O(3)$-structure, the construction is well defined. In this paper we show that in this case it is homeomorphic to Schmidt b-boundary.

1. Introduction

Let (M, g) be an m-dimensional space–time, $m \geq 2$. B. Schmidt introduced the notion of b-completion to attach a singular point to every inextensible timelike curve with finite fibre Length, [1]. This notion is relevant because it attaches a singularity to every inextensible timelike curve with finite proper time and bounded acceleration, in particular to inextensible timelike geodesics, [2]. The resulting set formed by the space–time and its singularities is equipped with a topology. The construction is based on the introduction of a Riemannian metric g^{ω} in a connected component of the frame bundle LM, which depends only on the space–time data. Right translations by elements of the structure group are uniformly continuous, thus the Cauchy completion $\overline{LM}$ of the associated metric space inherits a natural action of the structure group. The orbit space with the quotient topology is the b-completion $\overline{M}_b$ of the space–time. The same construction can be made from a connected component of the orthonormal frame bundle OM and we have the same result, [3,4].

In general b-completion is not Hausdorff. This happens in some classical space–times, for example in Friedmann spaces. Moreover in the closed Friedmann space, Big bang and Big crunch are identified as the same singular point in its b-completion, [5–7].

An attempt to improve those drawbacks can be done introducing a Riemannian metric directly in the space–time such that it depends only on the space–time data. The Cauchy singular boundary introduced in [8] is an example of it. The authors use the Sachs metric g_s^+ on the unitary bundle UM and a section $\xi : M \to UM$ to define a Riemannian metric $g^+ = \xi^* g_s^+$ in M. They found a sufficient condition such that the construction is unambiguous.

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It seems interesting to know possible relationships between different notions of singularities in a space–time. In this note we show that when the above construction is well defined, then the Cauchy singular completion is homeomorphic to Schmidt b-completion.

A survey on singularity theory can be found in [9] and the references therein.

2. Cauchy singular boundary

We recall the construction of the Cauchy singular boundary. Let (M, g) be a space–time, and $\pi^U : UM \to M$ the timelike unit bundle, where $UM = \{X \in TM/g(X, X) = -1\}$. It is a regular submanifold of TM and can be seen as an associated bundle of the orthonormal frame bundle $\pi : OM \longrightarrow M$. In fact, take the Minkowski space $(\mathbb{R}^m, \eta)$ and define $\aleph = \{h \in \mathbb{R}^m/\eta(h, h) = -1\}$ which is diffeomorphic to $O_1(m-1)/O(m-1)$. The group $O_1(m-1)$ acts in an obvious way on OM on the right and on $\aleph$ on the left, thus it defines a right action on $OM \times \aleph$ whose quotient is denoted $OM \times_{O_1(m-1)} \aleph$. The map $\psi : OM \times_{O_1(m-1)} \aleph \to UM$ given by $\psi([u, h]) = u(h)$ is an isomorphism, where $u \in OM$ is interpreted as the isometry $u : (\mathbb{R}^m, \eta) \to (T_{\pi(u)}M, g_{\pi(u)})$.

The Levi-Civita connection ∇ of (M, g) defines a connection ∇^U over the map π^U, $\nabla^U : \mathfrak{X}(UM) \times \mathfrak{X}(\pi^U) \to \mathfrak{X}(\pi^U)$, being $\mathfrak{X}(\pi^U)$ the module of vector fields over π^U, [10]. Let $\{E_0, \dots, E_{m-1}\}$ be a local basis of vector fields on an open set V of M. An element $A \in \mathfrak{X}(\pi^U)$ has a local expression $A = \sum_{i=0}^{m-1} A^i E_i \circ \pi^U$ on $\left(\pi^U\right)^{-1}(V)$, that is, $\{E_0 \circ \pi^U, \dots, E_{m-1} \circ \pi^U\}$ is a local basis for $\mathfrak{X}(\pi^U)$ and the component functions A^i are differentiable on $\left(\pi^U\right)^{-1}(V)$. Take $Y \in T_aUM$, and $a \in \left(\pi^U\right)^{-1}(V)$, then

$$\nabla^U_Y A = \sum_{i=0}^{m-1} Y(A^i) E_{i|\pi^U(a)} + \sum_{i=0}^{m-1} A^i(a) \nabla_{\pi^U_{*a}(Y)} E_i.$$

In particular, the canonical inclusion $I : UM \to TM$ is an element of $\mathfrak{X}(\pi^U)$, thus it defines a map

$$\begin{aligned} \Theta &: TUM \to TM \\ Y &\mapsto \nabla^U_Y I. \end{aligned}$$

The Sachs metric is the Riemannian metric on UM defined by

$$g_s^+(Y, Z) = g(\pi^U_{*a}Y, \pi^U_{*a}Z) + 2g(\pi^U_{*a}Y, I(a))g(\pi^U_{*a}Z, I(a)) + g(\Theta(Y), \Theta(Z)),$$

for $Y, Z \in T_aUM$, [11].

Fixed a section $X : M \to UM$, we have a Riemannian metric on M given by $g^+ = X^* g_s^+$. The Cauchy completion with the induced distance is the Cauchy singular completion $\overline{M_c}$ of the space–time. The Cauchy singular boundary is $\partial M_c = \overline{M_c} - M$.

3. Equivalence with Schmidt *b*-completion

There is a 1:1 correspondence between reductions of a principal bundle to a subfibre bundle with closed structure group, and sections of the associated bundle. In our case, a section $X : M \to UM$ has an associated $O(m-1)$-structure. The Cauchy singular completion depends on the choice of this section X.

Suppose that the Levi-Civita connection of (M, g) is reducible to an $O(m-1)$-structure. In this case we can chose its associated section to do the construction. We must be sure that if the connection is reducible to another $O(m-1)$-structure (i.e. associated to another section $Y : M \to UM$), the Cauchy singular completion is the same. In this case, the construction is well defined in the sense that it only depends on the space–time data.

The key point is that a connection on a principal bundle is reducible to a subbundle with closed structure group if and only if its associate section is parallel, [12]. This allow us to prove the following result.

Theorem 1. *Let (M, g) be an m-dimensional space–time such that its Levi-Civita connection is reducible to an $O(m-1)$-structure. Then the Cauchy singular completion induced by the associated section does not depend on the particular $O(m-1)$-structure. Moreover it is homeomorphic to b-completion.*

Proof. We compute $g^+ = X^* g_s^+$. Let $x \in M$ be a fixed point and $\{X_0, \dots, X_{m-1}\}$ an orthonormal basis of local vector fields in an open set which contains the point x, and such that $X_0 = X$, the parallel vector field associated to the $O(m-1)$-structure where the connection is reducible. Take $V \in T_xM$, then

$$\begin{aligned} \Theta(X_{*x}V) &= \sum_{k=0}^{m-1} X_{*x}V(I^k) X_{k}{}_{|_x} + \sum_{k=0}^{m-1} I^k(X(x)) \nabla_{(\pi^U)_{*X(x)} X_{*x} V} X_k \\ &= \nabla_V X = 0, \end{aligned}$$

where we use the fact that $I^k(X(x)) = \delta_0^k$ is the k-component of $X_{|x}$ in the basis $\{X_0, \dots, X_{m-1}\}$.

Thus if $V, W \in \mathfrak{X}(M)$, we have

$$g^+(V, W) = g(V, W) + 2g(V, X)g(W, X),$$

with X timelike unitary and parallel. It is well know that two metrics g and g^+ with the above relation have the same Levi-Civita connection, thus they have the same b-completion. On the other hand, (M, g^+) is a Riemannian manifold and its b-completion is just its Cauchy completion $(\overline{M}, d)$ computed with its associated distance d, [1]. Thus $\overline{M}_c = (\overline{M}, d) = \overline{M}_b$ and this proves both claims. □

Observe that there are two kinds of reductions. First the fibre bundle must be reducible to an $O(m-1)$-structure. This does not impose any restriction because it is equivalent to the existence of a timelike vector field, and this is an usual hypothesis in the definition of space–time. On the other hand, the Levi-Civita connection must be reducible to an $O(m-1)$-structure, and this is a too strong restriction. In fact, the existence of a timelike and parallel vector field in a space–time implies that locally the space–time is a direct product, and there are no classic space–times with such a vector field.

Observe that if the space–time is globally a direct product, it is possible to compute its b-completion as a product of its factors, [3,4].

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