

Totally umbilic null hypersurfaces in generalized Robertson-Walker spaces*

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Abstract

We show that there is a correspondence between totally umbilic null hypersurfaces in generalized Robertson-Walker spaces and twisted decompositions of the fibre. This allows us to prove that nullcones are the unique totally umbilic null hypersurfaces in the closed Friedmann Cosmological model. We also apply this kind of ideas to static spaces, in particular to Reissner-Nordström and Schwarzschild exterior spacetimes.

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1 Introduction

A hypersurface in a Lorentzian manifold is null if the induced metric tensor is degenerate on it. There is an increasing interest on these hypersurfaces both from a physical and a geometrical point of view. Black hole horizons are one of the most remarkable examples [2, 5]. On the other hand, nullcones play a central role in causality theory and its regularity is of key importance in the propagation properties of linear and nonlinear waves, [8]. Specific techniques are needed to study these hypersurfaces since it is not possible to define an orthogonal projection over them, so neither the induced connection nor the second fundamental form can be defined in the usual way.

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In this paper, we focus on geometrical properties of null hypersurfaces in a generalized Robertson-Walker space. The main result is Theorem 4.2, where we show that a totally umbilic null hypersurface gives rise to a local decomposition of the fibre as a twisted product and viceversa, providing a deep insight of twisted decompositions in Lorentzian geometry. For example, not all manifold admits such a decomposition [7, 10], hence there are spacetimes that do not admit totally umbilic null hypersurfaces. Moreover, given a totally umbilic null hypersurface, we can construct a dual one, considering the same induced twisted decomposition in the fibre but reversing the base. In particular, totally umbilic null hypersurfaces through each point in generalized Robertson-Walker spaces appear in pairs. Surprisingly, there are cases in which the dual construction is not trivial, see Example 4.6.

One of the most important examples of null hypersurfaces are nullcones, so we dedicate Section 3 to study them. We show that nullcones in Robertson-Walker spaces are totally umbilic. Nullcones with this property in generalized Robertson-Walker spaces have special importance because if there is a null geodesic starting at the vertex of a totally umbilic nullcone and it has a conjugate point along it, its multiplicity is maximum, Proposition 3.6. This is potentially interesting in Cosmology and extends a previous result for Robertson-Walker spaces, [3]. We also give necessary and sufficient conditions for a null hypersurface to be an open set of a nullcone, which, jointly with Theorem 4.2, allows us to show in Section 4 that any totally umbilic null hypersurface in a Robertson-Walker space $I \times_f \mathbb{S}^{n-1}$ with $\int_I \frac{1}{f} > \pi$ is an open set of a nullcone, Theorem 4.11. As a corollary, we can apply this theorem to the closed Friedmann Cosmological model.

Finally, in Section 5, we adapt Theorem 4.2 to the case of standard static spaces and we apply it to an important family of static spacetimes. We prove a uniqueness result for dual pairs of totally umbilic null hypersurfaces in Reissner-Nordström and De Sitter-Schwarzschild spacetimes.

2 Preliminaries

Take I an open interval of \mathbb{R} , $f \in C^\infty(I)$ a positive function and (F, g_F) a Riemannian manifold. The manifold $I \times F$ furnished with the Lorentzian metric $g = -dt^2 + f(t)^2 g_F$ is called a generalized Robertson-Walker (GRW) space and is denoted by $I \times_f F$. When F has constant curvature, it is called a Robertson-Walker (RW) space and if $f \equiv 1$, then it is simply denoted by $I \times F$. The vector field $\zeta = f \partial_t$ is timelike, closed and conformal. It locally characterizes these spaces and, under certain conditions, it is the unique vector field with these properties, [6]. If we consider the same construction as above, but with dt^2 instead of $-dt^2$, the resulting product metric is called warped product and if we consider a positive function $f \in C^\infty(I \times F)$ instead of $f \in C^\infty(I)$, then it is called a twisted product.

Given L a null hypersurface in a GRW space, we take the unique null vector field $\xi \in \mathfrak{X}(L)$ such that $g(\zeta, \xi) = 1$ and we call \mathcal{S} the distribution in L given

by $\zeta^\perp \cap TL$.

The vector field ξ is geodesic and $\nabla_X \xi$ is a section of TL for all $X \in \mathcal{S}$. In fact, $\nabla_X \xi \in \mathcal{S}$, since $g(\nabla_X \xi, \zeta) = -g(\xi, \nabla_X \zeta) = 0$. The null second fundamental form of L is defined by $B(X, Y) = -g(\nabla_X \xi, Y)$ for all $X, Y \in \mathcal{S}$. It is said that it is totally geodesic if $B \equiv 0$ and totally umbilic if $B = \rho g$ for certain $\rho \in C^\infty(L)$. The trace of B is the null mean curvature of L , explicitly given by

$$H_p = \sum_{i=3}^n B(e_i, e_i),$$

being $\{e_3, \dots, e_n\}$ an orthonormal basis of \mathcal{S}_p .

If L is totally umbilic, then the null sectional curvature respect to ξ of a null plane $\Pi = \text{span}(X, \xi)$, where $X \in \mathcal{S}$ is unitary, can be expressed as

$$\mathcal{K}_\xi(\Pi) = \xi(\rho) - \rho^2. \quad (1)$$

If it is totally geodesic, we have $\mathcal{K}_\xi(\Pi) = 0$ for any null tangent plane Π to L .

If θ is an open subset of F and $h : \theta \rightarrow I$ is a function, then the graph of h is a null hypersurface of $I \times_f F$ if and only if

$$|\nabla^F h|_F = f \circ h. \quad (2)$$

Locally, any null hypersurface L can be expressed in this way. If we call $\pi : I \times_f F \rightarrow F$ and $T : I \times_f F \rightarrow I$ the canonical projections, then $\pi : L \rightarrow F$ is a local diffeomorphism and thus, locally, L coincides with the graph of the function given by $h = T \circ \pi^{-1} : \theta \rightarrow I$, where $\theta \subset F$. Moreover, given $v \in TF$ we have

$$g_F(\nabla_{\nabla^F h}^F \nabla^F h, v) = \frac{1}{2} v((f \circ h)^2) = (f \circ h)(f' \circ h) g_F(v, \nabla^F h),$$

therefore

$$\nabla_{\nabla^F h}^F \nabla^F h = (f \circ h)(f' \circ h) \nabla^F h. \quad (3)$$

3 Nullcones in generalized Robertson-Walker spaces

If M is a Lorentzian manifold and Θ a normal neighborhood of a point $p \in M$, then we call $\hat{\Theta} = \text{exp}_p^{-1}(\Theta)$ and \hat{P} the position vector field in $T_p M$. The local position vector field at p is defined as $P_{\text{exp}_p(v)} = (\text{exp}_p)_* v(\hat{P}_v)$ for all $v \in \hat{\Theta}$ and the local future and past nullcones at p as

$$C_p^+ = \text{exp}_p(\hat{C}_p^+ \cap \hat{\Theta}), \quad C_p^- = \text{exp}_p(\hat{C}_p^- \cap \hat{\Theta}),$$

being \hat{C}_p^+, \hat{C}_p^- the future and past nullcone respectively in $T_p M$. In a GRW space, local nullcones can be characterized as follows.

Proposition 3.1. *Let $I \times_f F$ be a GRW space and fix $p_* = (t_*, x_*) \in I \times F$. If Θ is a normal neighborhood of p_* , then the local nullcones at p_* are given by*

$$C_{p_*}^+ = \{(t, x) \in \Theta : \int_{t_*}^t \frac{1}{f(r)} dr = d_F(x_*, x)\},$$

$$C_{p_*}^- = \{(t, x) \in \Theta : \int_t^{t_*} \frac{1}{f(r)} dr = d_F(x_*, x)\},$$

being d_F the Riemannian distance in F . Moreover, the local position vector field at p_* is given by

$$P_{(t,x)} = \int_{t_*}^t \frac{f(r)}{f(t)} dr \partial_t + \frac{\int_{t_*}^t \frac{f(r)}{f(t)} dr}{\int_{t_*}^t \frac{f(t)}{f(r)} dr} P_x^F,$$

for all $(t, x) \in C_{p_*}^+ \cup C_{p_*}^-$, where P^F is the local position vector field at x_* in F .

Proof. Given $(t, x) \in C_{p_*}^+$, it exists a null geodesic $\gamma : J \rightarrow C_{p_*}^+$ such that $\gamma(0) = p_*$, $g(\gamma'(0), \zeta_{p_*}) = -1$ and $\gamma(s^*) = (t, x)$ for certain $s^* \in J$. Since ζ is closed and conformal, $g(\gamma', \zeta)$ is constant and therefore, if $\gamma(s) = (\alpha(s), \beta(s))$, we have $\alpha'(s)f(\alpha(s)) = 1$. Hence $\alpha(s) = a^{-1}(s)$, being $a(s) = \int_{t_*}^s f(r)dr$. On the other hand, β is a pregeodesic in F which holds

$$\beta''(s) = -2 \frac{d}{ds} \left(\ln f(\alpha(s)) \right) \beta'(s),$$

so β is given by

$$\beta(s) = \exp_{x_*}^F(b(s)u),$$

where $b(s) = \int_0^s \frac{1}{f(\alpha(r))^2} dr$ and $u \in T_{x_*}F$ with $g_F(u, u) = 1$. Therefore,

$$d_F(x_*, x) = b(s^*) = \int_{t_*}^t \frac{1}{f(r)} dr.$$

Conversely, take $(t, x) \in \Theta$ such that $d_F(x_*, x) = \int_{t_*}^t \frac{1}{f(r)} dr$. If we call

$$a(s) = \int_{t_*}^t f(r)dr, \quad b(s) = \int_0^s \frac{1}{f(a^{-1}(r))^2} dr,$$

$s^* = a(t)$, $\alpha(s) = a^{-1}(s)$ and $\beta(s) = \exp_{x_*}^F(b(s)u)$ where $u \in T_{x_*}F$ is such that $\exp_{x_*}^F(d_F(x_*, x)u) = x$, then it is easy to show that $\gamma(s) = (\alpha(s), \beta(s))$ is a future null geodesic in M with $\gamma(0) = p_*$ and $\gamma(s^*) = (t, x)$. Therefore, $(t, x) \in C_{p_*}^+$.

To compute the local position vector field over $C_{p_*}^+$ we observe that for any manifold and any geodesic γ with $\gamma(0) = p_*$, the local position vector field is given by $P_{\gamma(s)} = s\gamma'(s)$. In our particular case, $P_{(t,x)} = s^*\gamma'(s^*)$. Since

$\alpha(s^*) = t$, we have

$$\begin{aligned} s^* &= \int_{t_*}^t f(r) dr, \\ \alpha'(s^*) &= \frac{1}{f(t)}, \\ \beta'(s^*) &= b'(s^*) (exp_{x_*}^F)_{*b(s^*)u} (u) = \frac{b'(s^*)}{b(s^*)} P_x^F, \end{aligned}$$

where P^F is the local position vector field at x_* in F . Since $b(s^*) = \int_{t_*}^t \frac{1}{f(r)} dr$ and $b'(s^*) = \frac{1}{f(t)^2}$, we have

$$P_{(t,x)} = \int_{t_*}^t \frac{f(r)}{f(t)} dr \partial_t + \frac{\int_{t_*}^t \frac{f(r)}{f(t)} dr}{\int_{t_*}^t \frac{f(t)}{f(r)} dr} P_x^F.$$

□

The following lemma will be helpful to compute the null second fundamental form of a nullcone in a Robertson-Walker space.

Lemma 3.2. *Let (F, g_F) be a semi-Riemannian manifold, fix $x_* \in F$ and take $\theta \subset F$ a normal neighborhood of x_* . Call $P^F \in \mathfrak{X}(\theta)$ the local position vector field at x_* . If $w \in T_x F$, being $x = exp_{x_*}^F(v) \in \theta$, then*

$$g_F(\nabla_w^F P^F, w) = \frac{1}{2} \frac{d}{ds} g(J, J)|_{s=1},$$

where J is the unique Jacobi vector field over $exp_{x_*}^F(sv)$ with $J(0) = 0$ and $J(1) = w$.

Proposition 3.3. *Local nullcones in a Robertson-Walker space are totally umbilic.*

Proof. Fix $p_* = (t_*, x_*) \in I \times_f F$ and consider $p = (t, x) \in C_{p_*}^+$. From Proposition 3.1, $\xi_{(t,x)} = \frac{-1}{\int_{t_*}^t \frac{1}{f(r)} dr} P_{(t,x)}$. If $w \in \mathcal{S}_{(t,x)}$, then $g(w, \partial_t) = 0$ and so $w \in T_x F$ with $g(P_x^F, w) = 0$. Therefore,

$$\nabla_w \xi = \frac{-1}{\int_{t_*}^t f(r) dr} \nabla_w P = -\frac{f'(t)}{f(t)^2} w - \frac{1}{f(t)^2 \int_{t_*}^t \frac{1}{f(r)} dr} \nabla_w P^F.$$

Now, we use Lemma 3.2 and that F has constant curvature k to compute $g_F(\nabla_w^F P^F, w)$. Take $u \in T_{x_*} F$ such that $exp_{x_*}^F(d_F(x, x_*)u) = x$. The Jacobi vector field over $exp_{x_*}^F(s d_F(x, x_*)u)$, $0 \leq s \leq 1$, with $J(0) = 0$ and $J(1) = w$ is given by $J(s) = \varphi(s)W(s)$, where W is parallel with $W(1) = w$ and

$$\varphi(s) = \begin{cases} \frac{\text{Sin}(\sqrt{k}d_F(x, x_*)s)}{\text{Sin}(\sqrt{k}d_F(x, x_*))} & \text{if } k > 0, \\ s & \text{if } k = 0, \\ \frac{\text{Sinh}(\sqrt{-k}d_F(x, x_*)s)}{\text{Sinh}(\sqrt{-k}d_F(x, x_*))} & \text{if } k < 0. \end{cases}$$

Therefore, since $d_F(x, x_*) = \int_{t_*}^t \frac{1}{f(r)} dr$, we have

$$B(w, w) = \begin{cases} \frac{1}{f(t)^2} \left(f'(t) + \frac{\sqrt{k}}{\text{Tan}(\sqrt{k} \int_{t_*}^t \frac{1}{f(r)} dr)} \right) g(w, w) & \text{if } k > 0, \\ \frac{1}{f(t)^2} \left(f'(t) + \frac{1}{\int_{t_*}^t \frac{1}{f(r)} dr} \right) g(w, w) & \text{if } k = 0, \\ \frac{1}{f(t)^2} \left(f'(t) + \frac{\sqrt{-k}}{\text{Tanh}(\sqrt{-k} \int_{t_*}^t \frac{1}{f(r)} dr)} \right) g(w, w) & \text{if } k < 0. \end{cases}$$

□

Remark 3.4. *In view of the null second fundamental form of nullcones in Robertson-Walker spaces, it follows that they can not be totally geodesic, since the null mean curvature tends to infinity as the coordinate t approaches to t_* . This is a general fact in any Lorentzian manifold (see for example Proposition 2.1 in [11]).*

Example 3.5. *Since \mathbb{R}_1^n , \mathbb{S}_1^n and a suitable portion of \mathbb{H}_1^n can be expressed as a RW space (see the first column of Table 1), the above proposition shows the well-known fact that nullcones of a Lorentzian manifolds of constant curvature are totally umbilic.*

Proposition 3.6. *Let γ be a null geodesic in a GRW space. If γ is contained in a totally umbilic nullcone L , and J is a Jacobi vector field with $J \in \mathcal{S}$, then it holds*

$$J'' + \frac{\text{Ric}(\gamma', \gamma')}{n-2} J = 0.$$

In particular, if there exists a conjugate point of $\gamma(0)$ along γ , it has maximum multiplicity.

Proof. Using that L is totally umbilic, we have $\nabla_X \xi = -\rho X$ for all $X \in \mathcal{S}$. Since ξ is geodesic, after a suitable affine reparametrization, γ is an integral curve of ξ , thus

$$R_{J\gamma'} \gamma' = R_{J\xi} \xi = (\xi(\rho) - \rho^2) J.$$

Using Equation (1), we get the result. Finally observe that the Jacobi operator is proportional to the identity, so if there exists a conjugate point, then it has maximum multiplicity. □

This proposition is potentially interesting in Cosmology because the multiplicity in gravitational lens phenomena can be detected by astronomical observations. On the other hand, after Proposition 3.3, conjugate points in Robertson-Walker spaces have maximum multiplicity, see also [3].

We finish this section with a criterion to determine whenever a null hypersurface given by the graph of a function is contained in a nullcone.

Lemma 3.7. *Let $I \times_f F$ be a GRW space and $p_* = (t_*, x_*)$ a fixed point. Let $\theta \subset F$ with x_* in the closure of θ and $h : \theta \rightarrow I$ a function. If the graph of h is a null hypersurface, then it is contained in the local nullcone at p_* if and only*

if $\lim_{x \rightarrow x_*} h(x) = t_*$ and $\nabla^F h$ is proportional to P^F , the local position vector field at x_* in F .

Proof. Take a null geodesic γ such that $\gamma(0) = p_*$ and $\gamma(s) = (\alpha(s), \beta(s))$. We have

$$\begin{aligned} g_F(\nabla^F h, \beta'(s)) &= (h \circ \beta)'(s), \\ |\nabla^F h|_F |\beta'(s)|_F &= |\alpha'(s)|. \end{aligned}$$

Using the Cauchy-Schwarz inequality and that locally $C_{p_*}^\pm = \exp_{p_*}(\widehat{C}_{p_*}^\pm)$, the graph of h is contained in the nullcone at p_* if and only if $\nabla^F h$ and β' are proportional and $\lim_{x \rightarrow x_*} h(x) = t_*$, but observe that β' is proportional to P^F . \square

Remark 3.8. *If L is a null hypersurface in a GRW space given by the graph of a function h and $p_* \in L$, then for any null geodesic $\gamma(s) = (\alpha(s), \beta(s))$ with $\gamma(0) = p_*$ we have $h(\beta(s)) \leq \alpha(s)$ and the equality holds if and only if γ belongs to L . This implies that, near p_* , it holds $L \subset I^+(p_*)^c$. Geometrically, this means that the local nullcone at p_* is an extremal null hypersurface near p_* . This result is also true for any arbitrary Lorentzian manifold. Indeed, given a point $p_* \in L$ consider Θ a normal neighborhood of p_* and suppose there exists a point $q \in L \cap I^+(p_*, \Theta)$. Take a timelike plane $\Pi \subset T_{p_*} M$ with $0, \widehat{q} \in \Pi$, where $\widehat{q} = \exp_{p_*}^{-1}(q)$. The intersection $L \cap \exp_{p_*}(\Pi)$ is the trace of a curve from p_* to q which is null or spacelike in each of its points and is contained in the timelike surface $\exp_{p_*}(\Pi)$, but $q \in I^+(p_*, \exp_{p_*}(\Pi))$, which is a contradiction. Using a past nullcone sharing a null geodesic of L , with vertex near p_* , we can figure out the situation as L being a sheet between two millstones.*

4 Umbilic null hypersurfaces

In this section we prove the main result of this paper. It gives us a correspondence between totally umbilic null hypersurfaces and twisted decompositions of the fibre of a GRW space. Thus, it shows that only special types of GRW spaces can admit totally umbilic null hypersurfaces and it also provides a method to construct them. First, we need the following lemma.

Lemma 4.1. *Let $I \times_f F$ be a GRW space and L a null hypersurface given by the graph of a function h . Then*

$$\xi = -\frac{1}{f \circ h} \partial_t - \frac{1}{(f \circ h)^3} \nabla^F h, \quad (4)$$

$\mathcal{S} \simeq \{X \in TF : g_F(X, \nabla^F h) = 0\}$ and the null second fundamental form is given by

$$B(X, Y) = \frac{f' \circ h}{(f \circ h)^2} g(X, Y) + \frac{1}{f \circ h} \text{Hess}_h^F(X, Y),$$

for all $X, Y \in \mathcal{S}$.

Theorem 4.2. *Let $I \times_f F$ be a GRW space. If L is a totally umbilic null hypersurface, then for each $(t_0, x_0) \in L$ there exists a decomposition of F in a neighborhood of x_0 as a twisted product with one dimensional base*

$$(J \times S, ds^2 + \mu(s, z)^2 g_S),$$

where x_0 is identified with $(0, z_0)$ for some $z_0 \in S$ and L is given by

$$\{(t, s, z) \in I \times J \times S : s = \int_{t_0}^t \frac{1}{f(r)} dr\}.$$

Moreover, if H is the null mean curvature of L , then

$$\mu(s, z) = \frac{f(t_0)}{f(t)} \exp\left(\int_0^s \frac{H(t, r, z) f(t)^2}{n-2} dr\right)$$

for all $(t, s, z) \in L$.

Conversely, if F admits a twisted decomposition in a neighborhood of x_0 as above, then $L = \{(t, s, z) \in I \times J \times S : s = \int_{t_0}^t \frac{1}{f(r)} dr\}$ is a totally umbilic null hypersurface with null mean curvature

$$H = \frac{n-2}{f(t)^2} \left(f'(t) + \frac{\mu_s(s, z)}{\mu(s, z)} \right).$$

Proof. Suppose that L is given by the graph of certain map $h : \theta \rightarrow I$ in a neighborhood of (t_0, x_0) and $B = \rho g$. From Formula (3) and Lemma 4.1 it holds

$$\begin{aligned} \nabla_{\nabla^F h}^F \nabla^F h &= (f \circ h)(f' \circ h) \nabla^F h, \\ \nabla_v^F \nabla^F h &= (\rho(f \circ h)^3 - (f' \circ h)) v \end{aligned}$$

for all $v \perp \nabla^F h$. Since $|\nabla^F h|_F = f \circ h$, we have that $|\nabla^F h|_F$ is constant through the level hypersurfaces of h . Thus, if we call $E = \frac{1}{|\nabla^F h|_F} \nabla^F h$, it is easy to show that

$$\begin{aligned} \nabla_E^F E &= 0, \\ \nabla_v^F E &= (\rho(f \circ h)^2 - (f' \circ h)) v \end{aligned}$$

for all $v \perp \nabla^F h$. From these equations it follows that E is closed and

$$(L_E g_F)(v, w) = 2(\rho(f \circ h)^2 - (f' \circ h)) g_F(v, w) \quad (5)$$

for all $v, w \in TF$ with $v, w \perp \nabla^F h$.

The following argument is local, so for simplicity we can suppose without loss of generality that E is complete. Call S_z the leaf of E^\perp through $z \in F$. Being E closed, the flow ϕ of E is foliated, that is, $\phi_s(S_z) = S_{\phi_s(z)}$ for all $z \in L$ and $s \in \mathbb{R}$. Using this, it is easy to check that $\phi : \mathbb{R} \times S_{x_0} \rightarrow F$ is onto and

a local diffeomorphism. Moreover, from Equation (5), $\phi_s : S_{x_0} \rightarrow S_{\phi_s(x_0)}$ is a conformal diffeomorphism with conformal factor

$$\exp\left(2 \int_0^s (\rho(\phi_r(z))f(h(\phi_r(z)))^2 - f'(h(\phi_r(z)))) dr\right)$$

and it follows that $\phi^*(g_F) = ds^2 + \mu^2 g|_{S_{x_0}}$, being

$$\mu(s, z) = \exp\left(\int_0^s (\rho(\phi_r(z))f(h(\phi_r(z)))^2 - f'(h(\phi_r(z)))) dr\right).$$

Since E is identified with ∂_s , in this decomposition h only depends on s and $h'(s) > 0$. Thus, from Equation (2) we have that $h(s) = c^{-1}(s)$ being $c(t) = \int_{t_0}^t \frac{1}{f(r)} dr$. Moreover, $(\ln f \circ h)' = f' \circ h$, so the above expression for μ can be written as

$$\mu(s, z) = \frac{f(t_0)}{f(h(s))} \exp\left(\int_0^s \rho(\phi_r(z))f(h(r))^2 dr\right).$$

For the converse, Equation (2) can be directly checked and applying Lemma 4.1 we get the result. \square

Observe that $\mu(0, z) = 1$ for all $z \in S$. Moreover, this theorem can be applied to any null surface in a three dimensional GRW space, since they are always totally umbilic.

Example 4.3. *Totally geodesic null hypersurfaces in \mathbb{R}_1^n are given by null hyperplanes. In \mathbb{S}_1^n and \mathbb{H}_1^n we can obtain totally geodesic null hypersurfaces intersecting them with null planes through the origin of \mathbb{R}_1^{n+1} and \mathbb{R}_2^{n+1} respectively. Table 1 shows how Theorem 4.2 is fulfilled in these particular cases. In this table we call $A(s) = 2 \arg \tanh(\tan(\frac{s}{2}))$, $B(s) = 2 \arg \tan(\tanh(\frac{s}{2}))$ and \mathbb{H} the portion of \mathbb{H}_1^n given by $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos(t)} \mathbb{H}^{n-1}$.*

Table 1: Totally geodesic null hypersurfaces in space forms.

Space form	Fibre	Tot. geod. null hypersurface
$\mathbb{R}_1^n = \mathbb{R} \times \mathbb{R}^{n-1}$	$\mathbb{R} \times \mathbb{R}^{n-2}$	$\{(s, s, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}\}$
$\mathbb{S}_1^n = \mathbb{R} \times_{\cosh(t)} \mathbb{S}^{n-1}$	$(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos(s)} \mathbb{S}^{n-2}$	$\{(A(s), s, z) \in \mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{S}^{n-2}\}$
\mathbb{H}	$\mathbb{R} \times_{\cosh(s)} \mathbb{H}^{n-2}$	$\{(B(s), s, z) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R} \times \mathbb{H}^{n-2}\}$

Example 4.4. *From Proposition 3.3 and Theorem 4.2, nullcones in a Robertson-Walker space induce a twisted decomposition of the fibre. Indeed. Consider, for example, $M = I \times_f \mathbb{S}^n$ and fix $(t_0, x_0) \in C_{(t_*, x_*)}^+$. If we call $\delta = d_{\mathbb{S}^n}(x_*, x_0)$, then from Proposition 3.1 we have $\int_{t_*}^{t_0} \frac{1}{f(r)} dr = \delta$. Take the local decomposition of*

\mathbb{S}^n given by $(0, \pi) \times_{\sin(u)} \mathbb{S}^{n-1}$, where $d_{\mathbb{S}^n}(x_*, x) = u$. In this decomposition, the point x_0 is identified with (δ, z_0) for some $z_0 \in \mathbb{S}^{n-1}$. Thus, if we call $s = u - \delta$ we get the decomposition $(-\delta, \pi - \delta) \times_{\sin(s+\delta)} \mathbb{S}^{n-1}$, where x_0 is identified with $(0, z_0)$ and $C_{(t_*, x_*)}^+$ is given by

$$\{(t, s, z) \in I \times (-\delta, \pi - \delta) \times \mathbb{S}^{n-1} : \int_{t_0}^t \frac{1}{f(r)} dr = s\}$$

as Theorem 4.2 asserts.

Analogously, nullcones in $I \times_f \mathbb{R}^n$ induce the decomposition of \mathbb{R}^n given by $(-\delta, \infty) \times_{s+\delta} \mathbb{S}^{n-1}$, whereas nullcones in $I \times_f \mathbb{H}^n$ induce the decomposition of \mathbb{H}^n given by $(-\delta, \infty) \times_{\sinh(s+\delta)} \mathbb{S}^{n-1}$.

Given a totally umbilic null hypersurface L , we can construct another one, which we call dual of L , simply by changing the sign of the parameter in the base of the twisted decomposition of the fibre induced by L .

Corollary 4.5. *Let $I \times_f F$ be a GRW space and L a totally umbilic null hypersurface. For each $(t_0, x_0) \in L$ we can construct another totally umbilic null hypersurface $\tilde{L}_{(t_0, x_0)}$, which we call dual of L through (t_0, x_0) .*

Specifically, if L induces a twisted decomposition $J \times_\mu S$ of F in a neighborhood of x_0 where L is $\{(t, s, z) \in I \times J \times S : s = \int_{t_0}^t \frac{1}{f(r)} dr\}$, then $\tilde{L}_{(t_0, x_0)}$ is given by

$$\{(t, s, z) \in I \times J \times S : s = \int_t^{t_0} \frac{1}{f(r)} dr\}$$

and its null mean curvature is

$$\tilde{H} = \frac{n-2}{f(t)^2} \left(f'(t) - \frac{\mu_s(s, z)}{\mu(s, z)} \right).$$

Proof. Suppose $J = (-\varepsilon, \varepsilon)$ and consider the coordinate change

$$\psi : I \times_f (J \times_{\mu(s, z)} S) \rightarrow I \times_f (J \times_{\mu(-u, z)} S)$$

given by the isometry $\psi(t, s, z) = (t, -s, z)$. Using Theorem 4.2, in the codomain of ψ , the twisted decomposition $du^2 + \mu(-u, z)^2 g_S$ induces the totally umbilic null hypersurface given by $\{(t, u, z) \in I \times J \times S : u = \int_{t_0}^t \frac{1}{f(r)} dr\}$. The inverse image of this hypersurface is $\tilde{L}_{(t_0, x_0)} = \{(t, s, z) : s = \int_t^{t_0} \frac{1}{f(r)} dr\}$. Its null mean curvature can be easily computed using Lemma 4.1. \square

Example 4.6. *Consider $\mathbb{R}_1^n = \mathbb{R} \times \mathbb{R}^{n-1}$ and $L = C_{(0,0)}^+$. Fixed $(t_0, x_0) \in L$, from Example 4.4 and the above corollary, the dual hypersurface through (t_0, x_0) is given by*

$$\{(t, s, z) \in \mathbb{R} \times (-t_0, \infty) \times \mathbb{S}^{n-2} : s = t_0 - t\} = C_{(2t_0, 0)}^-.$$

Consider now $\mathbb{S}_1^n = \mathbb{R} \times_{\cosh(t)} \mathbb{S}^{n-1}$ and $L = C_{(0,x_*)}^+$. As before, fixed $(t_0, x_0) \in L$, $\tilde{L}_{(t_0, x_0)}$ is given by

$$\{(t, s, z) \in \mathbb{R} \times (-\delta, \pi - \delta) \times \mathbb{S}^{n-2} : s = \int_t^{t_0} \frac{1}{\cosh(r)} dr\},$$

where $\delta = \int_0^{t_0} \frac{1}{\cosh(r)} dr$ and $s = d_{\mathbb{S}^{n-1}}(x_*, x) - \delta$. Take $t_c > 0$ such that

$$\int_0^{t_c} \frac{1}{\cosh(r)} dr = \int_{t_c}^{\infty} \frac{1}{\cosh(r)} dr = \frac{\pi}{4}.$$

If $t_0 < t_c$, then there exists t_s such that $\delta = \int_{t_0}^{t_s} \frac{1}{\cosh(r)} dr$ and using Proposition 3.1, it is easy to show that $\tilde{L}_{(t_0, x_0)} = C_{(t_s, x_*)}^-$.

If $t_c < t_0$, using that $d_{\mathbb{S}^{n-1}}(x^*, x_*) = \pi$, where x^* is the antipodal of x_* , we write $s = \pi - d_{\mathbb{S}^{n-1}}(x^*, x) - \delta$. If we take t_l such that $\pi - \delta = \int_{t_l}^{t_0} \frac{1}{\cosh(r)} dr$, then it follows that $\tilde{L}_{(t_0, x_0)} = C_{(t_l, x^*)}^+$.

Finally, suppose that $t_0 = t_c$. In this case,

$$\tilde{L}_{(t_0, x_0)} = \{(t, s, z) \in \mathbb{R} \times \left(-\frac{\pi}{4}, \frac{3\pi}{4}\right) \times \mathbb{S}^{n-2} : s - \frac{\pi}{4} = -2 \arg \tan \left(\tanh \left(\frac{t}{2} \right) \right)\}.$$

Reparametrizing the s coordinate, it follows that it is the totally geodesic null hypersurface given in Table 1.

The above example shows that the dual construction in Minkowski space is a time reflection, in the sense that the dual of a future nullcone is a past nullcone. However, in the De Sitter space it is more involved, since the dual of a future nullcone through a given point on it can be a past nullcone, a totally geodesic null hypersurface, or even another future nullcone, depending on the situation of the given point.

As an immediate corollary of Theorem 4.2, we can give the following obstruction to the existence of totally umbilic (geodesic) null hypersurfaces.

Corollary 4.7. *If the fibre of a GRW space does not admit any local decomposition as a twisted (warped) product with one dimensional base, then it does not exist any totally umbilic (geodesic) null hypersurface.*

If $\mathcal{K}(\Pi) \neq 0$ for any null plane at a point p in an arbitrary Lorentzian manifold, then Equation (1) implies that it does not exist any totally geodesic null hypersurface through p . In a GRW space $I \times_f F$, the null sectional curvature of a null plane $\Pi = \text{span}(v, u = -\partial_t + w)$, where $v, w \in TF$ are unitary and orthogonal, is given by

$$\mathcal{K}_u(\Pi) = \frac{K^F(\text{span}(v, w)) + f'^2 - f f''}{f^2}.$$

Therefore, this obstruction to the existence of totally geodesic hypersurfaces involves both the curvature of the fibre and the warping function. However, the

obstruction given in Corollary 4.7 is more general because it includes totally umbilic null hypersurfaces and only depends on the fibre.

Example 4.8. *In a Riemannian twisted product manifold $J \times_{\mu} S$, the sectional curvature of any plane containing ∂_s is $\frac{-1}{\mu} \text{Hess}_{\mu}(\partial_s, \partial_s)$. Therefore, $\mathbb{S}^2 \times \mathbb{S}^2$ does not admit any local twisted product decomposition as above, since for any vector we can find two planes containing it with different sectional curvatures. Applying Corollary 4.7, in a GRW space $I \times_f (\mathbb{S}^2 \times \mathbb{S}^2)$ there are not totally umbilic null hypersurfaces.*

Example 4.9. *Consider the twisted product $F = \mathbb{R} \times_{\mu} \mathbb{R}^n$, where $\mu(s, z) = e^s + |z|^2$. A curvature analysis as before shows that it does not admit another local decomposition as a twisted nor warped product with a one dimensional base. Therefore, from Corollary 4.5, in a GRW space $I \times_f F$ there are exactly two totally umbilic null hypersurface through each point and using Corollary 4.7, it does not have any totally geodesic null hypersurface.*

Theorem 4.2 does not hold for timelike nor spacelike hypersurfaces. Indeed, in $\mathbb{R} \times \mathbb{S}^2 \times \mathbb{S}^2$ we can find totally geodesic timelike or spacelike hypersurfaces, although, as it was shown in Example 4.8, $\mathbb{S}^2 \times \mathbb{S}^2$ does not admit local decompositions as a twisted product. However, it is known that a totally umbilic timelike hypersurface in a GRW space must be itself a GRW space, [6].

The following lemma gives us another characterization of an open set of a nullcone near its vertex.

Lemma 4.10. *Let $I \times_f F$ be a GRW space with F complete, take $(t_0, x_0) \in I \times F$ and suppose that F decomposes in a neighborhood of x_0 as a twisted product*

$$((a, b) \times S, ds^2 + \mu(s, z)^2 g_S),$$

where $-\infty < a < 0 < b \leq \infty$ ($-\infty \leq a < 0 < b < \infty$), S is connected and x_0 is identified with $(0, z_0)$ for some $z_0 \in S$. The null hypersurface

$$L = \{(t, s, z) \in I \times (a, b) \times S : s = \int_{t_0}^t \frac{1}{f(r)} dr\}$$

is contained in a future (past) local nullcone if and only if

1. $\lim_{s \rightarrow a^+} \mu(s, z) = 0$ for all $z \in S$ ($\lim_{s \rightarrow b^-} \mu(s, z) = 0$ for all $z \in S$).
2. It exists $t_* \in I$ with $\int_{t_0}^{t_*} \frac{1}{f(r)} dr = a$ ($\int_{t_0}^{t_*} \frac{1}{f(r)} dr = b$).

Proof. Suppose that (1) and (2) hold. Since the integral curves of ∂_s are unitary geodesics and F is complete, it exists $\lim_{s \rightarrow a^+} (s, z)$ for all $z \in S$. Fix $z, z' \in S$ two distinct points such that there exists $\sigma(r)$ a unitary geodesic in S with $\sigma(0) = z$ and $\sigma(d) = z'$. If we call $\gamma_s(r) = (s, \sigma(r))$, then

$$d_F(\gamma_s(0), \gamma_s(d)) \leq \text{length}(\gamma_s) = \int_0^d \mu(s, \sigma(r)) dr$$

and so $\lim_{s \rightarrow a^+} d_F(\gamma_s(0), \gamma_s(d)) = 0$. Therefore, since S is connected, $\lim_{s \rightarrow a^+} (s, z)$ is the same for all $z \in S$, say $x_* \in F$, and the integral curves of ∂_s are radial geodesic from x_* . Thus, $d_F(x_*, (s, z)) = s - a$ and given $(t, x) \in L$ we have

$$d_F(x_*, x) = \int_{t_0}^t \frac{1}{f(r)} dr - a = \int_{t_*}^t \frac{1}{f(r)} dr.$$

By Proposition 3.1, L is contained in the nullcone $C_{(t_*, x_*)}^+$.
The converse is straightforward. \square

Theorem 4.11. *Any totally umbilic null hypersurface in a Robertson-Walker space $I \times_f \mathbb{S}^{n-1}$ ($n > 3$) with*

$$\int_I \frac{1}{f(r)} dr > \pi \tag{6}$$

is an open set of a nullcone. In particular, it cannot exist totally geodesic null hypersurfaces.

Proof. Let L be a totally umbilic null hypersurface and take $(t_0, x_0) \in L$. Using Theorem 4.2, \mathbb{S}^{n-1} can be decomposed in a neighborhood of x_0 as a twisted product. Since \mathbb{S}^{n-1} is Einstein, this decomposition is actually a warped product [4], and it is easy to show that it is

$$\left(-\frac{\pi}{2} - \theta, \frac{\pi}{2} - \theta\right) \times_{\mu} \mathbb{S}^{n-2}(|\cos(\theta)|),$$

where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\mu(s) = \frac{\cos(s+\theta)}{\cos(\theta)}$ and x_0 is identified with $(0, z_0)$ for some $z_0 \in \mathbb{S}^{n-2}$.

Using (6), it exists $t_* \in I$ such that $\int_{t_0}^{t_*} \frac{1}{f(r)} dr = \frac{\pi}{2} - \theta$ or $\int_{t_0}^{t_*} \frac{1}{f(r)} dr = -\frac{\pi}{2} - \theta$ and applying the above lemma, L is contained in a lightcone. The last claim follows from Remark 3.4. \square

The condition (6) can not be sharpened. For example, in $\mathbb{S}_1^n = \mathbb{R} \times_{\cosh(t)} \mathbb{S}^{n-1}$ there are totally geodesic null hypersurfaces which evidently are not contained in a nullcone.

We can also get the following immediate corollaries.

Corollary 4.12. *Nullcones are the unique totally umbilic null hypersurfaces in the closed Friedmann Cosmological model.*

Corollary 4.13. *Any totally umbilic null hypersurface in $\mathbb{R} \times \mathbb{S}^{n-1}$ ($n > 3$) is contained in a nullcone.*

Recall that both, Friedmann models and $\mathbb{R} \times \mathbb{S}^{n-1}$, can not possess totally geodesic null hypersurfaces due to Equation (1) and Lemma 5.2 of [6].

In [1] it is shown that totally umbilic null hypersurfaces in a Lorentzian manifold of constant curvature are contained in nullcones. The proof is based on their following claim: in a Lorentzian manifold any totally umbilic null hypersurfaces with zero null sectional curvature is contained in a nullcone. But the example below shows that this is not true in general.

Example 4.14. Let $Q \times_r \mathbb{S}^2$ be the Kruskal spacetime, [9]. The hypersurface $L_{u_0} = \{(u, v, x) \in Q \times \mathbb{S}^2 : u = u_0\}$ is totally umbilic and null. Moreover, if Π is a null tangent plane to L_{u_0} , then it is spanned by ∂_v and $w \in T\mathbb{S}^2$, so

$$\mathcal{K}_{\partial_v}(\Pi) = -\frac{\text{Hess}_r(\partial_v, \partial_v)}{r} = 0,$$

but L_{u_0} is not contained in a nullcone.

However, under completeness hypothesis it seems that the above claim is true. For clarity, we give an alternative proof of the following result using the technique presented in this paper.

Theorem 4.15. Any totally umbilic null hypersurface in a complete space of constant curvature and dimension greater than three is totally geodesic or is contained in a nullcone.

Proof. We can suppose that M is \mathbb{R}_1^n , \mathbb{S}_1^n or \mathbb{H}_1^n .

Suppose first that $M = \mathbb{R}_1^n = \mathbb{R} \times \mathbb{R}^{n-1}$, and L is a totally umbilic and non totally geodesic null hypersurface in M . From Theorem 4.2 and [4], it induces a decomposition of \mathbb{R}^{n-1} as a warped product, but since L is not totally geodesic, the only possible decomposition is $(-\frac{1}{\theta}, \infty) \times_{\theta s+1} \mathbb{S}^{n-2}(\frac{1}{\theta})$ for $\theta > 0$ or $(-\infty, -\frac{1}{\theta}) \times_{\theta s+1} \mathbb{S}^{n-2}(-\frac{1}{\theta})$ for $\theta < 0$. Applying Lemma 4.10, L is contained in a nullcone.

Suppose now that $M = \mathbb{S}_1^n = \mathbb{R} \times_{\cosh(t)} \mathbb{S}^{n-1}$. Without loss of generality, we can suppose that $(0, x_0) \in L$ for some $x_0 \in \mathbb{S}^{n-1}$. As in the proof of Theorem 4.11, there is a decomposition of \mathbb{S}^{n-1} as

$$\left(-\frac{\pi}{2} - \theta, \frac{\pi}{2} - \theta\right) \times_{\mu} \mathbb{S}^{n-2}(|\cos(\theta)|),$$

where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\mu(s) = \frac{\cos(s+\theta)}{\cos(\theta)}$. If $\theta = 0$, then L is totally geodesic (see Table 1). If $\theta \neq 0$, using Lemma 4.10, L is contained in a nullcone.

Finally, we consider $M = \mathbb{H}_1^n \subset \mathbb{R}_2^{n+1}$. Since it does not admit a global decomposition as a RW space, a little more work must be done in this case. We can suppose that L intersects an open set of \mathbb{H}_1^n isometric to $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos t} \mathbb{H}^{n-1}$ and $(0, x_0) \in L$ for some $x_0 \in \mathbb{H}^{n-1}$. Applying Theorem 4.2 and [4], there is a decomposition of \mathbb{H}^{n-1} as a warped product $J \times_{\mu} S$ in a neighborhood of x_0 and L is given by

$$\{(2 \arg \tan\left(\tanh \frac{s}{2}\right), s, z) : s \in J, z \in S\}.$$

The decomposition of \mathbb{H}^{n-1} can be of three different types.

- $J \times \frac{\sinh(s+\theta)}{\sinh \theta} \mathbb{S}^{n-2}(|\sinh \theta|)$ where $J = (-\theta, \infty)$ if $\theta > 0$ and $J = (-\infty, -\theta)$ if $\theta < 0$. In this case we can apply Lemma 4.10.

- $\mathbb{R} \times \frac{\cosh(s+\theta)}{\cosh \theta} \mathbb{H}^{n-2}(\cosh \theta)$ where $\theta \in \mathbb{R}$. If $\theta = 0$, then L is totally geodesic (see Table 1), so we suppose $\theta \neq 0$. The map

$$\Phi : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times_{\cos t} \left(\mathbb{R} \times \frac{\cosh(s+\theta)}{\cosh \theta} \mathbb{H}^{n-2}(\cosh \theta)\right) \rightarrow \mathbb{R}_2^{n+1}$$

given by

$$\Phi(t, s, z) = \left(\cos t \sinh(s + \theta), \frac{\cos t \cosh(s + \theta)}{\cosh \theta} z, \sin t\right)$$

is an isometric embedding into \mathbb{H}_1^n . If we call $t = 2 \arg \tan\left(\tanh \frac{s}{2}\right)$, then it holds $\cos t \cosh s = 1$ and $\sin t = \tanh s$ and it is easy to show that

$$\begin{aligned} \Phi(L) = \\ \left\{(\sinh \theta + \cosh \theta \tanh s, (1 + \tanh \theta \tanh s)z, \tanh s) : s \in \mathbb{R}, z \in \mathbb{H}^{n-2}\right\} \end{aligned}$$

is contained in the nullcone of \mathbb{H}_1^n at $\left(-\frac{1}{\sinh \theta}, 0, \dots, 0, -\frac{1}{\tanh \theta}\right)$.

- $\mathbb{R} \times_{e^s} \mathbb{R}^{n-2}$. The map

$$\Psi : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times_{\cos t} \left(\mathbb{R} \times_{e^s} \mathbb{R}^{n-2}\right) \rightarrow \mathbb{R}_2^{n+1}$$

given by

$$\begin{aligned} \Psi(t, s, z) \\ = \left(e^s \cos t \cdot z, \frac{\cos t (e^s (1 - |z|^2) - e^{-s})}{2}, \frac{\cos t (e^s (1 + |z|^2) + e^{-s})}{2}, \sin t\right) \end{aligned}$$

is an isometric embedding. As before, we have

$$\begin{aligned} \Psi(L) \\ = \left\{\left(\frac{e^s \cdot z}{\cosh s}, \tanh s - \frac{e^s |z|^2}{2 \cosh s}, 1 + \frac{e^s |z|^2}{2 \cosh s}, \tanh s\right) : s \in \mathbb{R}, z \in \mathbb{R}^{n-2}\right\}, \end{aligned}$$

which is contained in the nullcone of \mathbb{H}_1^n at $(0, \dots, 0, -1, 1, -1)$.

□

Example 4.16. *Since \mathbb{H}^{n-1} qualitatively admits different decompositions, we can show that an analogous result of Corollary 4.13 replacing \mathbb{S}^{n-1} with \mathbb{H}^{n-1} is not true. In fact, from Theorem 4.2 the null hypersurface given by $L = \{(s, s, z)\}$ in $\mathbb{R} \times (\mathbb{R} \times_{e^s} \mathbb{R}^{n-2}) \subset \mathbb{R} \times \mathbb{H}^{n-1}$ is totally umbilic but it is evidently not contained in a nullcone.*

5 Standard static spaces

Given $I \subset \mathbb{R}$, (F, g_F) a Riemannian manifold and $\phi \in C^\infty(F)$ a positive function, the manifold $F \times I$ furnished with the Lorentzian metric $g^* = g_F - \phi^2 dt^2$ is called a standard static space and is denoted by $F \times_\phi I$. Totally umbilic null hypersurfaces are preserved if we apply a conformal transformation to get a GRW space with constant warping function. In general, we have the following.

Lemma 5.1. *Let (M, g^*) be a Lorentzian manifold with $\dim M = n$, $\phi \in C^\infty(M)$ a positive function and $g = \frac{1}{\phi^2} g^*$. If L is a null hypersurface in (M, g^*) and B^* its null second fundamental form respect to a fixed null vector field $\xi \in \mathfrak{X}(L)$, then L is a null hypersurface in (M, g) with null second fundamental form B respect to ξ given by*

$$B = \frac{1}{\phi^2} (B^* + \xi(\ln \phi) g^*).$$

In particular, if L is totally umbilic in (M, g^) with null mean curvature H^* , then it is also totally umbilic in (M, g) with null mean curvature $H = H^* + (n - 2)\xi(\ln \phi)$.*

Proof. Just use that $\nabla_U V = \nabla_U^* V + \frac{1}{\phi} (g^*(U, V) \nabla^* \phi - U(\phi)V - V(\phi)U)$. \square

Theorem 5.2. *Let $F \times_\phi I$ be a n -dimensional standard static space. If L is a totally umbilic null hypersurface, then for each $(x_0, t_0) \in L$ there exists a local decomposition of $(F, \frac{1}{\phi^2} g_F)$ in a neighborhood of x_0 as a twisted product with one dimensional base*

$$(J \times S, ds^2 + \mu(s, z)^2 g_S),$$

where x_0 is identified with $(0, z_0)$ for some $z_0 \in S$ and L is given by

$$\{(s, z, s + t_0) \in J \times S \times I\}.$$

Moreover, if H^* is the null mean curvature of L , then

$$\mu(s, z) = \frac{\phi(0, z)}{\phi(s, z)} \exp\left(\int_0^s \frac{H^*(r, z, r + t_0)}{n - 2} dr\right).$$

Conversely, if $(F, \frac{1}{\phi^2} g_F)$ admits a twisted decomposition in a neighborhood of x_0 as above, then

$$L = \{(s, z, s + t_0) \in J \times S \times I\}$$

is a totally umbilic null hypersurface with null mean curvature

$$H^* = (n - 2) \frac{d}{ds} \ln(\mu\phi). \tag{7}$$

Proof. Take the conformal metric $g = \frac{1}{\phi^2}g^*$ and apply the above lemma and Theorem 4.2. For this, take into account that from Equation (4), $\xi(\ln \phi) = -\partial_s(\ln \phi)$. \square

Remark 5.3. *As in Corollary 4.5, if L is a totally umbilic null hypersurface in a standard static space, we can construct another totally umbilic null hypersurface through each point $(x_0, t_0) \in L$ which we call $\tilde{L}_{(x_0, t_0)}$, the dual of L through (x_0, t_0) . In fact, if L induces a twisted decomposition of $(F, \frac{1}{\phi^2}g_F)$ in a neighborhood of x_0 where L is given by $\{(s, z, s + t_0)\}$, then $\tilde{L}_{(x_0, t_0)}$ is given by $\{(s, z, -s + t_0)\}$. From Corollary 4.5 and Lemma 5.1, its null mean curvature is*

$$\tilde{H}^* = (n - 2) \frac{d}{ds} \ln \left(\frac{\phi}{\mu} \right). \quad (8)$$

Now, we consider the family of standard static spacetimes given by

$$\left(I \times \mathbb{S}^2 \times \mathbb{R}, \frac{1}{h(r)} dr^2 + r^2 g_0 - h(r) dt^2 \right), \quad (9)$$

where $I \subset \mathbb{R}$, $h \in C^\infty(I)$ is a positive function and g_0 is the canonical metric on \mathbb{S}^2 . This family includes important examples of spacetimes. If $h(r) = 1 - \frac{m^2}{r} + \frac{c^2}{r^2}$ for certain constant m and c , then we get the Reissner-Nordström spacetime (the Schwarzschild exterior in the case $c = 0$) and if $h(r) = 1 - \frac{m^2}{r} + kr^2$, then we obtain the De Sitter-Schwarzschild spacetime (Minkowski, De Sitter or anti-De Sitter if $m = 0$ and $k = 0$, $k > 0$ or $k < 0$ respectively).

We first need to know how many different twisted decomposition admits the spatial part of (9) to apply Theorem 5.2 to these spacetimes.

Lemma 5.4. *Let F be the warped product $(I \times \mathbb{S}^2, ds^2 + \mu(s)^2 g_0)$. If there exists a different decomposition of F as a twisted product in a neighborhood of a point, then F has constant curvature in this neighborhood.*

Proof. A twisted decomposition is characterized by the existence of a unitary, closed and orthogonally conformal vector field, [6]. Therefore, if there exists a different decomposition as a twisted product in a neighborhood $\theta \subset F$, there is a vector field E with these properties and linearly independent with ∂_s in θ . Hence, any plane containing E has the same sectional curvature. Suppose that $E = \alpha \partial_s + U$ with $U_{(s,z)} \in T_z \mathbb{S}^2$ for all $(s, z) \in \theta$ and consider the planes $\Pi_0 = \text{span}(E, \partial_s)$ and $\Pi_1 = (E, V)$ where $V_{(s,z)} \in T_z \mathbb{S}^2$ with $V \perp U$. A straightforward computation shows that

$$K(\Pi_0) = -\frac{\mu''}{\mu},$$

$$K(\Pi_1) = -\alpha^2 \frac{\mu''}{\mu} + \frac{(1 - \alpha^2)(1 - \mu'^2)}{\mu^2}.$$

Therefore, since $K(\Pi_0) = K(\Pi_1)$, it holds $\mu'' = \frac{\mu'^2 - 1}{\mu}$. The solutions to this differential equation are $\mu(s) = \frac{1}{k} \sinh(ks + s_0)$, $\mu(s) = \frac{1}{k} \sin(ks + s_0)$

or $\mu(s) = \pm s + s_0$ but for these warping functions the sectional curvature of $ds^2 + \mu(s)^2 g_0$ is $-k^2$, k^2 or 0 respectively. \square

Theorem 5.5. *If $h \in C^\infty(I)$ is a positive function such that $h''' \neq 0$ in any open subset of I , then the spacetime given by (9) has exactly two totally umbilic null hypersurface through each point.*

Proof. Call $(F, g_F) = \left(I \times \mathbb{S}^2, \frac{1}{h(r)} dr^2 + r^2 g_0 \right)$. If we take a function φ such that $\varphi'(s) = h(\varphi(s))$ and we make the coordinate change $r = \varphi(s)$, then $\frac{1}{h} g_F$ is written as $ds^2 + \mu(s)^2 g_0$, where $\mu(s) = \frac{\varphi(s)}{\sqrt{h(\varphi(s))}}$. Since $\left(\frac{\mu''}{\mu} \right)' = -\frac{h^2 h'''}{2}$, there is not any open neighborhood in $(F, \frac{1}{h} g_F)$ with constant curvature. Applying the above lemma, this warped decomposition is unique. Therefore, using Theorem 5.2 and Remark 5.3, for each point there are exactly two totally umbilic null hypersurfaces. \square

Corollary 5.6. *In a De Sitter-Schwarzschild with $m \neq 0$ and in a Reissner-Nordström spacetime there are exactly two totally umbilic non-totally geodesic null hypersurface through each point.*

Proof. From Equations (7) and (8), the null mean curvatures are $H^* = 2 \frac{d}{ds} \ln \varphi$ and $\tilde{H}^* = 2 \frac{d}{ds} \ln \left(\frac{h}{\varphi} \right)$, which are not identically zero. \square

Remark 5.7. *If we consider the Schwarzschild exterior embedded in the Kruskal spacetime $Q \times_r \mathbb{S}^2$, the totally umbilic null hypersurfaces claimed in the above corollary are given by*

$$\{(u, v, x) \in Q \times \mathbb{S}^2 : u = u_0\}$$

and

$$\{(u, v, x) \in Q \times \mathbb{S}^2 : v = v_0\}.$$

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