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Abstract The model of massless relativistic particle relying in a curvature-dependent action functional is considered in the framework of generalized Robertson–Walker 4-spacetimes. The discussion is based on the number of nonvanishing Frenet curvatures, and we obtain characterizations, examples, and nonexistence results for the critical points of the action functional, which are included in the fibers.

Keywords Action functional · Total curvature · Massless relativistic particle · Frenet curvatures · Generalized Robertson–Walker spacetimes

Mathematics Subject Classification (2000) 53C80 · 53C50

1 Introduction

Action functionals are powerful tools to connect Mathematics and Physics. Indeed, many authors have applied action functionals to study particles, [4,5,7,12,14–16], and models of n-dimensional relativistic objects with rigidity, such as point particles ($n = 0$), [12–16], strings ($n = 1$) [6,17], and membranes ($n = 2$) [11]. In this way, a natural variational problem can be defined on a space of curves Γ in a Lorentzian manifold M,

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$$\mathcal{A} : \Gamma \longrightarrow \mathbb{R}, \quad \mathcal{A}(\gamma) = -\int_{\gamma} (m + \lambda\kappa_1),$$

where κ_1 is the first Frenet curvature of $\gamma : [a, b] \to M$, m is the particle mass, λ is a real parameter, and $\int_\gamma h$ means $\int_a^b h(s)|\gamma'(s)|\mathrm{d}s$ for any differentiable function $h : [a, b] \to \mathbb{R}$. The space Γ is supposed to be composed of reasonable curves, i. e., Frenet curves with a causal character, either closed or clamped (satisfying certain first-order boundary conditions). M. S. Plyushchay used this model to describe the massless relativistic particles, [14–16] in the four-dimensional Minkowski space.

We recall the following construction, [1]. Given a Riemannian manifold (P, g_P) and a smooth positive function $f : I \to \mathbb{R}$, where I is a real interval, the manifold $\bar{P} = I \times P$, with the Lorentzian metric $h = -\mathrm{d}t^2 + f(t)^2 g_P$ is, by definition, a *Generalized Robertson–Walker spacetime* with *warping function* f. For each $t \in I$, the spacelike submanifold $\{t\} \times P$ is known as a *fiber* or a *slice* as well as the *rest space* at t. Whenever P is a connected space of constant curvature c, $\bar{P}$ is called a *Robertson–Walker spacetime*.

Barros et al. [2] studied those critical points of the total curvature action functional contained in the rest spaces of Generalized Robertson–Walker 3-spacetimes, as a *mathematical test*. Also, in the higher-dimensional case, Ferrández et al. [7] considered actions in $(d+1)$-dimensions associated with null curves, whose action functional is a linear function of the curvature of the particle path.

This paper deals with critical points of the total curvature action functional

$$\mathcal{L} : \Gamma \longrightarrow \mathbb{R}, \quad \mathcal{L}(\gamma) = \int_{\gamma} \kappa_1,$$

where we consider Frenet curves of a generalized Robertson–Walker 4-spacetime contained in the rest spaces. This is a further step with respect to [2] in the sense that we work in the four-dimensional case, which is the most relevant dimension in relativity. The Euler–Lagrange equation is more involved but richer, from a mathematical point of view, than in lower dimensions. On the other hand, the analysis in a rest space is a priori compatible with the model developed for the Minkowski space in [14], which has the peculiarity that its classical equations of motion are consistent only for superrelativistic motion of a particle. We work in a more general framework than the Minkowski space, so if it admits an interpretation *à la* Plyushchay, it is still to be shown.

Section 2 is devoted to revising the Frenet apparatus of Frenet curves, in general semi-Riemannian manifolds, and to determining the Euler–Lagrange equations for a Frenet curve of order greater or equal than 2.

In Sect. 3, we focus on four-dimensional generalized Robertson–Walker spacetimes. Namely, we take a Frenet curve γ with curvature κ and torsion τ in a three-dimensional Riemannian manifold M, and we construct a curve $\overline{\gamma_t}$ in a fiber $\{t\} \times M$ and its Frenet apparatus. With these tools, we study, in the following sections, the Euler–Lagrange equations case by case, solving most of them. Surprisingly, it is not enough to solve the differential equations arising in all cases. In this way, we introduce another frame along the Frenet curve $\overline{\gamma_t}$, which we call fiber-adapted frame, which allows us to solve the remaining cases.

In sects. 4 and 5, we study curves $\overline{\gamma_t}$, which are the solutions of the Euler–Lagrange equation of the total curvature action. Our results are based on the conditions involving κ, τ, the warping function f, and the curvature tensor R of M along the curve. With respect to the curvature conditions, we found essentially two cases: in the first one, two specific components of the curvature tensor along the curve are simple functions of the torsion of the curve and we can analyze this case successfully. In the other case, general conditions for these components are not found. Since we want to obtain nontrivial solutions, we impose natural conditions on them compatible with the Robertson–Walker framework, obtaining new solutions in a variety of interesting cases. We also prove that there are not solutions satisfying those natural conditions in the most general situation, corresponding to four-order Frenet curves with nonconstant curvature κ.

Section 4 is devoted to studying the case of critical slices, i. e., $\dot{f}(t) = 0$. In Theorem 1, we see that a curve $\overline{\gamma_t}$ is a critical point of $\mathcal{L}$ if and only if either it is a geodesic in $\bar{M}$ or two specific components of the curvature tensor R along γ are simple functions of the torsion and its first derivative.

Section 5 contains a discussion based on the order of the Frenet curve $\overline{\gamma_t}$. For the case of order two, the characterization of the critical points relies again on two conditions on the curvature tensor along the curve. In addition, the curve γ should also satisfy that κ is a constant and $\ddot{f}(t) = 0$ (see Theorem 2). For Frenet curves of order 3, we impose the above-mentioned natural conditions on the curvature tensor. Theorem 3 studies the case $\tau = 0$, whereas we pay attention to case $\tau \neq 0$ in Theorem 4. Remarkably, specific expressions of the curvature and the torsion of the curve γ are obtained. For Frenet curves of order 4, we have two cases. Firstly, when the curvature $\kappa > 0$ is constant, we obtain a characterization or our critical points in terms of the curvature conditions as simple functions of the torsion and its first derivative (see Theorem 5). However, when κ is not constant, by imposing the natural curvature conditions, we reach to a nonexistence result in Theorem 6.

In addition, each theorem is followed by a corollary for the case of Robertson–Walker 4-spacetimes, since they satisfy the natural curvature conditions. Finally, each case of this discussion is illustrated by an example.

We would like to point out that some of the computations have been made by using elementary features of symbolic computation software.

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2 Preliminaries

Let $(\bar{M}, \bar{g})$ be an oriented semi-Riemannian manifold of signature ν such that $0 \leq \nu \leq n = \dim M$ and let $\bar{\nabla}$ denote the Levi-Civita connection. We consider a smooth curve $\bar{\gamma} : J \longrightarrow \bar{M}$, where $J \subset \mathbb{R}$ is an interval. The derivative of $\bar{\gamma}$ will be denoted by $\bar{\gamma}'$.

The curve $\bar{\gamma}$ is called a *Frenet curve* of order $d \in \{1, ..., n-1\}$ if there exists an orthonormal system of vector fields along $\bar{\gamma}$, $\{E_1, \dots, E_d\}$, and smooth functions $\kappa_1, \dots, \kappa_{n-1}$ defined on J, where $E_1 = \bar{\gamma}'/\xi$ and $\xi = \sqrt{|\bar{g}(\bar{\gamma}', \bar{\gamma}')|}$ is the speed of $\bar{\gamma}'$ (not necessarily constant), and $\kappa_i > 0, i = 1, \dots, d-1$ and $\kappa_d = \kappa_{d+1} \dots = \kappa_{n-1} = 0$, such that they verify the Frenet equations

$$\begin{aligned}
\bar{\nabla}_{E_1} E_1 &= \varepsilon_2 \kappa_1 E_2, \\
\bar{\nabla}_{E_1} E_i &= -\varepsilon_{i-1} \kappa_{i-1} E_{i-1} + \varepsilon_{i+1} \kappa_i E_{i+1}, \quad i = 2, \dots, d-1, \\
\bar{\nabla}_{E_1} E_d &= -\varepsilon_{d-1} \kappa_{d-1} E_{d-1},
\end{aligned}$$

being $\varepsilon_i = g(E_i, E_i) = \pm 1, i = 1, \dots, d$.

If $d = n$, the same definition works except that we also impose the basis $\{E_1, \dots, E_n\}$ to be positively oriented, then $\kappa_i > 0$, $i = 1, \dots, n-2$, and function κ_{n-1} could be either positive or negative.

It is said that the Frenet curve has Frenet curvatures $\{\kappa_1, \ldots, \kappa_{d-1}\}$ and Frenet system $\{E_1, \ldots, E_d\}$. Both Frenet system and Frenet curvatures are unique, since the equations provide a constructive method to obtain them. In this way, in order to ensure that the vector fields E_i are everywhere well defined and smooth, it is necessary to suppose that the Frenet curvatures will not be zero at isolated points.

Observe that a Frenet curve $\bar{\gamma}$ of order 1 ($\kappa_1 = 0$) satisfies $\bar{\nabla}_{\bar{\gamma}'}\bar{\gamma}' = (\ln \xi)'\bar{\gamma}'$. Then, an arc length reparametrization of $\bar{\gamma}$ is a geodesic.

In this paper, we deal with four-dimensional Lorentzian manifolds, $(\bar{M}, \bar{g})$. Then, the Frenet equations of a Frenet curve of order 4 become

$$\begin{aligned} \bar{\nabla}_{E_1} E_1 &= \varepsilon_2\kappa_1 E_2, \\ \bar{\nabla}_{E_1} E_2 &= -\varepsilon_1\kappa_1 E_1 + \varepsilon_3\kappa_2 E_3, \\ \bar{\nabla}_{E_1} E_3 &= -\varepsilon_2\kappa_2 E_2 + \varepsilon_4\kappa_3 E_4, \\ \bar{\nabla}_{E_1} E_4 &= -\varepsilon_3\kappa_3 E_3 \end{aligned} \tag{1}$$

and we must arrange them in a obvious way to obtain the Frenet equations of a Frenet curve of order 2 or 3.

Given $p, q \in \bar{M}$ and $v \in T_p\bar{M}$, $w \in T_q\bar{M}$, we define

$$\Gamma = \{\bar{\gamma} : [a, b] \to \bar{M} \quad / \quad \bar{\gamma}'(a) = v, \bar{\gamma}'(b) = w\}. \tag{2}$$

In case $p = q$ and $v = w$, the curves are called *closed*. Otherwise, they are known as *clamped*.

We are interested in studying critical points of the *total curvature action functional*

$$\mathcal{L} : \Gamma \longrightarrow \mathbb{R}, \quad \bar{\gamma} \longmapsto \int_{\bar{\gamma}} \kappa_1. \tag{3}$$

A trivial family of critical points are Frenet curves of order 1, representing absolute minima for the functional $\mathcal{L}$. Thus, we study critical points among Frenet curves of greater order. To determine them, we take a tangent vector at $\bar{\gamma} \in \Gamma$, which is nothing but a vector field W along the curve $\bar{\gamma}$. The critical points of the variational problem are those curves $\bar{\gamma} \in \Gamma$ such that

$$\delta\mathcal{L}(\bar{\gamma})[W] = 0, \quad \forall W \in T_{\bar{\gamma}}\Gamma.$$

Some standard arguments allow us to compute the above expression. Indeed, if we call $\Phi = \Phi(s, r) : [a, b] \times (-\varepsilon, \varepsilon) \to \bar{M}$ the variation of $\bar{\gamma}$, with variational field $W = W(s) = (\partial\Phi/\partial r)_{(s,0)}$ and longitudinal field $V = \partial\Phi/\partial s = \xi\, E_1$, we have $[V, W] = 0$ (see [9], Lemma 1.1), and we obtain $\bar{\nabla}_{E_1} W(a) = \bar{\nabla}_{E_1} W(b) = 0$. Thus, we have

$$\begin{aligned} \delta\mathcal{L}(\bar{\gamma})[W] &= \int_a^b \left(\frac{\partial\kappa_1(s, r)}{\partial r}\xi(s, r) + \kappa_1(s, r)\frac{\partial\xi(s, r)}{\partial r} \right) \mathrm{d}s \\ &= \int_a^b \bar{g}(\Omega(\bar{\gamma}), W)\mathrm{d}s + \left[\mathcal{B}(\bar{\gamma}, W)\right]_a^b, \end{aligned}$$

where

$$\begin{aligned} \Omega(\bar{\gamma}) &= \xi\bar{\nabla}_{E_1}\bar{\nabla}_{E_1} E_2 + \varepsilon_1\kappa_1' E_1 + \varepsilon_1\kappa_1\xi\bar{\nabla}_{E_1} E_1 + \xi\bar{R}(E_2, E_1)E_1, \\ \mathcal{B}(\bar{\gamma}, W) &= \bar{g}(\bar{\nabla}_{E_1} W, E_2) - \bar{g}(W, \bar{\nabla}_{E_1} E_2) - \varepsilon_1\kappa_1\,\bar{g}(W, E_1), \end{aligned}$$

are, respectively, the Euler–Lagrange and the boundary operators, being $\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z$ the curvature operator of $\bar{g}$ and κ_i' the derivative of the function κ_i with respect to the curve parameter. Consequently, $[\mathcal{B}(\bar{\gamma}, W)]_a^b = 0$. By using Eq. (1), the Euler–Lagrange operator becomes

$$\Omega(\bar{\gamma}) = -\varepsilon_2\varepsilon_3\xi\kappa_2^2 E_2 + \varepsilon_3\kappa_2' E_3 + \varepsilon_3\varepsilon_4\xi\kappa_2\kappa_3 E_4 + \xi\bar{R}(E_2, E_1)E_1,$$

with the obvious meaning for a Frenet curve of order 2 or 3.

Finally, the Frenet curve $\bar{\gamma}$ of order ≥ 2 in Γ is a critical point of $\mathcal{L}$ if and only if it satisfies the Euler–Lagrange equation

$$\xi\bar{R}(E_2, E_1)E_1 = \varepsilon_2\varepsilon_3\xi\kappa_2^2 E_2 - \varepsilon_3\kappa_2' E_3 - \varepsilon_3\varepsilon_4\xi\kappa_2\kappa_3 E_4. \quad (4)$$

3 Generalized Robertson-Walker spacetimes

Let (M, g) be an oriented three-dimensional Riemanninan manifold with Levi-Civita connection ∇. Let γ be a unit curve on M with Frenet system $\{T = \gamma', N, B\}$, curvature function κ and torsion τ, and Frenet equations

$$\nabla_T T = \kappa N,$$
$$\nabla_T N = -\kappa T + \tau B,$$
$$\nabla_T B = -\tau N.$$

Given an open interval $I \subset \mathbb{R}$ and a positive smooth function $f : I \longrightarrow \mathbb{R}$, we consider the warped product

$$\bar{M} = I \times M, \quad \bar{g} = -\mathrm{d}t^2 + f^2 g,$$

which is usually called *Generalized Robertson–Walker Spacetime*, since it was introduced in [1]. Given a point $t \in I$, we define the space of curves

$$\Gamma_t = \{\overline{\gamma_t} \in \Gamma \,/\, \overline{\gamma_t}(s) = (t, \gamma(s)),\ \forall s \in [a,b] \text{ and } \gamma : [a,b] \longrightarrow M \text{ a unit curve}\}.$$

That is to say, we are embedding the curve γ in the slice $\{t\} \times M \subset \bar{M}$. The Frenet system of $\overline{\gamma_t}$ will be denoted as in Eq. (1). Anyway, a useful positive oriented orthonormal frame along $\overline{\gamma_t}$ is the *fiber-adapted frame*

$$E_1 = \frac{1}{f}T, \ \frac{1}{f}N, \ \frac{1}{f}B, \ \partial_t. \quad (5)$$

In our case, some well-known formulas in warped spaces allow us to express the geometry of $(\bar{M}, \bar{g})$ in terms of the warping function f and the geometries of $(I, \mathrm{d}t^2)$ and (M, g), [10]. In addition, the gradient of f is

$$\mathrm{grad}\, f = -\dot{f}\,\partial_t.$$

After some computations, we obtain

$$\bar{\nabla}_{E_1} E_1 = \frac{\kappa}{f^2} N + \frac{\dot{f}}{f}\partial_t. \quad (6)$$

Since γ is a unit curve in M, then $\overline{\gamma_t}$ is a Frenet curve with constant speed $f(t)$.

For a Frenet curve of order 2 or greater, $\bar{\nabla}_{E_1} E_1$ cannot be light-like because E_2 must be a unitary vector. After (6), this is equivalent to

$$\dot{f}^2(t) - \kappa^2(s) \neq 0, \quad for\ any\ t \in I\ and\ s \in [a, b].$$

We will call

$$\Delta = \sqrt{\varepsilon_2(\kappa^2 - \dot{f}^2)}.$$

Similarly, for a Frenet curve of order 3 or greater, E_3 cannot be light-like, thus

$$k^2\tau^2\Delta^2 - \varepsilon_2 \dot{f}^2 \kappa'^2 \neq 0, \quad for\ any\ t \in I\ and\ s \in [a, b],$$

and we will put

$$\Theta = \sqrt{\varepsilon_3 \left(k^2\tau^2\Delta^2 - \varepsilon_2 \dot{f}^2 \kappa'^2\right)}.$$

Given $\overline{\gamma_t}$ a Frenet curve of order 4, we have its Frenet apparatus

$$E_1 = \frac{1}{f} T, \quad E_2 = \frac{\varepsilon_2\, \kappa\, N + \varepsilon_2\, f\, \dot{f} \partial_t}{f \Delta},$$

$$E_3 = -\frac{\varepsilon_3 \dot{f}^2 \kappa'}{f \Theta \Delta} N + \frac{\varepsilon_2 \varepsilon_3 \kappa \tau \Delta}{f \Theta} B - \frac{\varepsilon_3 \kappa\, \kappa' \dot{f}}{\Theta \Delta} \partial_t,$$

$$E_4 = \frac{\varepsilon_2 \kappa \tau \dot{f}}{f \Theta} N - \frac{\varepsilon_2 \kappa' \dot{f}}{f \Theta} B + \frac{\varepsilon_2 f \kappa^2 \tau}{f \Theta} \partial_t, \tag{7}$$

$$\kappa_1 = \frac{\Delta}{f}, \quad \kappa_2 = \frac{\Theta}{f \Delta^2}, \quad \kappa_3 = -\frac{\varepsilon_3 \dot{f} \Delta (\kappa^2\tau^3 + 2\tau(\kappa')^2 + \kappa(\kappa'\tau' - \tau\kappa''))}{f \Theta^2}, \tag{8}$$

where $\{E_1, E_2, E_3, E_4\}$ is a positive frame.

The main target of this paper is to identify which of these curves $\overline{\gamma_t} \in \Gamma_t \subset \Gamma$ are critical points of the action functional $\mathcal{L}$ given in (3), where Γ is given in (2) and we will just consider the clamped case. Moreover, Eq. (6) points out that for $\overline{\gamma_t}$ to be a geodesic of $\bar{M}$, it is necessary and sufficient that $\kappa \equiv 0$ and $\dot{f}(t) = 0$. In the sequel, we will find some solutions whose associated γ will be a geodesic in M, although $\overline{\gamma_t}$ will not be a geodesics in $\bar{M}$.

4 Working in critical slices $\dot{f}(t) = 0$

In this section, we choose a point $t \in I$ such that $\dot{f}(t) = 0$. Only in this case, critical points of the functional $\mathcal{L}$, which are also Frenet curves of order 1, may appear. We denote $R_{NTTA} = g(R(N, T)T, A)$.

Theorem 1 *Assume that $t \in I$ is such that $\dot{f}(t) = 0$. Then, a Frenet curve $\overline{\gamma_t} \in \Gamma_t$ is a critical point of $\mathcal{L}$ if and only if either it is a geodesic in $\bar{M}$ or the following equations hold $R_{NTTN} = \tau^2$, $R_{NTTB} = -\tau'$.*

Proof Suppose that $\overline{\gamma_t}$ is a Frenet curve of order ≥ 2. In this case, the Frenet apparatus of $\overline{\gamma_t}$ is

$$E_1 = \frac{T}{f}, \quad E_2 = \frac{N}{f}, \quad E_3 = \frac{\delta B}{f}, \quad E_4 = \delta\, \partial_t, \quad \kappa_1 = \frac{\kappa}{f}, \quad \kappa_2 = \frac{\delta \tau}{f}, \quad \kappa_3 = 0,$$

where we must suppose that τ is identically null or it does not change its sign $\delta = \pm 1$. The first case corresponds to Frenet curves of order 2 and the second one to Frenet curves of order

3. Frenet curves of order 4 do not exist because $\kappa_3 = 0$. The Euler–Lagrange equation (4) becomes

$$\bar{R}(E_2, E_1)E_1 = \frac{\tau^2}{f^2}E_2 - \frac{\delta\tau'}{f^2}E_3.$$

Observe that $\bar{R}(E_2, E_1)E_2 = \frac{1}{f^3}R(N, T)T$. Therefore, we get

$$R(N, T)T = \tau^2 N - \tau' B$$

and the result follows. □

Example 1 Take $M = SO(3, \mathbb{R})$ the special orthogonal group with the bi-invariant metric g defined by the scalar product $g(X, Y) = -\frac{1}{2}trace(XY)$ in $\mathfrak{so}(3, \mathbb{R})$. We take a curve $\gamma : \mathbb{R} \longrightarrow SO(3, \mathbb{R})$ defined by

$$\gamma(s) = \exp\begin{pmatrix} 0 & a\cos\frac{s}{r} & a\sin\frac{s}{r} \\ -a\cos\frac{s}{r} & 0 & 0 \\ -a\sin\frac{s}{r} & 0 & 0 \end{pmatrix},$$

where we put $a \in (0, \pi)$ and $r = \sqrt{2 - 2\cos a}$ for convenience. It is easy to check that γ is a unit curve in M. Since left translations $L_{\gamma(s)}$ are isometries, given a vector field $X \in \mathfrak{X}(\gamma)$, we can compute its covariant derivative along γ as

$$\nabla_{\gamma'(s)}X = (L_{\gamma(s)})_*\frac{D}{ds}((L_{\gamma(s)^{-1}})_*X) = \gamma(s)\left(a_1'(s)e_1 + a_2'(s)e_2 + a_3'(s)e_3\right),$$

being $(L_{\gamma(s)})_*$ the derivative of $L_{\gamma(s)}$. We have written $(L_{\gamma(s)^{-1}})_*X = a_1(s)e_1 + a_2(s)e_2 + a_3(s)e_3$ with $\{e_1, e_2, e_3\}$ the standard basis of $\mathfrak{so}(3, \mathbb{R})$, and we have used that $\frac{De_i}{ds} = 0$, $i = 1, 2, 3$. A standard computation gives the Frenet system of γ,

$$\kappa = \frac{\sin a}{2 - 2\cos a}, \quad \tau = \frac{1}{2}, \quad T = \gamma',$$

$$N = \begin{pmatrix} \sin a & -\cos a\cos\frac{s}{r} & -\cos a\sin\frac{s}{r} \\ \cos a\cos\frac{s}{r} & \sin a\cos^2\frac{s}{r} & \frac{1}{2}\sin a\sin\frac{2s}{r} \\ \cos a\sin\frac{s}{r} & \frac{1}{2}\sin a\sin\frac{2s}{r} & \sin a\sin^2\frac{s}{r} \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -\sin\frac{a}{2}\sin\frac{s}{r} & \sin\frac{a}{2}\cos\frac{s}{r} \\ -\sin\frac{a}{2}\sin\frac{s}{r} & 0 & \frac{\sin a}{r} \\ \sin\frac{a}{2}\cos\frac{s}{r} & -\frac{\sin a}{r} & 0 \end{pmatrix}.$$

To compute R_{NTTN} and R_{NTTB}, we use left translations again. For example, $R_{NTTN} = g\left(R\left(\gamma^{\mathbf{t}}(s)N, \gamma^{\mathbf{t}}(s)T\right)\gamma^{\mathbf{t}}(s)T, \gamma^{\mathbf{t}}(s)N\right)$, where we recall $\gamma^{\mathbf{t}}(s)$ is the transpose of the matrix $\gamma(s)$, and use the formula $R(X, Y)Z = -\frac{1}{4}[[X, Y], Z]$ for every $X, Y, Z \in \mathfrak{so}(3, \mathbb{R})$ (see [8]). Following this procedure, we have $R_{NTTN} = \frac{1}{4}$, $R_{NTTB} = 0$.

Now, in the warped product $I \times_f M$ with metric $-dt^2 + f^2(t)g$, given $t \in I$ a critical point of f, the curve $\overline{\gamma_t}(s) = (t, \gamma(s))$ is in the hypothesis of Theorem 1.

Let us denote the unique complete, connected, simply connected, three-dimensional space of constant curvature $c \in \mathbb{R}$ by $M^3(c)$. In other words, for $c = 0$, this is the Euclidean 3-space, for $c > 0$, it is a round 3-sphere, and for $c < 0$, it is a hyperbolic 3-space. Thus, we can construct a *Robertson–Walker spacetime* $I \times_f M^3(c)$, for which it is clear that $R_{NTTN} = c$ and $R_{NTTB} = 0$. Then, we have the following.

Corollary 1 *Given $\bar{M} = I \times_f M^3(c)$ a Robertson–Walker spacetime, assume that $t \in I$ is such that $\dot{f}(t) = 0$. Then, a Frenet curve $\overline{\gamma_t} \in \Gamma_t$ is a critical point of $\mathcal{L}$ if and only if either*

1. *It is a geodesic in $\bar{M}$ or*
2. *The torsion τ of γ is a constant function such that $c = \tau^2$.*

Observe that $c \geq 0$ and the curvature κ is arbitrary, so we have a wide range of examples.

5 Working in regular slices $\dot{f}(t) \neq 0$

From now on, we will always take $t \in I$ such that $\dot{f}(t) \neq 0$, although we might not say it explicitly. We make the discussion on the order of the Frenet curves.

5.1 Frenet curves of order 2. $\kappa_1 \neq 0$, $\kappa_2 = 0$

Theorem 2 *Assume that $t \in I$ is such that $\dot{f}(t) \neq 0$. Then, a Frenet curve $\overline{\gamma_t} \in \Gamma_t$ of order 2 is a critical point of $\mathcal{L}$ if and only if γ is a curve in M satisfying the following conditions.*

1. *Either it is a geodesic or κ is a constant and $\tau \equiv 0$.*
2. *$\ddot{f}(t) = 0$, $R_{NTTB} = 0$ and $R_{NTTN} = -\dot{f}^2(t)$ along γ.*

Proof The Frenet system is

$$\kappa_1 = \frac{\Delta}{f}, \quad E_1 = \frac{1}{f}T, \quad E_2 = \frac{\varepsilon_2\,\kappa\,N + \varepsilon_2\,f\,\dot{f}\partial_t}{f\,\Delta},$$

and by (1), we have

$$0 = \bar{\nabla}_{E_1}E_2 + \kappa_1 E_1 = \frac{\varepsilon_2}{f^2\Delta}\left(\frac{\dot{f}^2\kappa'}{-\kappa^2 + \dot{f}^2}N + \kappa\tau B - \frac{\kappa f\,\dot{f}\,\kappa'}{\kappa^2 - \dot{f}^2}\partial_t\right).$$

Therefore, $\kappa' = 0$, and $\kappa\tau = 0$. Thus, $\overline{\gamma_t}$ is a Frenet curve of order 2 if and only if either γ is a geodesic in M, or we have $\kappa > 0$ a constant, and $\tau \equiv 0$.

Now, the Euler–Lagrange equation (4) becomes

$$0 = \bar{R}(E_2, E_1)E_1 = \frac{\varepsilon_2}{f^3\Delta}\left(\kappa(R(N,T)T + \dot{f}^2 N) + f^2\,\dot{f}\,\ddot{f}\,\partial_t\right).$$

Comparing coordinates with respect to the fiber-adapted frame, see (5), we get $\ddot{f}(t) = 0$ and $R_{NTTN} = -\dot{f}^2$. □

Example 2 Let $\mathbb{H}^2$ be the real hyperbolic plane with the standard metric g_0 of constant sectional curvature -1. We consider β a unit curve in $\mathbb{H}^2$ with constant curvature $\kappa_\beta > 0$ and normal vector N_β. Given a smooth positive function $h : \mathbb{H}^2 \longrightarrow \mathbb{R}$, we consider the warped product $M = \mathbb{H}^2 \times_h \mathbb{R}$, $g = g_0 + h^2\mathrm{d}r^2$. Now, given a constant $a \in \mathbb{R}$, we define the unit curve $\gamma(s) = (\beta(s), a)$. Its Frenet system is (up to lift to TM),

$$T = \gamma', \quad N = N_\beta, \quad B = \partial_r/h, \quad \kappa = \kappa_\beta, \quad \tau_\gamma = 0.$$

If $\hat{R}$ and R are the curvature tensors of $\mathbb{H}^2$ and M, respectively, $R(N,T)T$ is the lift of $\hat{R}(N_\beta, \gamma')\gamma'$ along γ. Thus, along γ, $R_{NTTN} = -1$ and $R_{NTTB} = 0$ (see [10]).

Finally, given $\bar{M} = \mathbb{R} \times_f M$ with $f : I \longrightarrow \mathbb{R}$ a positive function admitting a point $t \in I$ such that $\dot{f}(t) = 1$ and $\ddot{f}(t) = 0$, the curve $\overline{\gamma_t}(s) = (t, \gamma(s))$ is in the hypothesis of the above theorem.

Corollary 2 *Let $\bar{M} = I \times_f M^3(c)$ be a Robertson–Walker spacetime. Choose a point $t \in I$ such that $\dot{f}(t) \neq 0$. Then, a Frenet curve $\overline{\gamma_t} \in \Gamma_t$ of order 2 is a critical point of $\mathcal{L}$ if and only if γ is a curve in $M^3(c)$ satisfying the following conditions.*

1. *Either it is a geodesic or κ is a constant and $\tau \equiv 0$.*
2. *$\ddot{f}(t) = 0$ and $c = -\dot{f}^2(t)$.*

5.2 Frenet curves of order 3. $\kappa_1\kappa_2 \neq 0$, $\kappa_3 = 0$

From (8),

$$\kappa_3 = -\frac{\varepsilon_3 \dot{f} \Delta(\kappa^2\tau^3 + 2\tau(\kappa')^2 + \kappa(\kappa'\tau' - \tau\kappa''))}{f\Theta^2} = 0,$$

and we consider two cases.

5.2.1 Case $\tau \equiv 0$

Theorem 3 *Assume that $t \in I$ is such that $\dot{f}(t) \neq 0$ and $\overline{\gamma_t} \in \Gamma_t$ a Frenet curve of order 3 such that γ is a curve in M with $\tau = 0$. Suppose also that $R_{NTTN} = c$ for some constant $c \in \mathbb{R}$. Then, $\overline{\gamma_t}$ is a critical point of $\mathcal{L}$ if and only if,*

1. *$\ddot{f}(t) < 0$.*
2. *The curvature κ of γ is*

$$\begin{aligned} \kappa(s) &= \dot{f} \tanh\left(\pm\sqrt{-f\ddot{f}}\, s + C\right), \ |\kappa(0)| < \dot{f}(t), \\ \kappa(s) &= \dot{f} \coth\left(\pm\sqrt{-f\ddot{f}}\, s + C\right), \ |\kappa(0)| > \dot{f}(t), \end{aligned} \tag{9}$$

   where C is an integration constant.
3. *$R_{NTTB} = 0$ and $c = -\dot{f}(t)^2 + f(t)\ddot{f}(t) < 0$.*

Proof Note that $\varepsilon_3\left(k^2\tau^2\Delta^2 - \varepsilon_2\dot{f}^2\kappa'^2\right) > 0$, thus, if $\tau = 0$, we have $-\varepsilon_2\varepsilon_3\dot{f}^2\kappa'^2 > 0$, that is, $\varepsilon_2\varepsilon_3 = -1$. We define $\sigma = \pm 1$ as the sign of $\kappa'\dot{f}$, and compute the Frenet system [see (7)], for $\overline{\gamma_t}$,

$$\begin{aligned} &\kappa_1 = \frac{\Delta}{f}, \quad \kappa_2 = \frac{\sigma\dot{f}\kappa'}{f\Delta^2}, \quad \kappa_3 = 0, \\ &E_1 = \frac{1}{f}T, \ E_2 = \frac{\varepsilon_2\kappa}{f\Delta}N + \frac{\varepsilon_2\ \dot{f}}{\Delta}\partial_t, \ E_3 = \frac{\varepsilon_2\sigma\dot{f}}{f\Delta}N + \frac{\varepsilon_2\sigma\kappa}{\Delta}\partial_t. \end{aligned}$$

By taking components in the fiber-adapted frame for the Euler–Lagrange equation (4) and using that $R_{NTTN} = c$ in $M^3(c)$, we have

$$\begin{aligned} 0 &= \kappa^4(c + \dot{f}^2) - \kappa^2\dot{f}^2(c + \dot{f}^2 + f\ \ddot{f}) + \dot{f}^2((\kappa')^2 + f\dot{f}^2\ddot{f}), \\ 0 &= \kappa^3(c + \dot{f}^2 - f\ddot{f}) - \kappa(2(\kappa')^2 + \dot{f}^2(c + \dot{f}^2 - f\ddot{f}) + (\kappa^2 - \dot{f}^2)\kappa''. \end{aligned}$$

The first equation becomes

$$(\kappa')^2 = -\frac{1}{\dot{f}^2}((\kappa^2 - \dot{f}^2)(\kappa^2(c + \dot{f}^2) - f\dot{f}^2\ddot{f})$$

and introducing it in the second one, we get

$$\kappa'' = -\frac{1}{\dot{f}^2}\kappa((c + \dot{f}^2)(2\kappa^2 + \dot{f}^2) + 3\kappa f\,\ddot{f}.$$

Differentiating the first equation and subtracting it to $2\kappa'$ times the second one, we obtain $c = -\dot{f}^2 + f\ddot{f}$. Thus, the curvature of γ satisfies the differential equation

$$(\kappa')^2 = -\frac{f\ddot{f}}{\dot{f}^2}(\kappa^2 - \dot{f}^2)^2, \tag{10}$$

whose solutions are those given in (9). □

Observe that Eq. (10) implies $\ddot{f}(t) < 0$, in order to avoid $\kappa_2 = 0$.

Example 3 In Example (2), we make the following changes. We take a curve β with curvature $\kappa_\beta(s) = a\tanh(s\sqrt{1-a^2})$, where $0 < a < 1$. We consider $\bar{M} = \mathbb{R}\times_f M$ with $f : I \longrightarrow \mathbb{R}$ a positive function admitting a point $t \in I$ such that $\dot{f}(t) = a$, $\ddot{f}(t) < 0$ and $f(t)\ddot{f}(t) = a^2 - 1$. Then, the curve $\overline{\gamma_t}$ is in the hypothesis of Theorem 3.

Corollary 3 *Let $\bar{M} = I \times_f M^3(c)$ be a Robertson–Walker 4-spacetime. Choose a point $t \in I$ such that $\dot{f}(t) \neq 0$ and $\overline{\gamma_t} \in \Gamma_t$ a Frenet curve of order 3 such that γ is a curve in $M^3(c)$ with vanishing torsion. Then, $\overline{\gamma_t}$ is a critical point of $\mathcal{L}$ if and only if $\ddot{f}(t) < 0$, the curvature κ of γ satisfies (9) and $c = -\dot{f}(t)^2 + f(t)\ddot{f}(t) < 0$.*

5.2.2 Case $\tau \neq 0$

Theorem 4 *Assume that $t \in I$ is such that $\dot{f}(t) \neq 0$. Let $\overline{\gamma_t} \in \Gamma_t$ be a Frenet curve of order 3 such that the associated curve γ in M satisfies $\kappa\tau \neq 0$. Let us suppose that $R_{NTTN} = c$ and $R_{NTTB} = d$ along γ, for suitable constants $c, d \in \mathbb{R}$. Then, $\overline{\gamma_t}$ is a critical point of $\mathcal{L}$ if and only if the following statements hold:*

1. $\ddot{f}(t) \neq 0$.
2. $R_{NTTB} = d = 0$ *and* $R_{NTTN} = c = f(t)\ddot{f}(t) - \dot{f}^2(t)$.
3. *The curvature κ and torsion τ are given by*

$$\kappa(s)^2 = \dot{f}(t)^2\frac{e^{2A} + C^2\tan^2(C\,s + B)}{e^{2A} - C^2},$$
$$\tau(s) = \frac{e^A\,C^2\sec^2(C\,s + B)}{e^{2A} + C^2\tan^2(C\,s + B)}, \tag{11}$$

or

$$\kappa(s)^2 = \dot{f}(t)^2\frac{e^{2A} + C^2\tan\mathrm{h}^2(C\,s + B)}{e^{2A} + C^2},$$
$$\tau(s) = \frac{-e^A\,C^2\sec\mathrm{h}^2(C\,s + B)}{e^{2A} + C^2\tan\mathrm{h}^2(C\,s + B)}, \tag{12}$$

according to $\ddot{f}(t) > 0$ *or* $\ddot{f}(t) < 0$*, respectively, where* $A, B, C \in \mathbb{R}$ *and* A *and* C *satisfy*

$$\mathrm{e}^{2A} - f(t)\ddot{f}(t) > 0, \;\; C = \sqrt{|f(t)\ddot{f}(t)|}.$$

Proof Since $\kappa_3 = 0$, Eq. (8) lead to $\kappa' \neq 0$ and

$$\kappa'' = \kappa\tau^2 + 2\frac{(\kappa')^2}{\kappa} + \frac{\kappa'\tau'}{\tau}. \tag{13}$$

Also, the Euler–Lagrange equation (4) becomes

$$\frac{\varepsilon_2\,\kappa\,R(N,T)T + \varepsilon_2\kappa\dot{f}^2N + \varepsilon_2 f^2\dot{f}\ddot{f}\partial_t}{f^2\Delta} = \varepsilon_2\varepsilon_3 f\kappa_2^2 E_2 - \varepsilon_3\kappa_2' E_3.$$

The nontrivial components with respect to the Frenet frame of $\overline{\gamma_t}$ are

$$\begin{aligned}
\varepsilon_3\kappa_2^2 &= \frac{\kappa^2(R_{NTTN} + \dot{f}^2) - f\dot{f}^2\ddot{f}}{f^2\Delta^2},\\
-\kappa_2' &= \frac{\varepsilon_3\kappa(\kappa\tau R_{NTTB}\Delta^2 - \varepsilon_2\dot{f}^2\kappa'(R_{NTTN} + \dot{f}^2 - f\ddot{f}))}{f\Delta^2\Theta},\\
0 &= -R_{NTTB}\kappa' + \kappa\tau(R_{NTTN} + \dot{f}^2 - f\ddot{f}).
\end{aligned}$$

By inserting the expressions of Δ and Θ in these equations, we have

$$\begin{aligned}
0 &= \kappa^4(R_{NTTN} - \tau^2 + \dot{f}^2) + \dot{f}^2(\kappa')^2 + f\dot{f}^4\ddot{f}\\
&\quad -\kappa^2\dot{f}^2(R_{NTTN} - \tau^2 + \dot{f}^2 + f\ddot{f}), \qquad (14)\\
0 &= \kappa^2\tau(\kappa^2 - \dot{f}^2)^2(R_{NTTB} + \tau') - \dot{f}^2(\kappa^2 - \dot{f}^2)\kappa'\kappa''\\
&\quad -\kappa^3\dot{f}^2\kappa'(R_{NTTN} + \tau^2 + \dot{f}^2 - f\ddot{f}) + \kappa\dot{f}^2\kappa'(2(\kappa')^2\\
&\quad +\dot{f}^2(R_{NTTN} + \tau^2 + \dot{f}^2 - f\ddot{f})), \qquad (15)\\
0 &= -R_{NTTB}\kappa' + \kappa\tau(R_{NTTN} + \dot{f}^2 - f\ddot{f}). \qquad (16)
\end{aligned}$$

Now, we assume that there exists two real constants c, d such that, along the curve γ, the curvature tensor satisfies $R_{NTTN} = c$ and $R_{NTTB} = d$. Equation (16) becomes $d\kappa' = \kappa\tau(c + \dot{f}^2 - f\ddot{f})$. In case $d \neq 0$, we call $H = c + \dot{f}^2 - f\ddot{f}$ and we take the derivative, getting $\kappa'' = (\kappa\tau)'H/d$. This readily implies $\kappa'' = \kappa'(\frac{\kappa'}{\kappa} + \frac{\tau'}{\tau})$, which is inserted in (13). As a consequence,

$$0 = \kappa\tau^2 + \frac{(\kappa')^2}{\kappa},$$

contradiction. Therefore, d has to be zero. According to (16), we have $c = f(t)\ddot{f}(t) - \dot{f}(t)^2$. By using all this information in Eqs. (14), (15), we obtain

$$(\kappa')^2 = \frac{(\kappa^2 - \dot{f}^2)}{\dot{f}^2}\left(\kappa^2\tau^2 - f\ddot{f}(\kappa^2 - \dot{f}^2)\right), \tag{17}$$

$$0 = 2\dot{f}^2(\kappa')^3\kappa + (\kappa^2 - \dot{f}^2)(-\dot{f}^2\tau^2\kappa\kappa' + \kappa^2\tau\tau'(\kappa^2 - \dot{f}^2) - \dot{f}^2\kappa'\kappa''). \tag{18}$$

We insert (13) in (18), deducing $\Theta^2(\kappa^3\tau' - \dot{f}^2(2\tau\kappa' + \kappa\tau')) = 0$. As $\kappa_2 \neq 0$, we know $\Theta \neq 0$ and therefore we get $\tau' = \frac{2\varepsilon_2\tau\dot{f}^2\kappa'}{\kappa\Delta^2}$ whose solution is

$$\tau = \mathrm{e}^A\frac{\varepsilon_2\Delta^2}{\kappa^2}, \tag{19}$$

where $A \in \mathbb{R}$ is an integration constant. We insert this in (17), so that

$$(\kappa')^2 = \frac{\Delta^4}{\kappa^2 \dot{f}^2}\left(\kappa^2(\mathrm{e}^{2A} - f\ddot{f}) - \mathrm{e}^{2A}\dot{f}^2\right).$$

We must take A such that $\mathrm{e}^{2A} - f(t)\ddot{f}(t) > 0$, otherwise the above equation has no solution. Now, by the change of variable $u = \kappa^2$, we deduce

$$(u')^2 \;=\; 4\frac{(u-\dot{f}^2)^2}{\dot{f}^2}\left((\mathrm{e}^{2A} - f\ddot{f})u - \mathrm{e}^{2A}\dot{f}^2\right).$$

We put $C = \sqrt{|f(t)\ddot{f}(t)|}$. We have three cases. The case $\ddot{f}(t) > 0$ leads to solution (11) and $\ddot{f}(t) < 0$ implies solution (12). Finally, we obtain for $\ddot{f}(t) = 0$,

$$u(s) = \kappa(s)^2 = \dot{f}(t)^2\left(1 + \frac{1}{(\mathrm{e}^A s + B)^2}\right), \quad \tau(s) = \frac{\mathrm{e}^A}{1 + (\mathrm{e}^A s + B)^2},$$

for a suitable integration constant $B \in \mathbb{R}$. In this cases $\varepsilon_2 = 1$ and a straightforward computation shows $\kappa_2 = 0$. Then, we have to discard this case. □

Corollary 4 *Let $\bar{M} = I \times_f M^3(c)$ be a Robertson–Walker 4-spacetime. Choose a point $t \in I$ such that $\dot{f}(t)\ddot{f}(t) \neq 0$. Let $\overline{\gamma_t} \in \Gamma_t$ be a Frenet curve of order 3 such that the associated curve γ in $M^3(c)$ satisfies $\kappa\tau \neq 0$. Then, $\overline{\gamma_t}$ is a critical point of $\mathcal{L}$ if and only if $c = f(t)\ddot{f}(t) - \dot{f}^2(t)$ and the curvature κ and torsion τ are given by (11) or (12), according to $\ddot{f}(t) > 0$ or $\ddot{f}(t) < 0$, respectively.*

Example 4 Take the open subset $\mathbb{R} \times_{e^t} \mathbb{R}^3$ of the de Sitter spacetime, and a curve $\gamma(s)$ in $\mathbb{R}^3$ with curvature and torsion given by Eq. (11). The curve $\overline{\gamma_t} = (t, \gamma) \in \Gamma_t$ satisfies the conditions of the corollary for any $t \in \mathbb{R}$.

5.3 Frenet curves of order 4. $\kappa_1\kappa_2\kappa_3 \neq 0$

5.3.1 Case $\kappa > 0$ is constant

Theorem 5 *Assume that $t \in I$ is such that $\dot{f}(t) \neq 0$. Let $\overline{\gamma_t} \in \Gamma_t$ be a Frenet curve of order 4 such that the curvature of the associated γ is constant. Then, $\overline{\gamma_t}$ is a critical point of $\mathcal{L}$ if and only if the following equations hold*

$$\ddot{f}(t) = 0, \quad R_{NTTN} = \tau^2 - \dot{f}(t)^2, \quad R_{NTTB} = -\tau'.$$

Proof In this case, since $\kappa_3 \neq 0$, by (8) we have $\tau \neq 0$. We define $\delta = \mathrm{sign}(\tau)$. Now, the Frenet apparatus becomes

$$\begin{aligned} E_1 &= \frac{1}{f}T, \quad E_2 = \frac{\varepsilon_2\kappa N + \varepsilon_2 f\dot{f}\partial_t}{f\Delta}, \quad E_3 = \frac{\delta\varepsilon_2}{f}B, \\ E_4 &= \frac{\delta\varepsilon_2\dot{f}N + \delta\varepsilon_2\kappa f\partial_t}{f\Delta}, \quad \kappa_1 = \frac{\Delta}{f}, \quad \kappa_2 = \frac{\delta\kappa\tau}{f\Delta}, \quad \kappa_3 = \frac{-\tau\dot{f}}{f\Delta}. \end{aligned} \tag{20}$$

Note that $\varepsilon_3 = 1$, then $\varepsilon_4 = -\varepsilon_2$. Also, for κ_2 to be positive and $\kappa_3 \neq 0$, it is necessary $\tau \neq 0$. The curvature is

$$\bar{R}(E_2, E_1)E_1 = \frac{\varepsilon_2\,\kappa\,R(N,T)T + \varepsilon_2\kappa\dot{f}^2N + \varepsilon_2 f^2\dot{f}\ddot{f}\partial_t}{f^3\Delta}. \tag{21}$$

By using (20) and (21), the Euler–Lagrange equation (4) becomes

$$\frac{\varepsilon_2\,\kappa\,R(N,T)T+\varepsilon_2\kappa\,\dot{f}^2N+\varepsilon_2 f^2\dot{f}\,\ddot{f}\,\partial_t}{f^2\Delta}=\frac{\varepsilon_2\kappa\,\tau^2}{f^2\Delta}N-\frac{\varepsilon_2\kappa\,\tau'}{f^2\Delta}B.$$

Taking components, we have

$$\ddot{f}=0,\quad R(N,T)T=(\tau^2-\dot{f}^2)N-\tau' B.$$

□

Corollary 5 *Let $\bar{M}=I\times_f M^3(c)$ be a Robertson–Walker 4-spacetime. Choose a point $t\in I$ such that $\dot{f}(t)\neq 0$. Let $\overline{\gamma_t}\in\Gamma_t$ be a Frenet curve of order 4 such that γ is a curve with κ and τ constant and $\tau\neq 0$. Then, $\overline{\gamma_t}$ is a critical point of $\mathcal{L}$ if and only if $\ddot{f}(t)=0$ and $c=\tau^2-\dot{f}(t)^2$.*

Note that, in particular, the helices in $M^3(c)$ with constant curvature provide a wide family of examples.

5.3.2 Case κ is not a constant

We should point out that Eq. (21) still holds. So the Euler–Lagrange equation becomes

$$\frac{\varepsilon_2\,\kappa\,R(N,T)T+\varepsilon_2\kappa\,\dot{f}^2N+\varepsilon_2 f^2\dot{f}\,\ddot{f}\,\partial_t}{f^2\Delta}=\varepsilon_2\varepsilon_3\kappa_2^2E_2-\varepsilon_3\kappa_2'\,E_3-\varepsilon_3\varepsilon_4\kappa_2\kappa_3E_4.$$

Now, by taking components with respect to the Frenet frame of $\overline{\gamma_t}$, we obtain

$$\begin{aligned}
0&=\varepsilon_2\Delta^2\left(\kappa^2(R_{NTTN}-\tau^2+\dot{f}^2)-f\dot{f}^2\ddot{f}\right)+\dot{f}^2(\kappa')^2,\\
0&=\kappa\dot{f}^2\kappa'(R_{NTTN}\dot{f}^2+\dot{f}^4+2(\kappa')^2+\dot{f}^2\tau^2-f\dot{f}^2\ddot{f})\\
&\quad-\kappa^3\dot{f}^2\kappa'(R_{NTTN}+\dot{f}^2+\tau^2-f\ddot{f})-\varepsilon_2\dot{f}^2\Delta^2\kappa'\kappa''+\tau\kappa^2\Delta^4(R_{NTTB}+\tau'),\\
0&=\kappa^2\tau(R_{NTTN}-\tau^2+\dot{f}^2-f\ddot{f})+\kappa(\tau\kappa''-\kappa'(R_{NTTB}+\tau'))-2\tau(\kappa')^2.
\end{aligned}\tag{22}$$

Thus, we obtain the following general result.

Proposition 1 *Assume $t\in I$ such that $\dot{f}(t)\neq 0$. Let $\overline{\gamma_t}\in\Gamma_t$ be a Frenet curve of order 4 such that the curvature of the associated curve γ satisfies $\kappa'\neq 0$. Then, $\overline{\gamma_t}$ is a critical point of the action functional $\mathcal{L}$ if and only if Eq. (22) hold.*

As in previous cases, if we assume that the curvature tensor along γ satisfies $R_{NTTN}=c$ and $R_{NTTB}=d$ for some constants $c,d\in\mathbb{R}$, the following theorem shows that there do not exist critical points of order 4 of the action functional.

Theorem 6 *Let us suppose $\dot{f}(t)\neq 0$. Then, there do not exist Frenet curves $\overline{\gamma_t}\in\Gamma_t$ of order 4, which are critical points of $\mathcal{L}$ such that*

1. *The associated curve γ satisfies $\kappa'\neq 0$.*
2. *The curvature tensor along γ is of the form $R_{NTTB}=d$, $R_{NTTN}=c$ for any constants $c,d\in\mathbb{R}$.*

Proof We will assume the existence of a curve as in the statement of the theorem and infer a contradiction.

Since $\kappa_1\kappa_2\kappa_3 \neq 0$, we have $\kappa\tau \neq 0$. Now, by manipulating the three equations (22), we have

$$(\kappa')^2 = \frac{-\varepsilon_2\Delta^2}{\dot f^2}\left(\kappa^2(c-\tau^2+\dot f^2) - f\dot f^2\ddot f\right), \tag{23}$$

$$\kappa'\kappa'' = \frac{\varepsilon_2\kappa}{\dot f^2\Delta^2}\left(\kappa\tau\Delta^4(d+\tau') + \dot f^2\kappa'\left(\dot f^4 + 2(\kappa')^2 + \dot f^2(c+\tau^2 - f\ddot f)\right)\right.$$
$$\left. -\kappa^2\dot f^2\kappa'(c+\dot f^2+\tau^2 - f\ddot f)\right), \tag{24}$$

$$\kappa'' = \frac{2(\kappa')^2}{\kappa} + \frac{\kappa'(d+\tau')}{\tau} + \kappa(\tau^2 - c - \dot f^2 + f\ddot f). \tag{25}$$

We differentiate Eq. (23) and subtract it to two times Eq. (24). Bearing in mind (23), after some computations, we deduce

$$d\kappa^3\tau - d\kappa\tau\dot f^2 - 2\dot f^2\kappa'(c+\dot f^2 - f\ddot f) = 0. \tag{26}$$

Firstly, let us suppose that $d \neq 0$. From the previous equation, we get

$$\tau = \frac{2\varepsilon_2\dot f^2 H\kappa'}{\mathrm{d}\kappa\Delta^2}, \quad \text{where } H = c + \dot f^2 - f\ddot f. \tag{27}$$

From $\kappa' \neq 0$ and $\tau \neq 0$, we have $H \neq 0$. We insert (27) in (25). There is no loss of generality if we take κ satisfying $d^2(\kappa^4 - \dot f^4) - 4\dot f^4H^2 \neq 0$. Therefore,

$$(\kappa')^2 = \frac{d^2\kappa^2\Delta^4(\varepsilon_2 d^2\Delta^2 - 2\dot f^2H^2)}{2H\dot f^2(d^2(\kappa^4-\dot f^4) - 4\dot f^4H^2)}. \tag{28}$$

Next, we insert (27) in (23). Again, there is no loss of generality if we also assume that κ satisfies $4\dot f^2H^2 - d^2(\kappa^2 - \dot f^2) \neq 0$. Then, we get

$$(\kappa')^2 = \frac{d^2\Delta^4(H\kappa^2 + \varepsilon_2 f\ddot f\Delta^2)}{4\dot f^4H^2 - \varepsilon_2 d^2\dot f^2\Delta^2}. \tag{29}$$

By (28) and (29), one has

$$\frac{d^2\kappa^2\Delta^4(\varepsilon_2 d^2\Delta^2 - 2\dot f^2H^2)}{2H\dot f^2(d^2(\kappa^4-\dot f^4) - 4\dot f^4H^2)} = \frac{d^2\Delta^4(H\kappa^2 + \varepsilon_2 f\ddot f\Delta^2)}{4\dot f^4H^2 - \varepsilon_2 d^2\dot f^2\Delta^2},$$

which is equivalent to

$$0 = -2H(d^2+4H^2)f\dot f^4\ddot f - d^2(d^2+4H^2)\dot f^2\kappa^2 + d^2(d^2+2H^2+2Hf\ddot f)\kappa^4.$$

By hypothesis, κ is not constant, so all coefficients of the previous polynomial have to vanish. This implies $d = H = 0$, contradiction.

Thus, we have $d = 0$. Equation (26) becomes $c + \dot f^2 - f\ddot f = 0$, which means $\kappa_3 \equiv 0$. This way, we have to discard this case. □

Corollary 6 *Given $\bar M = I \times_f M^3(c)$ a Robertson–Walker spacetime, assume that $t \in I$ is such that $\dot f(t) \neq 0$. Then, there do not exist Frenet curves $\overline{\gamma_t} \in \Gamma_t$ of order 4, which are critical points of $\mathcal{L}$ and such that its corresponding curve γ satisfies $\kappa' \neq 0$.*

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