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ORIGINAL PAPER

Curvature and conjugate points in Lorentz symmetric spaces

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Abstract We give some relations between conjugate points and curvature in a locally symmetric Lorentzian manifold. In the compact case, we show that the sectional curvature of timelike planes is non positive, and the lightlike sectional curvature of null planes is non negative. We also compute the lightlike conjugate loci of Cahen–Wallach manifolds, which are an important family of symmetric Lorentzian spaces.

Keywords Lorentz symmetric space · Cahen–Wallach space · Conjugate point · Curvature

Mathematics Subject Classification (2000) Primary 53C50 · Secondary 53C35 · 53C22

1 Introduction

In this article, we obtain geometric consequences from the study of conjugate points in Lorentz symmetric spaces. One of them is suggested by the classification theorem for simply connected Lorentz symmetric spaces. It is easy to see that there exist compact quotients of

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some spaces in this classification which have non-positive curvature for all timelike planes, so it is natural to ask: given M a compact (non necessarily simply connected) Lorentz symmetric space, does M have non-positive curvature for all its timelike planes? A first attempt to tackle the question is to use $\tilde{M}$, the universal covering space of M, which, by the classification theorem, is a direct product of a Riemannian symmetric space and a Lorentz factor, see Sect. 4. Now, M is a compact quotient of $\tilde{M}$ by a suitable group of isometries, but M may not be a product with compact factors nor $\tilde{M}$ compact in general, so we cannot apply known results on the curvature of compact Riemann symmetric spaces.

We give a positive answer to this question in Theorem 2, and apply it to show that in compact Lorentz symmetric spaces, there are not conjugate points along timelike geodesics if and only if they are flat. On the other hand, if furthermore dim $M \geq 3$, there are not conjugate points along lightlike geodesics if and only if they have constant curvature $k \leq 0$. This exhibits a subtle difference between conjugate points on a timelike and a lightlike geodesic.

We apply some of the above ideas to Cahen–Wallach manifolds, i.e., Lorentz symmetric spaces diffeomorphic to $\mathbb{R}^{n+2}$ and parameterized by a symmetric endomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$, which were introduced in [3] to classify simply connected Lorentz symmetric spaces. We establish a relationship between the eigenvalues of f and those of the Jacobi operator of any vector, which allows us to study the existence and location of conjugate points. Using this, we prove Theorem 3 where we show that Cahen–Wallach manifolds are geodesically connected and compute the lightlike conjugate loci of any point, obtaining that it must be empty or a paraboloid whose expression is explicitly given. We also show the existence of a foliation invariant by parallel transport whose leaves are complete, flat, totally geodesics and lightlike hypersurfaces.

2 Preliminaries

The curvature tensor of a Lorentzian manifold (M, g) is given by $R_{XY}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, where $X, Y, Z \in \mathfrak{X}(M)$. Given a plane $\sigma = \text{span}\{u, v\}$, we denote by $k(\sigma)$ the sectional curvature, if σ is non-degenerated whereas if σ is degenerated with v lightlike, then $\mathcal{K}_v(\sigma) = \frac{g(R_{uv}v,u)}{g(u,u)}$ is the *lightlike sectional curvature* associated to v [8]. Observe that the sign of $\mathcal{K}_v(\sigma)$ and $\mathcal{K}_{v'}(\sigma)$ are the same for lightlike $v, v' \in \sigma$.

Given $v \in T_pM$ the Jacobi operator is the endomorphism $R_v : T_pM \longrightarrow T_pM$ given by $R_v(u) = R_{uv}v$. We also denote by R_v to its associated matrix with respect to a fixed basis. Since the Jacobi operator is self-adjoint, R_v can be one of the following types

- Type I: diagonalizable with respect to an orthonormal basis.
- Type II: there exist $a, b \in \mathbb{R}$, $b \neq 0$, such that

$$R_v = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & D_{n-2} \end{pmatrix}$$

  with respect to an orthonormal basis, where D_k represents a diagonal matrix of order k.
- Type III: there exists $\lambda \in \mathbb{R}$ such that

$$R_v = \begin{pmatrix} \lambda & 0 & 0 \\ \epsilon & \lambda & 0 \\ 0 & 0 & D_{n-2} \end{pmatrix}, \quad \text{with } \epsilon = \pm 1,$$

  with respect to a pseudoorthonormal basis $\{u, w, e_3, ..., e_n\}$.

– Type IV: there exists $\lambda \in \mathbb{R}$ such that

$$R_v = \begin{pmatrix} \lambda & 0 & 1 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 0 & D_{n-3} \end{pmatrix}$$

with respect to a pseudoorthonormal basis $\{u, w, e_3, \ldots, e_n\}$.

In some cases, a little more can be said. For example, if v is timelike, R_v is necessarily of type I, whereas if v is lightlike, then R_v can be only of type I, III, and IV. Moreover, in this last case, if it is of type I, there exists an orthonormal basis such that $R_v = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 = \lambda_2 = 0$. On the other hand, if R_v is of type III or IV, we can choose the pseudo-orthonormal basis $\{u, w, e_3, \ldots, e_n\}$ such that the parameter λ in the upper box is zero and $v = w$ (resp. $v = u$) when R_v is of type III (resp. of type IV). If v is spacelike then any type is possible suggesting once more the difficulties in handling spacelike geodesics.

Given γ a geodesic in M, a conjugate point of $\gamma(0)$ is a point $\gamma(t_0)$ such that there is a non-trivial solution $J(t)$ to the Jacobi equation $J'' + R_{\gamma'} J = 0$ with boundary conditions $J(0) = 0$ and $J(t_0) = 0$.

A semi-Riemmanian manifold is locally symmetric, if the curvature tensor is parallel. Conjugate points in locally symmetric Lorentzian manifolds can be easily computed as the following theorem shows (although this result was known by the authors for some time, it should be attributed to [10], where it has been first published in the more general semi-Riemannian setting).

Theorem 1 *Let $\gamma_v : I \longrightarrow M$, $v \in T_pM$, be a geodesic in a Lorentz locally symmetric manifold. Then the conjugate points of $\gamma_v(0)$ along γ_v are $\gamma_v(\frac{\pi k}{\sqrt{\lambda}})$, where $k \in \mathbb{Z} - \{0\}$ with $\frac{k\pi}{\sqrt{\lambda}} \in I$, and λ is a real positive eigenvalue of R_v. The multiplicity of $\gamma_v(t_0)$ as a conjugate point is the number of eigenvalues $\lambda \in \mathbb{R}^+$ of R_v such that t_0 is a multiple of $\frac{\pi}{\sqrt{\lambda}}$.*

On the other hand, a semi-Riemannian manifold is symmetric if for each $p \in M$ there is an isometry $\xi_p : M \longrightarrow M$ with $\xi(p) = p$ and $\xi_{*p} = -id$. If γ is a geodesic in M with $\gamma(0) = p$, then $\tau_t : M \longrightarrow M$ given by $\tau_t(q) = \xi_{\gamma(\frac{t}{2})} \circ \xi_{\gamma(0)}(q)$ is a transvection along γ for each $t \in \mathbb{R}$ [13]. In the Riemannian case, it is well-known that the transvection τ_t induces a complete Killing vector field on M. The same is true in the semi-Riemannian setting. From the smooth dependence of solutions of an ordinary differential equation with respect to initial data, it is easy to see that in a semi-Riemannian symmetric manifold, the global symmetries induce a smooth map $\xi : M \times M \longrightarrow M$ defined by $\xi(p, q) = \xi_p(q)$.

Lemma 1 *In a semi-Riemannian symmetric manifold (M, g), the family $\{\tau_t\}_{t \in \mathbb{R}}$ is a one parameter group of isometries.*

Proof Consider $\tau : \mathbb{R} \longrightarrow I(M)$ given by $\tau(t) = \tau_t$, which is a Lie group morphism being $I(M)$ the isometry group of (M, g). Then it is enough to see that it is continuous. For this, observe that $\tau_t = \xi_{\gamma(\frac{t}{2})} \circ \xi_{\gamma(0)} = \mu(\xi_{\gamma(\frac{t}{2})}, \xi_{\gamma(0)})$, where $\mu : I(M) \times I(M) \longrightarrow I(M)$ is the product in $I(M)$. □

The following result has a proof identical to the Riemannian case [11].

Lemma 2 *Let (M, g) be a semi-Riemannian symmetric space, $\gamma : \mathbb{R} \to M$ a non-constant geodesic. Let $Y(t)$ be a Jacobi field along γ such that $Y'(0) = 0$. Then $Y(t)$ is the restriction to γ of a Killing vector field K. In fact, $K_p = \frac{\partial \tau_s(p)}{\partial s}|_{s=0}$ where τ_s is the one parameter group of transvections along the geodesic $b(s) = \exp_{\gamma(0)}(sY(0))$.*

3 Lorentz symmetric spaces

Analyzing the Jacobi operator, we can apply Theorem 1 to improve, in the category of Lorentz symmetric spaces, some Rauch comparison results in Lorentz geometry [1,8].

Proposition 1 *Let (M, g) be a Lorentz symmetric manifold, $p \in M$ and $v \in T_pM$.*

1. *If v is timelike, then p has not conjugate points along γ_v if and only if $k(\sigma) \geq 0$ where $\sigma \subset T_pM$ is any timelike plane containing v.*
2. *If v is lightlike, then p has not conjugate points along γ_v if and only if $\mathcal{K}_v(\sigma) \leq 0$ for any degenerate plane $\sigma \subset T_pM$ containing v.*

Proof 1. Being v timelike, there exists an orthonormal basis $\{E_1, \ldots, E_n\}$ with $v = aE_1$, such that the Jacobi operator is $R_v = \text{diag}\{0, \lambda_2, \ldots, \lambda_n\}$ with non positive eigenvalues λ_i by hypothesis. Let $u = \sum_{i=1}^n u_i E_i \in T_pM$ be such that $\sigma = \text{span}\{u, v\}$ is a timelike plane, then

$$k(\sigma) = \frac{\sum_{i=2}^n u_i^2 \lambda_i}{g(u,u)g(v,v) - g(u,v)^2} \geq 0.$$

Observe now that the sectional curvature $k(\sigma(t))$ is constant, where $\sigma(t)$ is the parallely propagated plane along $\gamma_v(t)$ of a timelike plane $\sigma \subset T_pM$ containing v, thus the sectional curvature of the timelike planes containing $\gamma_v'(t)$ is non-negative. If we call $J(t)$ a Jacobi field along γ_v, perpendicular to γ_v with $J(0) = 0$, it is easy to see that the function $h(s) = g(J(s), J(s))$ is zero only at $s = 0$. Thus p has not conjugate points along γ_v.

2. Suppose that p has not conjugate points along γ_v, with $v \in T_pM$ a lightlike vector. Then there exists a basis $\{E_1, \ldots, E_n\}$ such that the Jacobi operator is of type I, III, or IV. Let $u = \sum_{i=1}^n u_i E_i$ be a unit vector such that $\sigma = \text{span}\{u, v\}$ is a degenerate plane. For R_v of type I, it is immediate.
Suppose R_v is of type III. We take a pseudoorthonormal basis with $v = E_2$ such that on it, we can write

$$R_v = \begin{pmatrix} 0 & 0 & & & \\ \epsilon & 0 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_n \end{pmatrix}$$

with $\lambda_i \leq 0$. From $g(u, v) = 0$, we get $u_1 = 0$, then

$$\mathcal{K}_v(\sigma) = \sum_{i=3}^n u_i^2 \lambda_i \leq 0.$$

If R_v is of type IV, the basis is pseudoorthonormal with $v = E_1$ and

$$R_v = \begin{pmatrix} 0 & 0 & 1 & & & \\ 0 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ & & & \lambda_4 & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{pmatrix}$$

being $\lambda_i \leq 0$, obtaining $\mathcal{K}_v(\sigma) = \sum_{i=4}^{n} u_i^2 \lambda_i \leq 0$.

On the other hand, given $\sigma \subset T_pM$ any degenerate plane containing v, let $\sigma(t)$ be the parallely transported along γ_v of σ. For any t, $\sigma(t)$ is degenerate and $\mathcal{K}_{\gamma_v'}(\sigma(t))$ is constant, thus non-positive along γ_v. This means that the lightlike sectional curvature of any degenerate plane containing $\gamma_v'(t)$ is non-positive. The result follows as in the above item.

□

The "only if" parts in the above proposition are not true in general as the following example shows.

Example 1 Let $(\mathbb{R} \times \mathbb{R}^n, -dt^2 + f^2\eta)$ be a warped product being η the euclidean metric in $\mathbb{R}^n$. We know that non-spacelike geodesics have not conjugate points [5]. Let $\gamma_v(s) = (s, \gamma_2(s))$ be a non-spacelike geodesic. There are planes $\pi(s)$ containing $\gamma_v'(s)$ such that $g(R_{u(s)\gamma_v'(s)}\gamma_v'(s), u(s))$ have the sign of $f''(s)$, where $\pi(s) = \text{span}\{u(s), \gamma_v'(s)\}$. But the warping function f is an arbitrary positive function.

The following result shows that in a Lorentz symmetric space, the first conjugate point is a meeting point of any geodesic variation. Although the proof is like in the Riemannian case, we give it here to make precise the argument given in [11].

Proposition 2 *Let (M, g) be a Lorentz symmetric manifold, $\gamma : \mathbb{R} \to M$ a geodesic with $\gamma(t_1)$ the first conjugate point of $\gamma(0)$ along γ. Then any geodesic variation of $\gamma_{|[0,t_1]}$ has its ends fixed. Moreover, the length of the curves of the variation is constant.*

Proof Call $\{E_1(t), \ldots, E_n(t)\}$ the parallely propagated basis along γ in which $R_{\gamma'(t)}$ can be written as a constant matrix of one of the four types given in Sect. 2. By Theorem 1 there exists λ a positive eigenvalue of $R_{\gamma'(0)}$ satisfying $t_1 = \frac{\pi}{\sqrt{\lambda}}$. Observe that we have used that $\gamma(t_1)$ is the first conjugate point to ensure that the numerator in the expression of t_1 is just π. Let $Y(t)$ be a non trivial Jacobi field with $Y(0) = Y(t_1) = 0$, then $Y(t) = \sum_{j=1}^{r} a_j \sin\sqrt{\lambda} t E_{i_j}$ where r is the multiplicity of λ, $a_j \in \mathbb{R}$ and it verifies $Y'(\frac{t_1}{2}) = 0$. Let τ_s be the 1-parameter group generated by $Y(\frac{t_1}{2})$. Since $Y(t)$ is the Killing vector field of τ_s in $\gamma(t)$ (Lemma 2), we find that the geodesic $\tau_s \circ \gamma$ pass through $\gamma(0)$ and $\gamma(t_1)$. □

Example 2 Take $M = SL(2,\mathbb{R})$ endowed with the bi-invariant Lorentz structure induced by $\langle A, B\rangle = trace(AB)$, with $A, B \in \mathfrak{sl}(2,\mathbb{R})$. This defines on M a structure of Lorentz symmetric space. The curvature operator is given by $R_{XY}Z = -\frac{1}{4}[[X,Y],Z]$ for $X, Y, Z \in \mathfrak{sl}(2,\mathbb{R})$.

Consider the Jacobi operator $R_V(X) = -\frac{1}{4}XV^2 - \frac{1}{4}V^2X + \frac{1}{2}VXV$. If $V = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, the matrix of R_V in the standard basis of $\mathfrak{sl}(2,\mathbb{R})$ is

$$R_V = \begin{pmatrix} -bc & \dfrac{ac}{2} & \frac{ab}{2} \\ ab & \dfrac{-2a^2 - bc}{2} & \frac{b^2}{2} \\ ac & \dfrac{c^2}{2} & \frac{-2a^2-bc}{2} \end{pmatrix}$$

with eigenvalue $-(a^2 + bc) = -\frac{1}{2}\langle V, V\rangle$ associated to the eigenvectors $(c, -2a, 0)$ and $(0, -b, c)$, and 0 associated to V. Thus, if V is lightlike or spacelike, there are not conjugate points along the geodesic γ_V. If V is timelike then $\exp\left(\frac{n\pi}{\sqrt{\frac{-1}{2}<V,V>}}\right)$, with $n \in \mathbb{Z} - \{0\}$ are conjugate points with multiplicity 2, in virtue of Theorem 1.

We can compute explicitly the conjugate loci of $I \in SL(2,\mathbb{R})$. Call $\lambda = -(a^2 + bc) > 0$, then

$$\gamma_V(t) = \exp(tV) = \cos\sqrt{\lambda}tI + \frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}tV,$$

and the first conjugate point is $\gamma_V(\frac{\pi}{\sqrt{\lambda}}) = -I$. Thus the conjugate loci of I is $-I$, and due to the symmetry of the space, the conjugate loci of any point $A \in SL(2,\mathbb{R})$ is $-A$.

Note that we have computed the whole conjugate loci, which coincides with the timelike conjugate loci. On the other hand, lightlike geodesics from a point have not conjugate points, whereas timelike ones all pass through the antipode.

The ideas developed above have applications in the compact case. Take $(M, g) = (\mathbb{S}^1 \times N, -dt^2 + g_N)$ with (N, g_N) a compact Riemannian symmetric space. It is straightforward to see that the sectional curvature of a timelike plane is non-positive. If we change $\mathbb{S}^1$ with a compact Cahen–Wallach space (see the next section) in the above example, we have the same result. These examples are inspired in the classification of simply connected Lorentz symmetric spaces, but we cannot use it further to know if this is a general result for compact Lorentz symmetric spaces. Recall that $\mathcal{K}_v(\sigma)$ denotes the lightlike sectional curvature of the lightlike plane σ associated to the lightlike vector $v \in \sigma$.

Theorem 2 *Let (M, g) be a compact Lorentz symmetric space.*

1. *The sectional curvature verifies $k(\sigma) \leq 0$ for any timelike plane σ.*
2. *If $\dim M \geq 3$, then $\mathcal{K}_v(\sigma) \geq 0$ for any degenerate plane σ and any lightlike vector $v \in \sigma$.*

Proof 1. Let σ be a timelike plane such that $k(\sigma) > 0$. Take $\{v, w\}$ an orthonormal basis of σ with v a timelike vector and $\{E_1 = v, \ldots, E_m\}$ an orthonormal basis where the Jacobi operator R_v is

$$R_v = \begin{pmatrix} 0 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix}$$

Let us write $w = \sum_{i=1}^m w_i E_i$, then

$$0 < k(\sigma) = -\sum_{i,j\geq 2} w_i^2 \lambda_i,$$

thus there must be some eigenvalue $\lambda_i < 0$ of R_v. The rest of the proof is as in the Riemannian case but we include it for the sake of completeness. Consider the parallely propagated basis $\{E_1, \ldots, E_m\}$ along the geodesic γ_v, then $J(t) = \cosh\sqrt{-\lambda_i}tE_i(t)$ is a Jacobi field on γ_v with $J'(0) = 0$. Lemma 2 says that there exists a Killing vector field $K \in \mathfrak{X}(M)$ with $K_{\gamma_v} = J$, but M is compact and $|K|$ is continuous on M, thus $|J(t)|$ must be bounded above. Contradiction.

2. We follow the argument in the above proof, so it suffices to show that if there were a degenerate plane σ and a lightlike vector $v \in \sigma$ such that $\mathcal{K}_v(\sigma) < 0$, then there would exist a negative eigenvalue of R_v.

The Jacobi operator can be of type I, III, or IV because v is lightlike. We only prove the type IV because type III is simpler and type I is as in the previous proof.

We can take a pseudoorthonormal basis $\{v = E_1, \ldots, E_m\}$ such that on it the Jacobi operator R_v is

$$R_v = \begin{pmatrix} 0 & 0 & 1 & & & \\ 0 & 0 & 0 & & 0 & \\ 0 & 1 & 0 & & & \\ & & & \lambda_4 & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_m \end{pmatrix}$$

Let w be a unit spacelike vector such that $\{v, w\}$ is a basis of σ. Then $g(w, v) = 0$ implies $w = \sum_{i\neq 2}^{m} w_i E_i$, and we have

$$\mathcal{K}_v(\sigma) = w_3^2 g(R_v E_3, E_3) + 2\sum_{j\geq 4} w_3 w_j g(R_v E_3, E_j) + \sum_{i=4}^{m} w_i^2 \lambda_i$$

where the first two summands are zero. If $\dim M = 3$ then $\mathcal{K}_v(\sigma) = 0$ and the theorem follows, otherwise there exists a negative eigenvalue. □

The following corollary shows a difference between the role of conjugate points along timelike and lightlike geodesics which reflects the different role of sectional curvature and lightlike sectional curvature in Lorentz geometry. Both notions are related by the following fact: if a Lorentz manifold has zero lightlike sectional curvature, then it has constant curvature.

Corollary 1 *Let (M, g) be a compact Lorentz symmetric space, then*

1. *There are not conjugate points along timelike geodesics if and only if it is flat.*
2. *If* $\dim M \geq 3$, *then there are not conjugate points along lightlike geodesics if and only if it has constant sectional curvature* $k \leq 0$.

Proof In both cases, the "if" part of the proof are general results in Lorentz geometry, we see the "only if" part.

1. The hypothesis, Lemma 1 and Theorem 2 implies $k(\sigma) = 0$ for any timelike plane, thus $k = 0$ [13].
2. The hypothesis, Lemma 1 and Theorem 2 implies $\mathcal{K}_v(\sigma) = 0$ for any degenerate plane σ, thus the sectional curvature is constant, and is nonpositive due to the same Theorem 2.

□

In [7], it was shown that a compact Lorentz surface admitting a timelike Killing vector field with no conjugate point along its timelike geodesics must be flat.

We finish this section with an application to Riemannian geometry which we do not find in the literature. Recall that a simply connected Riemannian symmetric space (M, g) can be expressed as a product whose factors are compact, non-compact, or euclidean [13].

Proposition 3 *Let M be a simply connected Riemannian symmetric space which admits a compact quotient. Then M has not factors of non-compact type.*

Proof Call M/Γ the compact quotient. If N is a factor of M of non-compact type, there exists a plane σ in N with curvature $k(\sigma) < 0$ strictly. Take $\mathbb{S}^1 \times M$ with metric $-dt^2 + g_M$ which is a Lorentz symmetric space. Let $\pi = \text{span}\{u, v\}$ be a timelike plane in $\mathbb{S}^1 \times M$, with $u = e_1 + e_2$, being $e_1 \in T\mathbb{S}^1$ and $\{e_2, v\}$ a basis of σ, then

$$g(R_{uv}v, u) = g(R_{e_2 v}v, e_2) < 0,$$

thus $k(\pi) > 0$. This implies that there is a timelike plane with positive curvature in the compact quotient $(\mathbb{S}^1 \times M)/\{1\} \times \Gamma$, in contradiction with Theorem 2. □

4 Cahen–Wallach manifolds

It is well-known that a simply connected Lorentz symmetric space is isometric to a product of a simply connected Riemannian symmetric space and one of the following: $(\mathbb{R}, -dt^2)$, a complete simply connected Lorentz manifold of constant curvature or a Cahen–Wallach manifold [2].

Cahen–Wallach manifolds were introduced in [3], and they are defined as follows. Let $\mathfrak{g} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ with $n \geq 1$ be considered as a Lie algebra with Lie bracket

$$[(x, y, t, u), (x', y', t', u')] = (u'y - uy', u\, f(x') - u' f(x), \langle f(x'), y\rangle - \langle f(x), y'\rangle, 0)$$

where $\langle, \rangle$ is the euclidean metric on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a symmetric endomorphism.

The set $\mathfrak{h} = \{(x, 0, 0, 0) : x \in \mathbb{R}^n\} \subset \mathfrak{g}$ is an abelian subalgebra. Take G the simply connected Lie group generated by $\mathfrak{g}$, H the connected subgroup generated by $\mathfrak{h}$ which is a closed subgroup of G, and $\mathfrak{m} = \{(0, y, t, u) : y \in \mathbb{R}^n, t, u \in \mathbb{R}\} \subset \mathfrak{g}$ which is an Ad_H-invariant supplementary of $\mathfrak{h}$ in $\mathfrak{g}$. Let $q((0, y, t, u), (0, y', t', u')) = \langle y, y'\rangle - tu' - t'u$ be a Lorentz product on $\mathfrak{m}$. It induces an invariant metric on $M = G/H$. With this metric, it is a Lorentz symmetric manifold diffeomorphic to $\mathbb{R}^{n+2}$ which is called Cahen–Wallach manifold.

The Jacobi operator is given by

$$R_V(X) = -[[X, V], V] = (0, uf(uy' - u'y), \langle f(uy' - u'y), y\rangle, 0),$$

where $V = (0, y, t, u)$, $X = (0, y', t', u') \in \mathfrak{m}$.

A Cahen–Wallach manifold is flat if and only if $f = 0$, it is Ricci flat if and only if $\text{trace}(f) = 0$. On the other hand, there are not Cahen–Wallach manifolds of constant curvature other than flats. In fact, if $V = (0, y, t, u)$, $X = (0, y', t', u')$ form an orthonormal basis for a non-degenerate plane π in $\mathfrak{m}$ we have $k(\pi) = \varepsilon\langle f(w), w\rangle$, being $\varepsilon = q(V, V)q(X, X)$ and $w = uy' - u'y$. We may choose the above vectors V and X with $\{y, y'\}$ orthonormal and $u = u' = 0$, thus $k(\pi) = 0$.

The following lemma shows that the conjugate points of $\gamma(0)$ along γ are codified by the positives eigenvalues of f. The proof is straightforward.

Lemma 3 *Let γ be a geodesic through $\bar{e} \in M$ with $\gamma'(0) = (0, y, t, u)$. If $u = 0$, then γ has not conjugate point. Otherwise, the eigenvalues of $R_{\gamma'(0)}$ are zero with multiplicity* dim $Ker\ f + 2$, *and $\lambda_i = \mu_i u^2$ being μ_i the non-zero eigenvalues of f (both λ_i and μ_i with the same multiplicity).*

If f has a positive eigenvalue, then using Theorem 1 there are geodesics of any causal character with conjugate points. In particular, we have

Corollary 2 *In a Ricci flat but non-flat Cahen–Wallach manifold, there are geodesics of any causal character with conjugate points.*

The following theorem shows the geodesic connectivity and describes the lightlike conjugate loci of any point in a Cahen–Wallach manifold. Suppose that $\mu_1 \geq \cdots \geq \mu_n$ are the eigenvalues of f.

Theorem 3 *Cahen–Wallach manifold are geodesically connected. Moreover, the future (past) lightlike conjugate loci of any point is empty or a paraboloid whose dimension is the number of eigenvalues of f different from μ_1.*

Proof Consider M as $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$. It is enough to compute the geodesics starting at the origin.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$ formed with eigenvectors of f. Given a point $(y_0, t_0, u_0) \in M$, with $y_0 = \sum_{i=1}^n y_i e_i$, we can define a global chart $(x_1, \ldots, x_n, x_{n+1}, x_{n+2})$ on M which assigns to (y_0, t_0, u_0) the coordinates $(y_1, \ldots, y_n, t_0, u_0)$. The metric in this chart is

$$\begin{pmatrix} I & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & \sum_{i=1}^n -\mu_i x_i^2 \end{pmatrix}$$

where I is the identity matrix of order n, the Christoffel symbols are

$$\Gamma^{n+1}_{i\,n+2} = \mu_i x_i, \quad \Gamma^{i}_{n+2\,n+2} = \mu_i x_i \quad i = 1, \ldots, n.$$

Suppose that $\mu_1 \geq \cdots \geq \mu_p > 0 = \mu_{p+1} = \cdots = \mu_q > \mu_{q+1} \geq \cdots \geq \mu_n$, where $0 \leq p \leq q \leq n$. The geodesic γ with $\gamma(0) = 0 \in M$ is given by

$$\gamma_i = \begin{cases} a_i \sin(c_i s) & i = 1, \ldots, p \\ a_i s & i = p+1, \ldots, q \\ a_i \sinh(d_i s) & i = q+1, \ldots, n \end{cases}$$

$$\gamma_{n+1} = \sum_{i=1}^{p} \frac{\mu_i u a_i^2}{2c_i} \sin(c_i s)\cos(c_i s) - \sum_{i=q+1}^{n} \frac{\mu_i u a_i^2}{2d_i} \sinh(d_i s)\cosh(d_i s) + es$$

$$\gamma_{n+2} = us,$$

being $a_i \in \mathbb{R}$, $c_i = \sqrt{\mu_i u^2}$, and $d_i = \sqrt{-\mu_i u^2}$.

Given a point $(y_1, \ldots, y_n, t_0, u_0) \in M$ and a point $s_0 \in \mathbb{R}$ such that $\sin(c_i s_0) \neq 0 \neq \cos(c_i s_0)$, the above system for s_0 fixed and $(\gamma_1, \ldots, \gamma_{n+2}) = (y_1, \ldots, y_n, t_0, u_0)$ can be solved in the variables $a_1, \ldots, a_n, u, e$. Thus any point can be joined with a geodesic to $\overline{e} \in M$, and, by symmetry, to any other point. Then it is geodesically connected.

Let us now consider that γ is a lightlike geodesic. By Lemma 3 if $u = 0$ then γ has not conjugate points. Thus we take $u \neq 0$ and suppose $u > 0$ points in the future direction. The eigenvalues of $R_{\gamma'(0)}$ are zero and $\mu_j u^2$. If all the eigenvalues of f are non-positive then the conjugate loci of $\gamma(0)$ is empty by Theorem 1.

The geodesic γ is lightlike if and only if $2eu = \sum_{i=p+1}^{q} a_i^2$. Then using Theorem 1 again, the first conjugate point of $\gamma(0)$ along γ is

$$\gamma\left(\frac{\pi}{c_1}\right) = \Bigg(0, a_2 \sin\left(\frac{c_2}{c_1}\pi\right), \ldots, a_p \sin\left(\frac{c_p}{c_1}\pi\right), a_{p+1}\frac{\pi}{c_1}, \ldots, a_q \frac{\pi}{c_1}, a_{q+1} \sinh\left(\frac{d_{q+1}}{c_1}\pi\right), \ldots, a_n \sinh\left(\frac{d_n}{c_1}\pi\right), -\sum_{i=2}^{p} \frac{\mu_i u a_i^2}{2c_i} \sin\left(\frac{c_i}{c_1}\pi\right)\cos\left(\frac{c_i}{c_1}\pi\right) + \sum_{i=p+1}^{q} \frac{a_i^2 \pi}{2uc_1} + \sum_{i=q+1}^{n} \frac{\mu_i u a_i^2}{2d_i} \sinh\left(\frac{d_i}{c_1}\pi\right)\cosh\left(\frac{d_i}{c_1}\pi\right), u\frac{\pi}{c_1}\Bigg)$$

which is the announced paraboloid. □

Corollary 3 *In a Cahen–Wallach manifold with $f = \mu Id$, the future lightlike conjugate loci of any point is empty (if $\mu \le 0$) or another point.*

It is known that the universal anti-De Sitter space is not geodesically connected, thus the above theorem can not be extended to all Lorentz symmetric spaces, [4].

We have found a distinguished $n+1$-dimensional subspace $\mathfrak{p} = \{(0, y, t, 0) : y \in \mathbb{R}^n, t \in \mathbb{R}\}$ of $\mathfrak{m}$ such that every geodesic with initial velocity in it has not conjugate points. It is an abelian subalgebra of $\mathfrak{g}$ that verifies $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$. Let $p : G \to M = G/H$ be the canonical projection. We have $Ad_h(\mathfrak{p}) \subset \mathfrak{p}$ for every $h \in H$ which allows us to induce a distribution on M defined by $\bar{\mathfrak{p}}_{\bar{g}} = p_{*_g}(\mathfrak{p}_g)$, where $\mathfrak{p}_g = L_{g*e}\mathfrak{p}$ is the left invariant distribution on G defined by $\mathfrak{p}$. Observe that we can also write $p_{*_g}(\mathfrak{p}_g) = \bar{L}_{g*\bar{e}}\, p_{*e}\mathfrak{p}$, where $\bar{L} : G \times M \to M$ is the action of G on M, showing that the distribution $\{\bar{\mathfrak{p}}_{\bar{g}}\}$ is G-invariant.

There are other distinguished subalgebras of $\mathfrak{g}$ contained in $\mathfrak{m}$ but only $\mathfrak{r} = \{(0, 0, t, 0) : t \in \mathbb{R}\} \subset \mathfrak{p}$ verifies the same properties than $\mathfrak{p}$, defining another G-invariant distribution on M. Notice that $p_{*e} : \mathfrak{m} \to T_{\bar{e}}M$ is an isomorphism, then $\dim \bar{\mathfrak{p}}_{\bar{g}} = n + 1$, $\dim \bar{\mathfrak{r}}_{\bar{g}} = 1$, both $\bar{\mathfrak{r}}_{\bar{g}}$ and $\bar{\mathfrak{p}}_{\bar{g}}$ are degenerated for the Lorentz metric and $\bar{\mathfrak{r}}_{\bar{g}} \subset \bar{\mathfrak{p}}_{\bar{g}}$ for every $\bar{g} \in M$.

The subalgebra $\mathfrak{p}$ determines a complete, totally geodesic, and flat submanifold $\overline{P}$ through $\bar{e} \in M$, because it is a Lie triple system with $[[\mathfrak{p}, \mathfrak{p}], \mathfrak{p}] = 0$. The submanifold can be determined by the condition $T_{\bar{e}}\overline{P} = p_{*e}(\mathfrak{p})$, expressed as $\overline{P} = \exp_{\bar{e}} \bar{\mathfrak{p}}_{\bar{e}}$ [9,12], or alternatively as $\overline{P} = p(P)$, being P the connected subgroup of G induced by $\mathfrak{p}$, where we have used that $\exp : \mathfrak{p} \to P$ is onto because P is abelian.

The same is true for $\mathfrak{r}$ word-by-word.

Proposition 4 *Both distributions $\{\bar{\mathfrak{p}}_{\bar{g}}\}$ and $\{\bar{\mathfrak{r}}_{\bar{g}}\}$ are involutives. Moreover, the projection $p : G \to M$ and the action of G on M are foliated maps for both distributions.*

Proof Let $\bar{g} \in M$ be arbitrary, and $g \in G$ such that $p(g) = \bar{g}$. Using that $\bar{L}_g : M \to M$ is a diffeomorphism, we get that $\bar{S} = \bar{L}_g \bar{P}$ is a submanifold of M through $\bar{g}$. Let $\bar{a} \in \bar{S}$, this means that there exists $z \in P$ such that $\bar{a} = \overline{gz}$. Define $a = gz \in G$, then

$$T_{\bar{a}}\bar{S} = \bar{L}_{a*\bar{e}} T_{\bar{e}}\bar{P} = p_{*a} L_{a*e}\mathfrak{p} = \bar{\mathfrak{p}}_{\bar{a}}.$$

This shows that $\bar{S}$ is an integral submanifold of $\{\bar{\mathfrak{p}}_{\bar{a}}\}$ through $\bar{g} \in M$.

The submanifold $S = L_g P$ is a leaf of $\{\mathfrak{p}_a\}$ on G and $p(S) = \bar{S}$, thus the projection $p : G \to M$ takes leaves into integral submanifolds of $\{\bar{\mathfrak{p}}_{\bar{a}}\}$. Using that $\bar{P}$ is complete it is easy to see that they are maximal.

The action of G on M is foliated by construction.

The same argument works for $\{\bar{\mathfrak{r}}_{\bar{g}}\}$. □

The leaves of both distributions form two complete, flat, totally geodesics, and lightlike foliations on M of codimension 1 and dimension 1, respectively. To compute them, note that the group operation in P is given by

$$(0, y, t, 0)(0, y', t', 0) = (0, y, t, 0) + (0, y', t', 0)$$

(see [3]), thus the leaf of $\{\bar{\mathfrak{p}}_{\bar{g}}\}$ through $\bar{a} \in M$, where $a = (x, y, t, u) \in G$, is $\mathbb{R}^n \times \mathbb{R} \times \{u\}$, and that of $\{\bar{\mathfrak{r}}_{\bar{g}}\}$ is $\{0\} \times \mathbb{R} \times \{u\}$.

Both distributions $\{\bar{\mathfrak{p}}_{\bar{g}}\}$ and $\{\bar{\mathfrak{r}}_{\bar{g}}\}$ are also invariant by parallel transport, (for $n = 2$ see [2]). This is immediate from the fact that $A(\bar{\mathfrak{p}}) \subset \bar{\mathfrak{r}}$ for every A in the holonomy algebra. In fact, take $A = R_{ZT}$ for some $Z, T \in T_{\bar{e}}M$, where we identify $T_{\bar{e}}M$ with $\mathfrak{m}$ and write $Z = (0, y', t', u')$ and $T = (0, y'', t'', u'')$. Let $V \in \bar{\mathfrak{p}}$ be arbitrary, $V = (0, y, t, 0)$. Then

$$A(V) = (0, 0, -\langle f(u''y' - u'y''), y\rangle, 0) \in \bar{\mathfrak{r}}.$$

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