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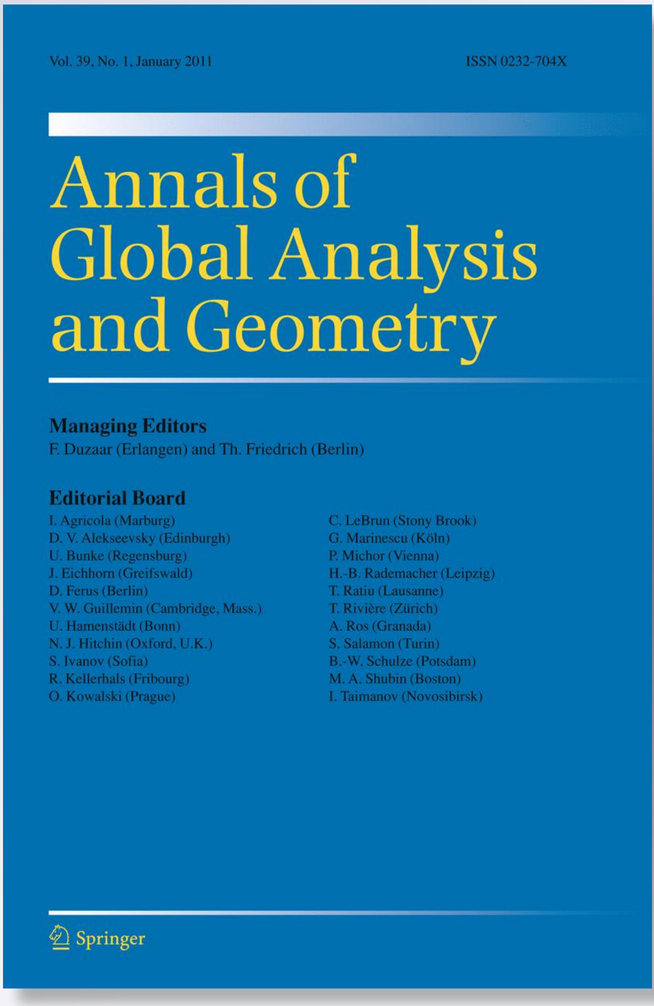

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ORIGINAL PAPER

Uniqueness of static decompositions

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Abstract We classify static manifolds which admit more than one static decomposition whenever a condition on the curvature is fulfilled. For this, we take a standard static vector field and analyze its associated one parameter family of projections onto the base. We show that the base itself is a static manifold and the warping function satisfies severe restrictions, leading us to our classification results. Moreover, we show that certain condition on the lightlike sectional curvature ensures the uniqueness of static decomposition for Lorentzian manifolds.

Keywords Static space · Static vector field · Isometric decomposition · Lightlike sectional curvature

Mathematics Subject Classification (2000) Primary 53C50; Secondary 53C80

1 Introduction

Given (L, g_L) a connected Riemannian manifold, $f \in C^\infty(L)$ a positive function and $\varepsilon = \pm 1$, we call *static manifold* to a product $L \times \mathbb{R}$ furnished with the metric $g_L + \varepsilon f^2 \mathrm{d}t^2$, which is denoted by $L \times_{\varepsilon f} \mathbb{R}$. To follow standard definitions [6], we refer to the case $\varepsilon = -1$ as *static space* instead of static manifold. Although static manifold is not a common term, it allows us to handle jointly the Riemannian and the Lorentzian case.

A vector field is irrotational if it has integrable orthogonal distribution. It is well known that a manifold furnished with an irrotational and Killing vector field can be decomposed as a static manifold, at least locally. Due to this, such a vector field is called *static* and when it gives rise to a global decomposition, it is called *standard static*, [7]. Obviously, the existence

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of two different static decompositions is equivalent to the existence of two standard static vector fields linearly independent at some point. Minkowski, Euclidean, hyperbolic, and a suitable portion of the anti De Sitter spaces provide us good examples of manifolds with different decompositions, but there are many other manifolds which can be also decomposed as a static manifold in several ways.

The decomposition uniqueness problem has been studied by several authors. The De Rham-Wu Theorem ensures the uniqueness of a direct product decomposition for a simply connected semi-Riemannian manifold (for the nonsimply connected case see [1] and [3]) and the uniqueness of Generalized Robertson-Walker decomposition was studied in [2], obtaining that the De Sitter space is the only complete space with several nontrivial decomposition, whereas Friedmann spaces have a unique decomposition even locally.

The uniqueness of static decomposition seems more complicated and only partial results have been obtained. In [7], it is shown that static spaces with compact base do not admit another global static decomposition and in [9] it is computed Riemannian three-dimensional metrics which can be used to construct Einstein static spacetimes in more than one way. On the other hand, it is a remarkable fact that for the exterior Schwarzschild spacetime the uniqueness can be directly shown. Indeed, any timelike Killing vector field is proportional to the canonical one, [6]. Apart from these results, no much more is known about this topic.

In this paper, we study manifolds with more than one static decomposition, with special emphasis on the Lorentzian case. For this, we consider a static manifold with a standard static vector field linearly independent to the canonical one at some point. The projection onto the base gives us a family of vector fields, which we use to decompose the base itself as a static manifold. This allows us to prove in Proposition 4.7 that these manifolds are a special type of warped product and, under a mild curvature hypothesis about the base or Einstein assumption, we classify them in Theorems 5.1 and 5.2, respectively. We particularize to Einstein spacetimes in Theorem 5.5 and finally, in Theorem 5.7, we show that in the Lorentzian case the uniqueness is guaranteed if the lightlike sectional curvature at a point is never zero.

2 Preliminaries

Given a product manifold $L \times \mathbb{R}$, the lift of ∂_t is still denoted by ∂_t, however, given $X \in \mathfrak{X}(L)$ we will denote by $\tilde{X}$ to its lift. We call $\pi : L \times \mathbb{R} \to L$ the canonical projection and $i_t : L \to L \times \mathbb{R}$ the injection given by $i_t(p) = (p, t)$. We will usually avoid π in the formulaes to lighten the notation.

If $V \in \mathfrak{X}(L \times \mathbb{R})$ we call $V_p^t = \pi_{*(p,t)}(V)$, which is a vector field on L for each $t \in \mathbb{R}$. Fixed $p \in L$, V_p^t is a curve in T_pL and thus we can consider the vectorial derivative $\frac{\mathrm{d}}{\mathrm{d}t} V_p^t \in T_pL$, obtaining in this way another vector field on L for each $t \in \mathbb{R}$. On the other hand, if $h \in C^\infty(L \times \mathbb{R})$ then we call h^t the function on L given by $h^t = h \circ i_t$, but h_t will mean the derivative respect to t.

Recall that any vector field on a two- or one-dimensional manifold is irrotational. By convention, in a one-dimensional manifold, the orthogonal leaf of a vector field through a point simply means this point. This will allow us to prove some results without distinguishing cases.

We denote by $\mathbb{R}^2_{[\varepsilon]}$ the Euclidean/Minkowski plane $(\mathbb{R}^2, \mathrm{d}s^2 + \varepsilon \mathrm{d}t^2)$ and $\mathbb{H}^2_{[\varepsilon]}(r)$ the hyperbolic/anti De Sitter plane $(\mathbb{R}^2, \mathrm{d}s^2 + \varepsilon \cosh^2(rs)\mathrm{d}t^2)$ of curvature $-r^2$. By $\widehat{\mathbb{H}}^2_{[\varepsilon]}(r)$ we denote the static manifold $(\mathbb{R}^2, \mathrm{d}s^2 + \varepsilon e^{2rs}\mathrm{d}t^2)$, which is another representation of the hyperbolic plane if $\varepsilon = 1$ and a piece of the anti De Sitter plane if $\varepsilon = -1$. To simplify this notation, we also denote $\mathbb{H}^2_{[1]}(r)$ by $\mathbb{H}^2(r)$ and $\mathbb{R}^2_{[1]}$ by $\mathbb{R}^2$.

If Π is a degenerate plane in a Lorentzian manifold of dimension greater than two, the lightlike sectional curvature of Π is defined as

$$\mathcal{K}_u(\Pi) = \frac{g(R(v, u, u), v)}{g(v, v)},$$

where $u, v \in \Pi$ with u lightlike and v spacelike. This curvature depends on the chosen u, but its sign does not depend on it. Thus, we can say zero lightlike sectional curvature without explicit mention of the chosen lightlike vector. We can easily compute the lightlike sectional curvature in a static space.

Lemma 2.1 *Let $M = L \times_{-f} \mathbb{R}$ be a static space with $dim L \geq 2$ and take $v, w \in TL$ unitary vectors with $v \perp w$. If $\Pi = span(v, u)$, where $u = w + \frac{1}{f}\partial_t$, then*

$$\mathcal{K}_u(\Pi) = K^L\left(span(v, w)\right) + \frac{g(\widetilde{\nabla_v \nabla f}, v)}{f},$$

being K^L the sectional curvature of L.

The following result is well known. We prove the precise version that we need which, jointly with the below remark, ensures the global decomposition as a static manifold.

Proposition 2.2 *Let M be a geodesically complete Lorentzian or Riemannian manifold of dimension greater that one. If V is a static vector field without zeros (timelike in the Lorentzian case), then there exists a normal semi-Riemannian covering map Φ : $\left(L \times \mathbb{R}, g_L + \varepsilon f^2 \mathrm{d}t^2\right) \to (M, g)$ with $\Phi_*(\partial_t) = V$.*

Proof Call $\Phi : M \times \mathbb{R} \to M$ the flow of V and L_p the orthogonal leaf of V through $p \in M$. Since Φ_t is an isometry which preserves V, it holds $\Phi_t(L_p) = L_{\Phi_t(p)}$ for all $t \in \mathbb{R}$, i.e., it is a foliated map for each $t \in \mathbb{R}$. Therefore, it is easy to show that the restriction $\Phi : L \times \mathbb{R} \to M$, where L is a fixed leaf, is a local diffeomorphism and so $\Phi\,(L \times \mathbb{R}) = \cup_{t\in\mathbb{R}} \Phi_t(L)$ is an open subset of M.

If $p \notin \cup_{t\in\mathbb{R}} \Phi_t(L)$, we can show in the same way as before that $\cup_{t\in\mathbb{R}} \Phi_t(L_p)$ is an open neighborhood of p, which is contained in the complementary of $\cup_{t\in\mathbb{R}} \Phi_t(L)$. Since M is supposed connected, Φ is onto.

Now, take the pull-back metric $\Phi^*(g)$ which makes Φ a local isometry. If $v \in T_pL \prec T_pM$ then

$$|(v_p, 0_t)| = |(\Phi_t)_{*p}(v_p)| = |v_p| = |(\Phi_0)_{*p}(v_p)| = |(v_p, 0_0)|,$$

where we have taken into account that $\Phi|_{L\times\{0\}} = id$. On the other hand, if we call $f(p, t) = |V_{\Phi(p,t)}|$ then $f(p, t) = f(p, 0)$ and therefore we can conclude that $\Phi^*(g) = g|_L + \varepsilon f^2 \mathrm{d}t^2$, where ε is the sign of V.

Now we show that it is a covering map. Let $\sigma : [0, 1] \to M$ be a geodesic and $(x_0, t_0) \in L \times \mathbb{R}$ a point such that $\Phi(x_0, t_0) = \sigma(0)$. We must show that there exists a lift $\alpha : [0, 1] \to L \times \mathbb{R}$ of σ through Φ starting at (x_0, t_0), [6]. There is a geodesic $\alpha : [0, s_0) \to L \times \mathbb{R}$, $\alpha(s) = (x(s), t(s))$, such that $\Phi \circ \alpha = \sigma$ and $\alpha(0) = (x_0, t_0)$ because Φ is a local isometry. If we suppose $s_0 < 1$, there is a geodesic $(x_1(s), t_1(s))$ such that $\Phi(x_1(s), t_1(s)) = \sigma(s)$ with $s \in (s_0 - \delta, s_0 + \delta)$. Then in the open interval $(s_0 - \delta, s_0)$ it holds $\Phi(x(s), t(s)) = \Phi(x_1(s), t_1(s))$. Differentiating we get for $s \in (s_0 - \delta, s_0)$ that

$$\Phi_{*(x(s),t(s))}\left(t'(s)\partial_t + x'(s)\right) = \Phi_{*(x_1(s),t_1(s))}\left(t_1'(s)\partial_t + x_1'(s)\right),$$

and since $\Phi_{*_{(x,t)}}(\partial_t) = V_{\Phi_t(x)}$

$$\left(t'(s) - t_1'(s)\right) V_{\sigma(s)} = \Phi_{*_{(x(s),t(s))}}\left(x'(s)\right) - \Phi_{*_{(x_1(s),t_1(s))}}\left(x_1'(s)\right).$$

But $\Phi_{*_{(x(s),t(s))}}\left(x'(s)\right) - \Phi_{*_{(x_1(s),t_1(s))}}\left(x_1'(s)\right)$ is orthogonal to V since Φ_t is a foliated map for each $t \in \mathbb{R}$. Therefore, $t_1(s) - t(s) = c \in \mathbb{R}$ and so it exists $\lim_{s\to s_0} t(s)$ and since $x(s) = \Phi_{-t(s)}(\sigma(s))$ it also exists $\lim_{s\to s_0} x(s)$ and $\lim_{s\to s_0} \alpha(s)$. Thus, the geodesic α is extendible.

It remains to show that the group of deck transformations acts transitively on the fiber. Fix $p \in L$ and take $(x_0, t_0) \in L \times \mathbb{R}$ such that $\Phi(x_0, t_0) = \Phi(p, 0) = p$. Since Φ_{-t_0} is a foliated map, in the sense that it preserves the foliation given by $V^{\perp}$, it follows that $\Phi_{-t_0}(L) = L$ and the map $L \times \mathbb{R} \to L \times \mathbb{R}$ given by $(x, t) \to (\Phi_{-t_0}(x), t + t_0)$ is a deck transformation and takes $(p, 0)$ to (x_0, t_0). □

Remark 2.3 In the above proposition, we can ensure the injectivity of Φ, and thus the global decomposition of M, supposing that integral curves of V only intersect each orthogonal leaf one time. Anyway, although we do not suppose completeness, we can still obtain a local decomposition.

3 Killing vector fields in static manifolds

In order to tackle the uniqueness problem, we start studying Killing vector fields in a static manifold.

Proposition 3.1 *Let $M = L \times_{\varepsilon f} \mathbb{R}$ be a static manifold and $V = a\partial_t + W$, where $a \in C^\infty(M)$ and $W \in \mathfrak{X}(M)$ with $W \perp \partial_t$. Then V is a Killing vector field if and only if V^t is a Killing vector field on L for each $t \in \mathbb{R}$ and the following equations hold*

$$\frac{\mathrm{d}}{\mathrm{d}t} V^t = -\varepsilon f^2 \nabla^L a^t, \tag{1}$$

$$V^t(\ln f) = -a_t. \tag{2}$$

Proof Since $L \times \{t\}$ is a geodesic hypersurface of M, it is straightforward that V^t is a Killing field on L for all $t \in \mathbb{R}$. We state Eqs. 1 and 2.

Given $X \in \mathfrak{X}(L)$, from $g(\nabla_{\partial_t} V, \tilde{X}) = -g(\nabla_{\tilde{X}} V, \partial_t)$, we get

$$\begin{aligned} &ag(\nabla_{\partial_t}\partial_t, \tilde{X}) + g(\nabla_{\partial_t} W, \tilde{X}) \\ &\quad = -\varepsilon f^2 \tilde{X}(a) - ag(\nabla_{\tilde{X}}\partial_t, \partial_t) - g(\nabla_{\tilde{X}} W, \partial_t). \end{aligned}$$

Since ∂_t is Killing and $g(\nabla_{\tilde{X}} W, \partial_t) = -g(W, \nabla_{\tilde{X}}\partial_t) = 0$, the above simplifies to

$$g(\nabla_{\partial_t} W, \tilde{X}) = -\varepsilon f^2 \tilde{X}(a).$$

We compute each term at a point $(p_0, t_0) \in L \times \mathbb{R}$. The projection $\pi : L \times \{t_0\} \to L$ is an isometry, so we have

$$\begin{aligned} g(\nabla_{\partial_t} W, \tilde{X})_{(p_0,t_0)} &= \partial_t|_{(p_0,t_0)} g(W, \tilde{X}) = \frac{\mathrm{d}}{\mathrm{d}t} g(W, \tilde{X})_{(p_0,t)}|_{t=t_0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} g_L\left(\pi_{*(p_0,t)}(W), X_{p_0}\right)|_{t=t_0} = \frac{\mathrm{d}}{\mathrm{d}t} g_L\left(W^t_{p_0}, X_{p_0}\right)|_{t=t_0} \\ &= g_L\left(\frac{\mathrm{d}}{\mathrm{d}t} W^t_{p_0}|_{t=t_0}, X_{p_0}\right) = g_L\left(\frac{\mathrm{d}}{\mathrm{d}t} V^t_{p_0}|_{t=t_0}, X_{p_0}\right). \end{aligned}$$

On the other hand, if γ is an integral curve of X, then

$$\tilde{X}(a)_{(p_0,t_0)} = \frac{\mathrm{d}}{\mathrm{d}t}a(\gamma(t),t_0)|_{t=t_0} = \frac{\mathrm{d}}{\mathrm{d}t}a^{t_0}(\gamma(t))|_{t=t_0}$$
$$= X_{p_0}(a^{t_0}) = g(\nabla a^{t_0}, X_{p_0}).$$

Therefore, $\frac{\mathrm{d}}{\mathrm{d}t}V^t = -\varepsilon f^2\nabla^L a^t$ for all $t \in \mathbb{R}$.

From $g(\nabla_{\partial_t}V, \partial_t) = 0$ it follows $g(\nabla_{\partial_t}W, \partial_t) = -\varepsilon f^2 a_t$. But

$$g(\nabla_{\partial_t}W, \partial_t) = -g(W, \nabla_{\partial_t}\partial_t) = \varepsilon f g(W, \widetilde{\nabla f})$$
$$= \varepsilon f W^t(f) = \varepsilon f V^t(f),$$

and therefore $V^t(\ln f) = -a_t$ for all $t \in \mathbb{R}$. The "only if" part is clear. □

Observe that $\frac{\mathrm{d}}{\mathrm{d}t}V^t$ is the vectorial derivative of V^t, thus it is also a Killing vector field.

Theorem 3.2 *Let $M = L \times_{\varepsilon f} \mathbb{R}$ be a static manifold with L complete. If $V \in \mathfrak{X}(M)$ is a Killing vector field, then one of the following holds.*

(1) *$V = (a_1 t + a_2)\partial_t + \widetilde{W}$ where $a_1, a_2 \in \mathbb{R}$ and $W \in \mathfrak{X}(L)$ is a Killing vector field with $W(\ln f) = -a_1$.*
(2) *L decomposes as a static manifold $\left(N \times \mathbb{R}, g_N + \lambda(x)^2\mathrm{d}s^2\right)$ and $f(x,s) = \lambda(x)c(s)$ for certain $c \in C^\infty(\mathbb{R})$.*

Proof Decompose $V = a\partial_t + W$ where $a \in C^\infty(M)$ and $W \perp \partial_t$. Using Proposition 3.1, V^t is a Killing field on L and $\frac{\mathrm{d}}{\mathrm{d}t}V^t = -\varepsilon f^2\nabla^L a^t$. Therefore, $\frac{\mathrm{d}}{\mathrm{d}t}V^t$ is a Killing and irrotational vector field for each $t \in \mathbb{R}$.

Fix $t \in \mathbb{R}$, take γ a geodesic in L with $\gamma(0) = p$ and call $y(s) = a^t(\gamma(s))$. Then $y_s = g_L(\nabla^L a^t, \gamma')$ and

$$y_{ss} = g_L(\nabla^L_{\gamma'}\nabla^L a^t, \gamma') = \varepsilon\frac{2(f\circ\gamma)_s}{(f\circ\gamma)^3}g_L\left(\frac{\mathrm{d}}{\mathrm{d}t}V^t, \gamma'\right) = -\frac{2(f\circ\gamma)_s}{f\circ\gamma}g_L(\nabla^L a^t, \gamma')$$
$$= -2(\ln f\circ\gamma)_s y_s.$$

Therefore, $y_s(s) = y_s(0)\left(\frac{f(p)}{f(\gamma(s))}\right)^2$. If $\frac{\mathrm{d}}{\mathrm{d}t}V^t_p = 0$ then $\nabla^L a^t = 0$ at p, hence $y_s(0) = 0$ and thus $y_s \equiv 0$. Therefore, a^t is constant in a neighborhood of p, which implies that $\frac{\mathrm{d}}{\mathrm{d}t}V^t = 0$ in this same neighborhood. Since L is supposed connected, for each $t \in \mathbb{R}$ there are two possibilities: $\frac{\mathrm{d}}{\mathrm{d}t}V^t \equiv 0$ or it does not have any zero.

If $\frac{\mathrm{d}}{\mathrm{d}t}V^t \equiv 0$ for all $t\in\mathbb{R}$, then $V = a\partial_t + \widetilde{W}$ where $W \in \mathfrak{X}(L)$ is a Killing field. Moreover, by Eq. 1, a only depends on t and Eq. 2 implies that there is a constant $a_1 \in \mathbb{R}$ with $a_t = -W(\ln f) = a_1$, from which the first assertion follows.

Suppose on the contrary that there is some $t_0 \in \mathbb{R}$ such that $\frac{\mathrm{d}}{\mathrm{d}t}V^{t_0}$ has no zeros. If $\alpha : \mathbb{R} \to L$ is an integral curve of $\frac{\mathrm{d}}{\mathrm{d}t}V^{t_0}$, then $\frac{\mathrm{d}}{\mathrm{d}s}a^{t_0}(\alpha(s)) \neq 0$ for all $s \in \mathbb{R}$. Therefore, since a is constant through the orthogonal leaves of $\frac{\mathrm{d}}{\mathrm{d}t}V^{t_0}$, α only intersects them one time, which implies that L decomposes as a static manifold $N \times_\lambda \mathbb{R}$, where $\partial_s = \frac{\mathrm{d}}{\mathrm{d}t}V^{t_0}$ (see Remark 2.3). Since $\partial_s = -\varepsilon f^2\nabla a^{t_0}$, then a^{t_0} only depends on s and a direct computation gives us $\nabla a^{t_0} = \frac{a^{t_0}_s}{\lambda^2}\partial_s$. Replacing in the above equation, $f^2 = -\varepsilon\frac{\lambda^2}{a^{t_0}_s}$ and thus $f(x,s) = \lambda(x)c(s)$, where $c(s) = \sqrt{\frac{-\varepsilon}{a^{t_0}_s}}$. □

Remark 3.3 Recall that in the above theorem, as in other results in this paper, when L is one-dimensional the factor N is a point and therefore can be removed.

Corollary 3.4 *If $M = L \times_{\varepsilon f} \mathbb{R}$ is a static manifold with L compact, then any Killing vector field is of the form $a\partial_t + \widetilde{W}$ where $a \in \mathbb{R}$ and $W \in \mathfrak{X}(L)$ is a Killing vector field with $W(f) = 0$.*

Proof Since L is compact, only the first case of the above theorem holds. Moreover, f must have a critical point and so $a_1 = 0$. □

A condition on the lightlike sectional curvature also gives us information about Killing vector fields.

Corollary 3.5 *Let $M = L \times_{-f} \mathbb{R}$ be a static space with L complete and dimension greater than one. If there is a point $(p_0, t_0) \in M$ such that $\mathcal{K}(\Pi) \neq 0$ for any degenerate plane Π of $T_{(p_0,t_0)}M$, then any Killing vector field is of the form $(a_1 t + a_2)\partial_t + \widetilde{W}$, where $a_1, a_2 \in \mathbb{R}$ and $W \in \mathfrak{X}(L)$ is a Killing vector field with $W(\ln f) = -a_1$.*

Proof Suppose that L can be decomposed as $\left(N \times \mathbb{R}, g_N + \lambda(x)^2 \mathrm{d}s^2\right)$, being p_0 identified with (x_0, s_0), and $f(x, s) = \lambda(x)c(s)$. Take a unitary vector $v \in T_{x_0}N$ and Π the degenerate plane spanned by v and $u = \frac{1}{\lambda(x_0)}\partial_s - \frac{1}{f(s_0,x_0)}\partial_t$. We know that $\mathcal{K}_u(\Pi) = K^L\left(span(v, \partial_s)\right) + \frac{g(\nabla_v \widetilde{\nabla f}, v)}{f}$, but being L also a static manifold,

$$K^L\left(span(v, \partial_s)\right) = -\frac{g(\nabla_v \widetilde{\nabla \lambda}, v)}{\lambda},$$
$$g(\nabla_v \widetilde{\nabla f}, v) = cg(\nabla_v \widetilde{\nabla \lambda}, v).$$

Thus, $\mathcal{K}_u(\Pi) = 0$, which is a contradiction. Applying Theorem 3.2 we get the conclusion. □

4 Standard static vector fields

In this section, we show that if a static manifold M admits two different static decompositions, then the base itself is a static manifold and M can be viewed as a special type of warped product. A standard static vector field in a Lorentzian manifold will be always supposed timelike.

Proposition 4.1 *Let $M = L \times_{\varepsilon f} \mathbb{R}$ be a static manifold. If $V = a\partial_t + W$, where $a \in C^\infty(M)$ and $W \perp \partial_t$, is a static vector field, then V^t is a static vector field on L for each $t \in \mathbb{R}$. Moreover, if* $\dim L \geq 2$, *in the open set $\{p \in L : a^t \neq 0, V^t \neq 0\}$ it holds*

$$X\left(\ln(a^t f)\right) = X\left(\ln\sqrt{g(V^t, V^t)}\right) \tag{3}$$

for all $X \in \mathfrak{X}(L)$ with $X \perp V^t$.

Proof The first assertion follows easily because $L \times \{t\}$ are geodesic hypersurfaces. Suppose that $dimL \geq 2$ and take $\xi = -\varepsilon g(W, W)\partial_t + af^2 W$, which is a vector field orthogonal to V. Since V is static, $g(\nabla_{\tilde{X}} V, \xi) = 0$ for all $X \in \mathfrak{X}(L)$ with $X \perp W$. But

$$\begin{aligned} g(\nabla_{\tilde{X}} V, \xi) &= \left(\tilde{X}(a) + a\tilde{X}(\ln f)\right) g(\partial_t, \xi) + g(\nabla_{\tilde{X}} W, \xi) \\ &= -\left(\tilde{X}(a) + a\tilde{X}(\ln f)\right) f^2 g(W, W) + \frac{af^2}{2}\tilde{X}\left(g(W, W)\right) \\ &= -\left(X(a^t) + a^t X(\ln f)\right) g(V^t, V^t) f^2 + \frac{a^t f^2}{2} X\left(g(V^t, V^t)\right). \end{aligned}$$

Therefore, $\left(X(a^t) + a^t X(\ln f)\right) g(V^t, V^t) = \frac{a^t}{2} X(g(V^t, V^t))$. Where $a^t \neq 0$ and $V^t \neq 0$, we can write

$$X\left(\ln(a^t f)\right) = X\left(\ln\sqrt{g(V^t, V^t)}\right).$$

□

Proposition 4.2 *Let $M = L \times_{\varepsilon f} \mathbb{R}$ be a static manifold with L complete and V a static vector field linearly independent to ∂_t at some point. Then it holds the following.*

(1) *For each $t \in \mathbb{R}$, V^t is identically zero or it does not have zeros. In fact, there exists a dense open subset $\Theta \subset \mathbb{R}$ such that the second statement holds for all $t \in \Theta$.*
(2) *If moreover V is standard, then V^t is standard static in L for each $t \in \Theta$. So, fixed $t \in \Theta$, L decomposes as $\left(N \times \mathbb{R}, g_N + \lambda(x)^2 \mathrm{d}s^2\right)$ where V^t is identified with ∂_s.*

Proof (1) Fixed $t \in \mathbb{R}$, call $A = \{p \in L : V \text{ and } \partial_t \text{ are l.i. at } (p, t)\}$ and $B = A^c$. It is clear that A is open and since the orthogonal leaf of V and ∂_t are geodesic, B is also open. Therefore, $A = L$ and thus V^t does not have zeros or $B = L$ and $V^t \equiv 0$. Now, call $\Theta = \{t \in \mathbb{R} : V^t \text{ does not have zeros}\}$, which obviously is open. If $V^t \equiv 0$ for all $t \in (-\delta, \delta)$, then V and ∂_t are linearly dependent in $L \times (-\delta, \delta)$, but since they are Killing vector fields, they must be linearly dependent in the whole M, which is a contradiction. Therefore Θ is dense.

(2) Given $t \in \Theta$, using the above point and Proposition 4.1, we know that V^t is a static vector field without zeros in L. We show that it gives rise to a global decomposition. Call $F_{(p,t)}$ the orthogonal leaf of V through (p, t) and N the orthogonal leaf of V^t through p, which is inside $\pi\left(L \times \{t\} \cap F_{(p,t)}\right)$. Take α an integral curve of V^t with $\alpha(0) \in N$ and suppose that there is $s > 0$ with $\alpha(s) \in N$. Since V is standard, there is a global projection $P : M \to \mathbb{R}$ such that it is constant through the orthogonal leaves of V and $P_{*(p,t)}(v)$ gives the component in the direction of $V_{(p,t)}$ of any vector $v \in T_{(p,t)}M$. If we call $\gamma(s) = (\alpha(s), t)$, then $P(\gamma(s))$ has a critical point $s_1 \in (0, s)$, because $\gamma(0), \gamma(s) \in F_{(p,t)}$. But then

$$\begin{aligned} g\left(V^t_{\alpha(s_1)}, V^t_{\alpha(s_1)}\right) &= g\left(\gamma'(s_1), V_{\gamma(s_1)}\right) \\ &= P_{*\gamma(s_1)}\left(\gamma'(s_1)\right) g\left(V_{\gamma(s_1)}, V_{\gamma(s_1)}\right) = 0, \end{aligned}$$

which is a contradiction. Using Remark 2.3, L can be decomposed as

$$\left(N \times \mathbb{R}, g_N + \lambda(x)^2 \mathrm{d}s^2\right),$$

where ∂_s is identified with V^t. □

As a consequence of the above proposition, we can prove the main result of [7] in a different way.

Theorem 4.3 *Let $M = L \times_{\varepsilon f} \mathbb{R}$ be a static manifold with L compact. Then any other standard static vector field is proportional to ∂_t and so M admits a unique decomposition as a static manifold.*

The following Proposition will be the key to prove our main results.

Proposition 4.4 *Let $M = L \times_{\varepsilon f} \mathbb{R}$ be a static manifold with L complete and V a standard static vector field linearly independent to ∂_t at a point (p_0, t_0). If V^t is proportional to V^{t_0} for all $t \in \mathbb{R}$, then M is isometric to one of the following.*

(1) $(N \times \mathbb{R}^2, g_N + \lambda(x)^2 \mathrm{d}s^2 + \varepsilon f(x)^2 \mathrm{d}t^2)$ *and* $V = \partial_s$.

(2) $\left(N \times \mathbb{H}^2_{[\varepsilon]}(r), g_N + \lambda(x)^2(\mathrm{d}s^2 + \varepsilon \cosh^2(rs)\mathrm{d}t^2)\right)$ *and*

$$V = \left(-\frac{\varepsilon}{r} h_t(t) \tanh(rs) + \gamma\right) \partial_t + h(t)\partial_s,$$

where $h(t) = \alpha \sin(rt + \beta)$ *if* $\varepsilon = -1$ *or* $h(t) = \alpha e^{rt} + \beta e^{-rt}$ *if* $\varepsilon = 1$ *and* $\alpha, \beta, \gamma \in \mathbb{R}$.

(3) $\left(N \times \widehat{\mathbb{H}}^2_{[\varepsilon]}(r), g_N + \lambda(x)^2(\mathrm{d}s^2 + \varepsilon e^{2rs}\mathrm{d}t^2)\right)$ *and*

$$V = \left(\frac{\varepsilon\alpha}{2r} e^{-2rs} - \frac{r\alpha}{2} t^2 - r\beta t + \gamma\right) \partial_t + (\alpha t + \beta)\partial_s,$$

where $\alpha, \beta, \gamma \in \mathbb{R}$.

(4) $\left(N \times \mathbb{R}^2_{[\varepsilon]}, g_N + \lambda(x)^2(\mathrm{d}s^2 + \varepsilon \mathrm{d}t^2)\right)$ *and* $V = \gamma\partial_t + \partial_s$, *where* $\gamma \in \mathbb{R}$.

Proof Suppose that $V^t = h(t)V^{t_0}$ for some $h \in C^\infty(\mathbb{R})$ with $h(t_0) = 1$. Proposition 4.2 ensures that L decomposes as $\left(N \times \mathbb{R}, g_N + \lambda(x)^2\mathrm{d}s^2\right)$ where ∂_s is identified with V^{t_0}.

If $X \in \mathfrak{X}(N)$, using Eq. 1 of Proposition 3.1 we get $g(X, \nabla a^t) = 0$ and thus a only depends on s and t. On the other hand, multiplying by V^{t_0} in Eq. 1 we get

$$a_s(t, s) = -\varepsilon \frac{h_t(t)\lambda(x)^2}{f(x, s)^2}, \tag{4}$$

and Eq. 2 of Proposition 3.1 can be written as

$$a_t(s, t) = -h(t)(\ln f)_s. \tag{5}$$

Now we consider two possibilities:

(1) $a \equiv 0$. Then above equations give us that $h \equiv 1$, and f only depends on x, i.e., M is isometric to $(N \times \mathbb{R}^2, g_N + \lambda(x)^2\mathrm{d}s^2 + \varepsilon f(x)^2\mathrm{d}t^2)$ and $V = \partial_s$.

(2) There is a point (s_0, t_0) with $a(s_0, t_0) \neq 0$. Take

$$A = \{(s, t) \in \mathbb{R}^2 : a(s, t) \neq 0\}.$$

Equation 3 of Proposition 4.1 reduces to $X(\ln f) = X(\ln \lambda)$ in $N \times A$ for all $X \in \mathfrak{X}(N)$, which implies that there is certain function c such that $f(x, s) = \lambda(x)c(s)$ for all $(x, s, t) \in N \times A$.

Take $B = \left(\overline{A}\right)^c$. If $B = \emptyset$, then by continuity $f(x, s) = \lambda(x)c(s)$ for all $(x, s, t) \in N \times \mathbb{R}^2$. If $B \neq \emptyset$, then $a \equiv 0$ in $N \times B$ and so f only depends on x, i.e. $f(x, s) = F(x)$ for all $(x, s, t) \in N \times B$ where F is certain function. Since $\lambda(x)c(s) = F(x)$ for all $(x, s, t) \in Fr(N \times A) = N \times Fr(A)$, it it easy to show that c can be extended to the whole $\mathbb{R}$ and, with this extension, it holds $f(x, s) = \lambda(x)c(s)$ for all $(x, s) \in N \times \mathbb{R}^2$.

If we call $S = \mathbb{R} \times_{\varepsilon c} \mathbb{R}$, then M is the warped product $N \times_\lambda S$. We now classify this surface S. Equations 4 and 5 reduce to

$$a_s = -\varepsilon \frac{h_t(t)}{c(s)^2},$$
$$a_t = -h(t)\left(\ln c(s)\right)_s,$$

and using the Schwarz's Theorem we get the differential equations

$$c_{ss}(s)c(s) - c_s(s)^2 = k, \qquad h_{tt}(t) = \varepsilon k h(t) \tag{6}$$

for some constant $k \in \mathbb{R}$. The solutions of (6) are

- $c(s) = \frac{\sqrt{-k}}{r} \sinh(rs + b)$ or $c(s) = \frac{\sqrt{-k}}{r} \sin(rs + b)$ if $k < 0$.
- $c(s) = e^{rs+b}$ if $k = 0$.
- $c(s) = \frac{\sqrt{k}}{r} \cosh(rs + b)$ if $k > 0$.

Since $c(s) > 0$ for all $s \in \mathbb{R}$ we should discard the case $k < 0$. In the case $k > 0$, solving the above differential equations, there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$V = \left(\frac{r\alpha}{\sqrt{k}} \cos(\sqrt{k}t + \beta) \tanh(rs + b) + \gamma\right) \partial_t + \alpha \sin(\sqrt{k}t + \beta)\partial_s$$

if $\varepsilon = -1$ or

$$V = \left(-\frac{r}{\sqrt{k}} \left(\alpha e^{\sqrt{k}t} - \beta e^{-\sqrt{k}t}\right) \tanh(rs + b) + \gamma\right) \partial_t + \left(\alpha e^{\sqrt{k}t} + \beta e^{-\sqrt{k}t}\right) \partial_s$$

if $\varepsilon = 1$. Now, we obtain point (2) rescaling with $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\Phi(s,t) = (s + \frac{b}{r}, \frac{\sqrt{k}}{r} t)$.

If $k = 0$, we can suppose $b = 0$ rescaling s, and thus $c(s) = e^{rs}$ with $r \neq 0$ or $c(s) = 1$. In the first case, $S = \widehat{\mathbb{H}}^2_{[\varepsilon]}(r)$ and

$$V = \left(\frac{\varepsilon\alpha}{2r} e^{-2rs} - \frac{r\alpha}{2} t^2 - r\beta t + \gamma\right) \partial_t + (\alpha t + \beta)\partial_s.$$

In the second case $S = \mathbb{R}^2_{[\varepsilon]}$, $a(s,t) = -\varepsilon\alpha s + \gamma$ and $h(t) = \alpha t + \beta$. But $\alpha = 0$ and $\beta = 1$ because V does not have zeros and $h(t_0) = 1$. □

Remark 4.5 The case $N \times_\lambda \mathbb{H}^2(r)$ and $N \times_\lambda \widehat{\mathbb{H}}^2(r)$ are equivalent since $\mathbb{H}^2(r)$ and $\widehat{\mathbb{H}}^2(r)$ are isometric spaces. However, $N \times_\lambda \mathbb{H}^2_{[-1]}(r)$ and $N \times_\lambda \widehat{\mathbb{H}}^2_{[-1]}(r)$ are not equivalent because $\mathbb{H}^2_{[-1]}(r)$ is complete and $\widehat{\mathbb{H}}^2_{[-1]}(r)$ is not.

As an immediate consequence we obtain the following.

Corollary 4.6 *Let $M = L \times_{\varepsilon f} \mathbb{R}$ be a complete two dimensional static manifold. If there exists V a non-identically zero Killing vector field linearly independent to ∂_t at some point, then M is isometric to $\mathbb{R}^2_{[\varepsilon]}$ or $\mathbb{H}^2_{[\varepsilon]}(r)$.*

Proof Since $dim L = 1$, V is also irrotational, V^t is linearly dependent to a fixed V^{t_0} for all $t \in \mathbb{R}$ and the proof of Proposition 4.4 works with N reduced to a point, although V is not necessarily standard and, maybe, with zeros. □

Now, we show that a manifold with more than one static decomposition is a particular type of warped product.

Proposition 4.7 *Let $M = L \times_{\varepsilon f} \mathbb{R}$ be a static manifold with L complete. If there exists V a standard static vector field linearly independent to ∂_t at some point, then M decomposes as $(N \times \mathbb{R}^2, g_N + \lambda(x)^2 ds^2 + \varepsilon f(x)^2 dt^2)$ and $V = \partial_s$ or L decomposes as $\left(N \times \mathbb{R}, g_N + \lambda(x)^2 ds^2\right)$ and $f(x,s) = \lambda(x)c(s)$, i.e., M is the warped product $N \times_\lambda (\mathbb{R} \times_c \mathbb{R})$.*

Proof Using Theorem 3.2, M decomposes as a warped product $N \times_\lambda (\mathbb{R} \times_c \mathbb{R})$ or $V = (a_1 t + a_2)\partial_t + \widetilde{W}$ where $a_1, a_2 \in \mathbb{R}$ and $W \in \mathfrak{X}(L)$. But in this last case, we can apply Proposition 4.4 to obtain that M may also be decomposed as $(N \times \mathbb{R}^2, g_N + \lambda(x)^2 \mathrm{d}s^2 + \varepsilon f(x)^2 \mathrm{d}t^2)$ and $V = \partial_s$ or again as $N \times_\lambda (\mathbb{R} \times_c \mathbb{R})$. □

Remark 4.8 As we said in the introduction, euclidean, Minkowski, hyperbolic and a portion of the anti De Sitter spaces have different static decompositions and therefore they must fulfill the above proposition. But this can be easily checked taking into account that the hyperbolic space can be viewed as

$$\mathbb{H}^n = \left(\mathbb{R}^n, dx_1^2 + e^{2x_1}\left(\sum_{i=2}^{n} dx_i^2\right)\right)$$

and the above mentioned portion of anti De Sitter as

$$\left(\mathbb{R}^n, dx_1^2 + e^{2x_1}\left(\sum_{i=2}^{n-1} dx_i^2 - dx_n^2\right)\right).$$

Corollary 4.9 *Let $M = L \times_\varepsilon \mathbb{R}$ be a static manifold with L complete and with constant warping function. If V is a standard static vector field which is linearly independent to ∂_t at some point, then M is isometric to $(N \times \mathbb{R}^2, g_N + \lambda(x)^2 \mathrm{d}s^2 + \varepsilon \mathrm{d}t^2)$ where $V = \partial_s$ or to a direct product $N \times \mathbb{R}^2_{[\varepsilon]}$.*

5 Main results

We are already able to classify, under a curvature hypothesis, manifolds with more than one static decomposition. We start assuming that the base has a point with positive curvature.

Theorem 5.1 *Let $M = L \times_{\varepsilon f} \mathbb{R}$ be a static manifold with L complete and dimension greater than one. Suppose that there exists a standard static vector field V linearly independent to ∂_t at some point and there is $p \in L$ with $K^L(\Pi) > 0$ for any plane Π of $T_p L$. Then M is isometric to*

(1) $(N \times \mathbb{R}^2, g_N + \lambda(x)^2 \mathrm{d}s^2 + \varepsilon f(x)^2 \mathrm{d}t^2)$ *and* $V = \partial_s$.

(2) $\left(N \times \mathbb{H}^2_{[\varepsilon]}(r), g_N + \lambda(x)^2(\mathrm{d}s^2 + \varepsilon \cosh^2(rs)\mathrm{d}t^2)\right)$ *and*

$$V = \left(-\frac{\varepsilon}{r} h_t(t) \tanh(rs) + \gamma\right)\partial_t + h(t)\partial_s,$$

where $h(t) = \alpha \sin(rt + \beta)$ *if* $\varepsilon = -1$ *or* $h(t) = \alpha e^{rt} + \beta e^{-rt}$ *if* $\varepsilon = 1$ *and* $\alpha, \beta, \gamma \in \mathbb{R}$.

(3) $\left(N \times \widehat{\mathbb{H}}^2_{[\varepsilon]}(r), g_N + \lambda(x)^2(\mathrm{d}s^2 + \varepsilon e^{2rs}\mathrm{d}t^2)\right)$ *and*

$$V = \left(\frac{\varepsilon\alpha}{2r} e^{-2rs} - \frac{r\alpha}{2} t^2 - r\beta t + \gamma\right)\partial_t + (\alpha t + \beta)\partial_s,$$

where $\alpha, \beta, \gamma \in \mathbb{R}$.

(4) $\left(N \times \mathbb{R}^2_{[\varepsilon]}, g_N + \lambda(x)^2(\mathrm{d}s^2 + \varepsilon \mathrm{d}t^2)\right)$ *and* $V = \gamma\partial_t + \partial_s$, *where* $\gamma \in \mathbb{R}$.

Proof We proceed by induction over $dimL$. Suppose first that $dimL = 2$. Using Proposition 4.2, V^t is standard static for t in a dense open set Θ. Fix $t_0 \in \Theta$ and decompose L as $\left(N \times \mathbb{R}, g_N + \lambda(x)^2 \mathrm{d}s^2\right)$ where $\partial_s = V^{t_0}$. If there is a $t_1 \in \Theta$ with V^{t_1} and ∂_s linearly independent at some point, then Corollary 4.6 ensures $L = \mathbb{H}^2$ or $L = \mathbb{R}^2$, which contradicts the curvature hypothesis. Therefore V^t is linearly dependent to ∂_s for all $t \in \mathbb{R}$ and Proposition 4.4 proves the statement.

Now, assuming the statement for $dimL = n - 1$, we will prove it for $dimL = n$. As before, $L = \left(N \times \mathbb{R}, g_N + \lambda(x)^2 \mathrm{d}s^2\right)$, where $\partial_s = V^{t_0}$. If there is $t \in \Theta$ with V^t linearly independent to ∂_s at some point, applying the induction hypothesis, L is isometric to a warped product $S \times_\mu \mathbb{H}^2_{[\varepsilon]}(r)$, $S \times_\mu \widehat{\mathbb{H}}^2_{[\varepsilon]}(r)$, $S \times_\mu \mathbb{R}^2_{[\varepsilon]}$ or to $(S \times \mathbb{R}^2, g_S + \mu(z)^2 \mathrm{d}u^2 + \lambda(z)^2 \mathrm{d}s^2)$ where $V^t = \partial_u$ and $V^{t_0} = \partial_s$. In the first three cases, any tangent plane to the fiber has nonpositive curvature and should be discarded. In the last case, V^t is orthogonal to V^{t_0} and we can obtain a contradiction using the continuity of V^t respect to t. Therefore, V^t must be linearly dependent to ∂_s for all $t \in \mathbb{R}$ and applying Proposition 4.4 we get the result. □

We also obtain the same classification if the manifold is Einstein.

Theorem 5.2 *Let $M = L \times_{\varepsilon f} \mathbb{R}$ be an Einstein static manifold with L complete and dimension greater than one. If there exists a standard static vector field V linearly independent to ∂_t at some point, then M is isometric to $\left(N \times \mathbb{R}^2, g_N + \lambda(x)^2 \mathrm{d}s^2 + \varepsilon f(x)^2 \mathrm{d}t^2\right)$ where $V = \partial_s$ or to a warped product $N \times_\lambda \mathbb{H}^2_{[\varepsilon]}(r)$, $N \times_\lambda \widehat{\mathbb{H}}^2_{[\varepsilon]}(r)$ or $N \times_\lambda \mathbb{R}^2_{[\varepsilon]}$.*

Proof First, note that for any static space $\left(P \times \mathbb{R}, g_P + \varepsilon h^2 \mathrm{d}t^2\right)$ it holds

$$Ric(X, Y) = Ric^P(X, Y) - \frac{1}{h} H_h(X, Y) \quad \text{for } X, Y \in \mathfrak{X}(P), \tag{7}$$

$$Ric(\partial_t, \partial_t) = -\varepsilon h \Delta^P h, \tag{8}$$

where H_h is the hessian of h and Δ^P the laplacian operator in P.

Proposition 4.7 says that M is $\left(N \times \mathbb{R}^2, g_N + \lambda(x)^2 \mathrm{d}s^2 + \varepsilon f(x)^2 \mathrm{d}t^2\right)$ and $V = \partial_s$ or L decomposes as $\left(N \times \mathbb{R}, g_N + \lambda(x)^2 \mathrm{d}s^2\right)$ and the warping function f also decomposes as $f(x, s) = \lambda(x) c(s)$, where $\lambda \in C^\infty(N)$ and $c \in C^\infty(\mathbb{R})$. Thus, it is enough to analyze the second case and show that $c(s)$ is e^s, $\cosh s$ or a positive constant.

Using that M is static, Einstein and $dimM \geq 3$, there is $\delta \in \mathbb{R}$ such that Eq. 7 can be written as

$$Ric^L(X, Y) = \frac{1}{f} H_f(X, Y) + \delta g(X, Y) \quad \text{for } X, Y \in \mathfrak{X}(L). \tag{9}$$

Since L is also the static manifold $N \times_\lambda \mathbb{R}$, we have from Eq. 8

$$Ric^L(\partial_s, \partial_s) = -\lambda \Delta^N \lambda.$$

Moreover, since $f(x, s) = \lambda(x) c(s)$,

$$H_f(\partial_s, \partial_s) = \lambda \left(c g(\nabla\lambda, \nabla\lambda) + c_{ss}\right).$$

Notice that $\partial_s \in \mathfrak{X}(L)$, thus we can replace the above two equations in Eq. 9 to get

$$-\lambda \Delta^N \lambda = g(\nabla\lambda, \nabla\lambda) + \frac{c_{ss}}{c} + \delta\lambda^2. \tag{10}$$

The left hand side and the first and third summands of the right hand side of this equation do not depend on the parameter s, thus we conclude that $\frac{c_{ss}}{c} = a$ for certain constant $a \in \mathbb{R}$. Moreover, since L is complete and c has not zeros, necessarily $a = r^2 \geq 0$ and therefore $c(s) = e^{rs+\beta}$ or $c(s) = \alpha \cosh(rs + \beta)$ with $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}$. Rescaling we obtain that M is isometric to $N \times_\lambda \mathbb{H}^2_{[\varepsilon]}(r)$, $N \times_\lambda \widehat{\mathbb{H}}^2_{[\varepsilon]}(r)$ or $N \times_\lambda \mathbb{R}^2_{[\varepsilon]}$. □

Remark 5.3 If L is the static manifold $N \times_\lambda \mathbb{R}$, then Eq. 7 says

$$Ric^L(X,Y) = Ric^N(X,Y) - \tfrac{1}{\lambda} H_\lambda(X,Y) \quad \text{for } X, Y \in \mathfrak{X}(N),$$

and since $f(x,s) = \lambda(x)c(s)$,

$$H_f(X,Y) = cH_\lambda(X,Y) \quad \text{for } X, Y \in \mathfrak{X}(N).$$

Replacing the above two equations in (9), we get the equation

$$Ric^N(X,Y) = \tfrac{2}{\lambda} H_\lambda(X,Y) + \delta g(X,Y) \quad \text{for } X, Y \in \mathfrak{X}(N), \tag{11}$$

which will be used in the next theorem.

Remark 5.4 The arguments used in Theorems 5.1 and 5.2 also work locally, so we can avoid the completeness of the base obtaining similar conclusions. More concretely, we would obtain that a neighborhood of the point where ∂_t and V (being V static but nonnecessarily standard) are linearly independent, is locally isometric to $(N \times \mathbb{R}^2, g_N + \lambda(x)^2 \mathrm{d}s^2 + \varepsilon f(x)^2 \mathrm{d}t^2)$ where $V = \partial_s$ or to a warped product $N \times_\lambda S$, where S is a surface of constant curvature.

Now, we particularize the above to the case of a (four-dimensional) spacetime. In the following theorem, it is shown that a fundamental component of static Einstein spacetimes with different decompositions are the Riemannian surfaces

$$\frac{1}{k_1 + \frac{k_2}{u} + k_3 u^2} \mathrm{d}u^2 + \left(k_1 + \frac{k_2}{u} + k_3 u^2\right) \mathrm{d}v^2, \tag{12}$$

where $k_1, k_2, k_3 \in \mathbb{R}$. Observe that surfaces of constant curvature $1, -1$, and 0 are included in this family for an appropriate choice of the constants.

Theorem 5.5 *Let $M = L \times_{-f} \mathbb{R}$ be an Einstein static spacetime. If there exists a timelike static vector field linearly independent to ∂_t at some point, then almost every point in M has a neighborhood isometric to a direct product of two surfaces with the same constant curvature or to a warped product $\Gamma \times_u S$, where Γ is a surface as the one given in (12) and S has constant curvature.*

Proof As it is said in Remark 5.4, since L is not supposed complete, from Theorem 5.2 we only obtain that M is locally a warped product $N \times_\lambda S$, where S has constant curvature (the other case in the remark should be discarded because a standard static vector field in a Lorentzian manifold is supposed timelike). Since $\dim N = 2$, we have $Ric^N = Kg$, being K the curvature of N, and thus Eq. 11 transforms to

$$H_\lambda = \frac{\lambda(K - \delta)}{2} g. \tag{13}$$

Taking trace we get

$$\Delta^N \lambda = \lambda(K - \delta) \tag{14}$$

and replacing in Eq. 10 we have

$$-\lambda^2 K = g(\nabla\lambda, \nabla\lambda) + a. \tag{15}$$

If λ is constant, then N has constant curvature $K = \frac{-a}{\lambda^2}$ and rescaling the metric of S, M is locally the direct product of two surfaces with the same constant curvature.

Suppose now that λ is not constant. Differentiating the above equation, $-\lambda \mathrm{d}K = (3K - \delta)\mathrm{d}\lambda$ and thus $\lambda^3 K = b + \frac{\delta}{3}\lambda^3$, for certain $b \in \mathbb{R}$. Now, Eq. 15 can be written as

$$-\frac{b}{\lambda} - \frac{\delta}{3}\lambda^2 = g(\nabla\lambda, \nabla\lambda) + a. \tag{16}$$

Equation 13 means that $\nabla\lambda$ is conformal and, since it is not constant, its critical points are isolated. Moreover, in a neighborhood of any point where $\nabla\lambda \neq 0$, N is a warped product $\left(\mathbb{R}^2, \mathrm{d}u^2 + \lambda_u^2 \mathrm{d}v^2\right)$ being $\partial_u = \frac{\nabla\lambda}{|\nabla\lambda|}$, [5,8,4]. In this coordinates, Eq. 16 is

$$\lambda_u^2 = -a - \frac{b}{\lambda} - \frac{\delta}{3}\lambda^2, \tag{17}$$

and reparametrizing with $(u, v) \mapsto (\lambda(u), v)$, N is locally isometric to

$$\frac{1}{h(u)}\mathrm{d}u^2 + h(u)\mathrm{d}v^2,$$

where $h(u) = -a - \frac{b}{u} - \frac{\delta}{3}u^2$. □

Remark 5.6 The same conclusion of this Theorem is obtained in [9], but with slightly different hypothesis. The author starts with a fixed Riemannian 3-dimensional metric and he supposes that different Einstein static spacetimes can be constructed using it. Then, he proves that the metric is locally the one given in (12). Observe that if we have a static manifold with two different static decompositions, a priori, we do not know if the respective bases are isometric.

Finally, we show that a condition on the lightlike sectional curvature ensures the uniqueness of the static decomposition in the Lorentzian case.

Theorem 5.7 *Let $M = L \times_{-f} \mathbb{R}$ be a static space with $dim L \geq 2$. If there exists a point $(p, t) \in M$ such that $\mathcal{K}(\Pi) \neq 0$ for any degenerate plane Π of $T_{(p,t)}M$, then M admits an unique decomposition as static space.*

Proof Let V be a (timelike) standard static vector field on M linearly independent to ∂_t at some point and suppose first that L is complete. Using Proposition 4.7, $L = \left(N \times \mathbb{R}, g_N + \lambda(x)^2 \mathrm{d}s^2\right)$ and $f(x, s) = \lambda(x)c(s)$ and we can show as in Corollary 3.5 that there is a degenerate plane at (p, t) with zero lighlike sectional curvature, which is a contradiction.

Now, although L is not necessarily complete, Proposition 4.7 is valid locally and we can still use the above arguments to show that in a neighborhood U of (p, t) there is a unique static decomposition. Therefore V and ∂_t are linearly dependent in U, but being Killing vector fields, this implies that in fact they are linearly dependent in the whole M. □

It would be interesting to have a more accurate classification of static manifolds admitting more than one standard static vector field at least in dimension 3 and 4.

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