

Compact Lorentzian holonomy ^{*}

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Abstract

We consider (compact or noncompact) Lorentzian manifolds whose holonomy group has compact closure. Among other results, we obtain that this property is equivalent to admitting a parallel timelike vector field. We also derive some properties of the space of all such metrics on a given manifold.

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1 Introduction

It is well known that a structure group reduction of the frame bundle encodes the existence of a geometric structure on the manifold. If, moreover, it contains the holonomy group of a given connection ∇ , the geometric structure is ∇ -parallel. The most familiar example is the existence of a semi-Riemannian metric which is equivalent to a reduction of the structure group to $O_\nu(n)$, and if the holonomy group of a given connection is contained in $O_\nu(n)$, this means that the semi-Riemannian metric is parallel, that is, the connection is metric. Another classical example is a $2n$ -dimensional Kähler manifold. It has holonomy group contained in $U(n)$. In fact, $U(n) = GL(n, \mathbb{C}) \cap O(2n) \cap Sp(n, \mathbb{R})$, and this means that the manifold has a complex structure and a parallel symplectic structure adapted to a Riemannian metric.

In (oriented) Riemannian geometry, the generic holonomy is the (special) orthogonal group, so noncompact (i.e., non-closed) holonomy implies the presence of a parallel geometric structure. Simply connected Riemannian manifolds have compact holonomy group because it coincides with its restricted holonomy group, which is well known to be compact, [4]. On the other hand, the question

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of the existence of a compact Riemannian manifold with noncompact holonomy was solved in [21] where the author showed the existence of such manifolds and studied their structures. In fact, a compact Riemannian manifold with noncompact holonomy has a finite cover that is the total space of a torus bundle over a compact manifold. Moreover they must have dimension greater or equal than 5.

The situation in Lorentzian manifolds is similar but slightly different because the generic holonomy is the Lorentz group which is noncompact. It is natural to ask the analogous question: can we describe the Lorentzian manifolds which have compact holonomy?

Noncompactness of the holonomy group is responsible for noncompleteness in some compact Lorentzian manifolds, as the Clifton-Pohl torus. The relationship between holonomy and completeness is in general not well known. See for example [17] where the authors study the case of compact pp-waves. In a space-time it is related to undesirable identifications of singular points in b-singularity theory, [20]. In fact, in [1] it was shown that in the four dimensional Friedmann closed model of the Universe, having noncompact holonomy, big bang and big crunch are the same point in the b-boundary. On the other hand, compactness of the holonomy group has been used in [7] to define the so-called Cauchy singular boundaries in space-times. Later, one of the authors of this article (M.G.), using the fundamental observation that both g and the flip metric of g around a parallel vector field induce the same Levi-Civita connection, proved that the compactness of the holonomy group implies that the Cauchy singular boundary of the manifold is homeomorphic to its b-boundary, [10].

In this article we identify Lorentzian manifolds (compact or not) whose holonomy groups have compact closure, and draw some consequences.

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2 Compact holonomy

We assume all manifolds to be connected. First of all, we recall some basic facts on principal bundles following [15, Chapter II.]. Let $\pi : P(M, G) \rightarrow M$ and $\pi' : P'(M', G') \rightarrow M'$ be two principal bundles. When there are no possible confusion, they are denoted $P(M, G)$ and $P'(M', G')$ or even P and P' . A **bundle map** is a differentiable map $f : P \rightarrow P'$ such that $f(ua) = f(u)f(a)$ for every $u \in P$ and $a \in G$, where $f : G \rightarrow G'$ is a group morphism (we use the same letter f for the induced morphism on the structure groups). It is said that P and P' are **equivalent** or **isomorphic as fibre bundles** if there exists a bijection $f : P \rightarrow P'$ such that both f and f^{-1} are bundle maps. The map f itself is called a **fibre bundle isomorphism**.

The structure group of $P(M, G)$ is said to be **reducible to a subgroup** $H \subset G$ if there exists a principal bundle $P'(M, H)$ and an embedding $f : P' \rightarrow P$ compatible to a monomorphism on groups $f : H \rightarrow G$ and inducing the identity $i : M \rightarrow M$. With the natural identifications, $P'(M, H)$ is a subfibre bundle of $P(M, G)$, sometimes called the **reduced bundle of** $P(M, G)$. Note that right

translations by an element $a \in G$ is a fibre bundle isomorphism $R_a : P \rightarrow P$. If $f : P' \rightarrow P$ is a reduction of the structure group, the map $f_a = R_a \circ f$ defines an equivalent subfibre bundle in P which is the translation of P' by $a \in G$. Given a point $x \in M$, the choice of a point $u \in \pi^{-1}(x)$ is equivalent to localize the identity in the fibre containing p (which is diffeomorphic to G), so if $u \in P'(M, H)$ we can think of $P'(M, H)$ as a subfibre bundle of $P(M, G)$ through u , and this fixes the position of $P'(M, H)$ as a subset of $P(M, G)$. Similarly, $f_a(P') = P'a$ is a subfibre bundle through ua . Thus, a section of a G -principal bundle is nothing but a reduction to the the subgroup $\{Id\}$ of G .

Remark 1 *Using transition functions, it is clear that if the structure group of $P(M, G)$ is reducible to a subgroup H and S is a subgroup of G with $H \subset S \subset G$, then G is also reducible to S and S is reducible to H . We have special interest in the following case: suppose that G is reducible to a subgroup H . If $a \in G$, then G is also reducible to aHa^{-1} (their transition functions are related by $\psi_{\beta\alpha} = a\varphi_{\beta\alpha}a^{-1}$ [15, Proposition 5.3]). Moreover in the special case in which $a \in S$, both subfibre bundles $P'(M, H)$ and $P''(M, aHa^{-1})$ can be seen as subfibre bundles of $Q(M, S)$.*

On the other hand, a connection Γ on P is called **reducible to a subbundle** Q if it has a connection Γ' and $i : Q \rightarrow P$ maps horizontal subspaces of Γ' to horizontal subspaces of Γ . Let $u \in P$ and $P(u)$ the set of points in P that can be joined to u by a horizontal curve. The Reduction Theorem (Th. II.7.1. in [15]) states that $P(u)$ is a reduced bundle of $P(M, G)$ with structure group the holonomy group of P . It is called the **holonomy subbundle through** u , and the connection Γ in P is always reducible to a connection in $P(u)$.

The structure group is reducible to a closed subgroup H if and only if the associated fibre bundle $P \times_G (G/H)$ admits a cross section as explained in the following. Let $\mu : P \rightarrow P \times_G (G/H)$ be the projection given by $\mu(u) = uH$ where we have identified $P \times_G (G/H)$ with P/H as usual, that is $(u, aH) \leftrightarrow uaH$. The following diagram summarize the situation,

$$\begin{array}{ccc} Q & \xrightarrow{i} & P \\ \pi' \downarrow & & \downarrow \mu \\ M & \xrightarrow{X} & P/H \end{array}$$

where $Q = X^*P$ is the pull back of P . Q is a closed submanifold of P and a subfibre bundle $Q(M, H)$ in $P(M, G)$. The correspondence between subfibre bundles and cross sections of the associated bundle is bijective (Remark after Prop. I.5.6 in [15]). A connection on P is reducible to Q if and only if its associated section $X : M \rightarrow P/H$ is parallel (Prop. II.7.4 in [15]).

The following example illustrates the kind of applications we are interested in.

Example 2 *Let (M, g) be a m -dimensional Riemannian manifold. The holonomy group is a subgroup of $O(m-1)$ if and only if it admits a parallel vector field. In fact, if OM is the orthonormal frame bundle with structure group*

$O(m)$, and $OM \times_{O(m)} (O(m)/O(m-1))$ its associated bundle, the hypothesis means that the structure group and the Levi-Civita connection are reducible to an $O(m-1)$ -structure, thus the associated bundle admits a parallel cross section, but this bundle is just the unit bundle SM of M .

Let (M, g) be a semi-Riemannian manifold with signature ν . We denote Hol^M and Hol_0^M its holonomy group and its restricted holonomy group, respectively. We drop the superindex M if no confusion is possible.

Observe that unlike in the Riemannian case, there are examples of simply connected Lorentzian manifolds with non-closed holonomy group, [3].

The following theorem identifies Lorentzian manifolds with holonomy contained in a compact group. One could try to prove it via Thomas Leistner's classification theorem [16]. Here, however, we want to use other methods, which will lead us moreover to two extensions of the theorem, namely Theorem 8 and Theorem 9. We use the following well known lemma, whose proof we want to include nevertheless for the sake of completeness.

Lemma 3 *The map*

$$\pi_1(M, p) \xrightarrow{j} Hol/Hol_0$$

given by $j([\gamma]) = [p_\gamma]$ (where p_γ is the parallel transport along γ) is a surjective group morphism.

Proof: Consider the usual assignment $J : \Omega(M, p) \rightarrow Hol$ (where $\Omega(M, p)$ is the space of smooth loops at p) given by parallel transport. We can take the quotients on both sides. The right-hand side remains a Lie group due to Theorem II.4.2 in [15], thus we only have to show that J descends to the quotient. But that follows easily from the fact that if c_1 and c_2 are two different representatives of the same class $\alpha \in \pi_1(M, p)$, then their curve concatenation $c := c_1 \cdot c_2^{-1}$ is contractible, thus $J(c_1) = J(c) \circ J(c_2)$ and $J(c) \in Hol_0$. ■

Theorem 4 *Let (M, g) be an oriented and time oriented m -dimensional Lorentzian manifold.*

1. *The holonomy group is relatively compact (and thereby contained in a compact group) if and only if it admits a timelike parallel vector field.*
2. *If (M, g) admits a timelike parallel vector field and $\pi_1(M, p)$ is finite, then its holonomy group is compact.*

Remark 5 *Of course, a parallel timelike vector field induces a local product splitting of the manifold. But a local or global splitting does not suffice to imply that the holonomy is contained in a compact group, see Remark (10) below for a counterexample.*

It is also not true that the holonomy is always compact if there is a parallel timelike vector field, see Remark (7).

Proof: 1. Using orientation and time-orientation, the principal fibre bundle of oriented and time oriented frames on M , which we call OM , is reducible to a $SO_1^+(m)$ structure, where $SO_1^+(m)$ is the identity component of the Lorentz group. Let K be a maximal compact subgroup of $SO_1^+(m)$. The uniqueness part of Iwasawa's Theorem [13] states that it coincides with the standard embedded $SO(m-1)$ up to conjugation. Moreover, K contains Hol as Hol is contained in a compact subgroup and K is a maximal compact subgroup. Therefore, OM and the Levi-Civita connection are reducible to a K -structure, see Remark (1). The associated bundle $OM \times_{SO_1(m)} (SO_1^+(m)/K)$ is SM , the bundle formed by the future timelike unitary vector fields on M . Thus the section $V : M \rightarrow SM$ associated to the K -structure is a timelike unitary and parallel vector field.

To see the identification of the associated bundle, take $e_0 \in \mathbb{R}^m$ the timelike future and unitary vector of the canonical basis in the Minkowski space (\mathbb{R}^m, η) . Suppose that $K = aSO(m-1)a^{-1}$ for some $a \in SO_1^+(m)$. The group $SO_1^+(m)$ acts transitively on $U = \{v \in \mathbb{R}^m / \eta(v, v) = -1, v \text{ future}\}$ and K is the isotropy group of $ae_0 \in U$, so U is diffeomorphic to the homogeneous space $SO_1^+(m)/K$.

Conversely, if (M, g) admits a timelike parallel vector field V , let g_R be the associated Riemannian metric (flip metric)

$$g_R(X, Y) = g(X, Y) + 2g(X, V)g(Y, V) \quad (1)$$

where we suppose $|V| = 1$. It has the same holonomy group Hol than (M, g) and Example 2 shows that it is contained in $O(m-1)$.

2. Lemma 3 implies that Hol has a finite number of connected components. As seen above, the restricted holonomy group Hol_0 is that of (M, g_R) , so it is compact, and Hol is also compact.

We give a second proof using a Haar measure average. Later, it will have the advantage to be a bit more flexible in the sense that one can restrict to the parallel transport along contractible curves, curves lying completely in prescribed open sets or in a foliation.

As Hol is contained in a compact subset, its closure C , which is a subgroup as well, is compact, and thus carries a bi-invariant Haar measure μ . Now let a point $p \in M$ be given. We want to construct a timelike vector $v \in T_p M$ invariant under Hol_p . To that purpose, choose a future timelike vector $v_0 \in T_p M$ at will and define $v := \int_{Hol_p} h(v_0) d\mu(h)$. The integral exists as the Haar measure of the compact group C is finite and the action is continuous. Now, given any $k \in Hol_p$, we compute

$$kv = \int_{Hol_p} k \circ h(v_0) d\mu(h) = \int_{Hol_p} h(v_0) k^* d\mu(h) = \int_{Hol_p} h(v_0) d\mu(h) = v,$$

so indeed v is invariant, and it is timelike, as the integrand consists in timelike future vectors and those form a convex set. And now, using the parallel transport P_c along a curve c , we have that $P_c(v) = P_k(v)$ if $c(0) = p = k(0)$ and $c(1) = k(1)$, because P_k is an isomorphism and $P_k^{-1} \circ P_c(v) = P_{ck^{-1}}(v) = v$ as

$P_{ck-1} \in Hol_p$. Thus there is a well-defined way to extend v to a parallel future timelike vector field V . ■

Remark 6 *Note that both proofs do not work in higher signature. The first one does not, because the unit timelike vectors can not be identified with a quotient of the type above, and the second one does not, as the set of timelike vectors is not convex anymore. On the other hand, the local version of the De Rham-Wu Theorem, [23], can only be applied once we know there is a parallel vector field, but in some sense this is itself a stronger conclusion.*

Remark 7 *The hypothesis on the orientability in Theorem (4) is necessary. In fact, we can take M a Möbius strip with flat metric such that there is no globally defined timelike vector field. It is clear that its holonomy group is \mathbb{Z}_2 . The hypothesis on the finiteness of $\pi_1(M, p)$ is also necessary to ensure compact holonomy. A counterexample is the direct product $(\mathbb{S}^1, -dt^2) \times (T, g_0)$ whose holonomy group is that of (T, g_0) and we can choose it with noncompact holonomy.*

To show that the method above can in fact be applied in a more general situation, let us show an statement analogous to the previous theorem adapted to a distribution. We define, for a distribution D in a submanifold of M ,

$$Hol_{D,x}^{(M,g)} := \{P_c / c(0) = x = c(1), c'(s) \in D \forall s \in [0, 1]\}$$

and call $Hol_D^{(M,g)}$ the **holonomy related to D** . The equivalence class of this representation does not depend on the point, just as in the classical case. Then we can conclude

Theorem 8 *If S is a totally geodesic spacelike submanifold of a spacetime (M, g) and if $Hol_{TS}^{(M,g)}$ is contained in a compact set, but $Hol^{(S,g)}$ does not fix a vector (and correspondingly if both are replaced by the connected component of Id in the respective groups), then the normal bundle of S contains an invariant one-dimensional subbundle.*

Proof: We conclude exactly as above that $Hol_{TS}^{(M,g)}$ fixes a temporal vector field V which we can assume to be future. We want to show that it is a normal vector to S . As $D := TS$ is fixed by $Hol_{TS}^{(M,g)}$, we know that $W := pr_D^g V$ is fixed as well. And as $Hol_{TS}^{(M,g)}|_D = Hol^{(S,g)}$, we can conclude that V is normal, which proves the claim. ■

Whereas the Haar measure method in the proof of 4 is appropriate to prove the above theorem, the first method used there is the right one for the following theorem.

Theorem 9 *Let (M, g) be an oriented and time oriented m -dimensional Lorentzian manifold with $Hol \subset SO(m-2)$. Then it admits an orthonormal system $\{V_1, V_2\}$ formed by parallel vector fields, with V_1 timelike.*

Proof: After Theorem (4), we need only prove the existence of V_2 . If (e_0, e_1, \dots, e_m) is the canonical basis of \mathbb{R}^m , $SO_1^+(m)$ acts transitively on $L = \{v \in \mathbb{R}^m / \eta(v, v) = 1\}$ and $SO(m-2)$ is the isotropy group of $e_1 \in L$, so $SO_1^+(m)/SO(m-2) = L$. The associated bundle $OM \times_{SO_1^+(m)} (SO_1^+(m)/SO(m-2))$ is the bundle of unitary spacelike vectors in M . So the corresponding section to the group reduction to $SO(m-2)$ is a unitary spacelike vector field V_2 , and the hypothesis $Hol \subset SO(m-2)$ implies that it is parallel. It remains to show the orthogonality, but changing V_2 by a suitable combination of both if necessary, the system claimed is $\{V_1, V_2\}$. ■

Remark 10 *The existence of a Lorentzian manifolds with a timelike parallel vector field and Hol noncompact is clear in the noncompact case because we turn the question in a well-known Riemannian one using the above flip metric (1). The compact case is more involved, but it can be solved using the results in [21]. We consider three cases*

- $\dim M \geq 6$, the above example $M = \mathbb{S}^1 \times T$ with T compact and Hol^T noncompact shows that they exist, but in this case we know that $\dim T \geq 5$.
- $\dim M \leq 4$, the presence of a parallel timelike vector field allows us construct the Riemannian flip metric on M , and this implies that the holonomy is compact.
- $\dim M = 5$. We can not apply neither of the above direct arguments, but the Wilking example provides one. Let $a \in (0, 2\pi)$ be a irrational number and $b \in \mathbb{R}$, $b > 0$. Define

$$X = \begin{pmatrix} 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & -b \end{pmatrix} \in M(4, \mathbb{R}),$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 2 \\ 0 & -1 & 0 & -5 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

There is a matrix $C \in GL(4, \mathbb{R})$ such that $\exp X = CBC^{-1}$. The matrix $\exp X$ leaves the lattice $L = C \cdot \mathbb{Z}^4$ invariant.

Consider the semidirect product $S = \mathbb{R}^4 \rtimes \mathbb{R}$, $(v, s)(w, t) = (v + \exp(sX)w, s + t)$ and the discrete cocompact subgroup $\Lambda = L \rtimes \mathbb{Z}$. The group Λ acts as deck transformation group of the covering $p : S \rightarrow \Lambda \backslash S$ by left translations. The group S admits a left invariant metric $g = \langle \cdot, \cdot \rangle \times g_1$ where $\langle \cdot, \cdot \rangle$ is the euclidean metric in $\mathbb{R}^2 \times \{0\}$ and g_1 is a left invariant metric on $\{0\} \times \mathbb{R}^3$. The quotient $\Lambda \backslash S$ with the induced metric has non compact holonomy group. Take $V \in \mathfrak{X}(S)$ the left invariant vector field defined by $(1, 0, 0, 0, 0) \in \mathfrak{s}$. It is invariant by Λ , so it define a vector field $V \in \mathfrak{X}(\Lambda \backslash S)$. It is clear that both vector field are parallel.

Using the flip metric in (1) we get a Lorentzian metric on a compact manifold $\Lambda \setminus S$ with V a timelike parallel vector field and non compact holonomy.

As an application of Theorem 4, we see directly that some kind of manifolds do not admit Lorentzian metrics with relatively compact holonomy, for example odd spheres (which *do* admit Lorentzian metrics because of vanishing Euler number but which are not direct products).

To see another application, we need the following lemma that compares the holonomy groups in a covering space. Let $\pi : M \rightarrow B$ be a semi-Riemannian covering, so both M and B have the same restricted holonomy group.

Lemma 11 *Let $\pi : M \rightarrow B$ be a semi-Riemannian covering map.*

1. *The map $\pi^\# : Hol^M \rightarrow Hol^B$ given by $\pi^\#(P_\gamma) = P_{\pi \circ \gamma}$, is a Lie group monomorphism.*
2. *If Hol^B is compact, then Hol^M is also compact.*
3. *If Hol^M is compact and $\pi_1(B, p)$ is finite, then Hol^B is also compact.*

Proof: Observe that $P_\gamma P_\beta = P_{\beta\gamma}$ and $P_{\pi(\beta\gamma)} = P_{\pi(\gamma)} P_{\pi(\beta)}$ for any couple of lassos γ, β at p . On the other hand, π is a local isometry so $\pi_{*p} P_\gamma = P_{\pi(\gamma)} \pi_{*p}$. Thus if $P_\gamma = P_\beta \in Hol^M$, we have $e = P_{\beta^{-1}\gamma}$ being e the identity element, and applying π_{*p} implies that $P_{\pi(\gamma)} = P_{\pi(\beta)}$. This shows that $\pi^\#$ is well defined.

1. We see that it is a morphism using $P_\gamma P_\beta = P_{\beta\gamma}$ and $P_{\pi(\beta\gamma)} = P_{\pi(\gamma)} P_{\pi(\beta)}$. To see that it is injective use $\pi_{*p} P_\gamma = P_{\pi(\gamma)} \pi_{*p}$.
2. Observe that $Hol_0^B = Hol_0^M$ is compact, so the connected components of Hol^M are diffeomorphic to Hol_0^B and Hol^M itself can be identified to its image by $\pi^\#$ in Hol^B . Finally, Hol^B has a finite number of connected components because it is compact.
3. Note that the hypothesis implies $\#Hol^B / Hol_0^B < \infty$ by Lemma 3, and $Hol_0^B = Hol_0^M$ is compact, thus Hol^B is also compact.

■

Given $u, v \in T_p M$ where u is a null vector, it is defined the null sectional curvature of the degenerate plane $\pi = span\{u, v\}$ as

$$\mathcal{K}_u(\pi) = \frac{g(R_{uv}v, u)}{g(v, v)}.$$

It depends on the null vector $u \in T_p M$, but once it is fixed, it is a map on degenerate planes in $T_p M$. If we fix a null vector field U , we can see \mathcal{K}_U as a map on the subset of degenerate planes in the Grassmannian of planes in TM . There are examples where \mathcal{K}_U is in fact a map from M , that is, it

does not depend on the choice of degenerate plane $\pi \subset T_p M$ but just on the point p itself. In this case we say that it is a pointwise function. It is a strong condition, in some sense similar to the same condition (without degeneracy) in the Riemannian case. The sign of \mathcal{K}_u does not depend on the chosen null vector, so it is reasonable to speak of positive null sectional curvature for all degenerate planes, [11], [12].

The following result shows that null curvature can determine a Lorentzian manifold via its holonomy.

Proposition 12 *Let (M, g) be a complete and non-compact Lorentzian manifold with $m = \dim M \geq 4$ such that the null sectional curvature is a positive pointwise function. If the holomy group is contained in a compact group, then $Hol^M = SO(m-1)$ or $O(m-1)$.*

Proof: A suitable finite covering \widetilde{M} of M is orientable and time orientable, so Lemma 11 and Theorem 4 tell us that \widetilde{M} admits a timelike parallel vector field. Then we can deduce that \widetilde{M} is a direct product $\mathbb{R} \times L$ where the second factor is a quotient of the usual sphere \mathbb{S}^{m-1} of constant positive curvature, see [8, Proposition 5.4]. The fact that L is a quotient of \mathbb{S}^{m-1} and Lemma 11 again implies both $Hol^{\widetilde{M}}$ contains a copy of $SO(m-1)$, and $Hol^{\widetilde{M}} \subset Hol^M$. By hypothesis, Hol^M is contained in a compact group, in particular in a maximal compact one, that is, in a copy of $O(m-1)$. ■

Let us consider another consequence of Theorem 4. It is a well-known result by Marsden [18] that a compact homogeneous semi-Riemannian manifold is geodesically complete (whereas the same is not true omitting the condition of homogeneity). Here we can conclude

Corollary 13 *Let M be a compact manifold and let g be a Lorentzian metric on M with compact holonomy. Then (M, g) is geodesically complete.*

Proof: Take the oriented and time-oriented finite Lorentzian cover $(\widetilde{M}, \widetilde{g})$ of (M, g) . As the cover is finite, $(\widetilde{M}, \widetilde{g})$ is compact as well. By Theorem 4 there is a parallel timelike vector field X on $(\widetilde{M}, \widetilde{g})$. Then a theorem by Romero and Sánchez [19] states that $(\widetilde{M}, \widetilde{g})$ is geodesically complete. Thus (M, g) is geodesically complete as well. ■

3 Uniqueness of direct product decompositions

Inspired by [9], we use the associated fiber bundle argument to direct products to show that Euclidean and Minkowski plane are the only examples with the property that they admit more than one direct product decomposition with non degenerate properties.

A semi-Riemannian manifold is called **weakly irreducible** if it does not admit non-trivial and nondegenerate invariant subspaces by the holonomy group

in any tangent space. In the Riemannian case this notion coincides with the usual notion of irreducibility. Given a manifold $M = M_1 \times M_2$, we call $M_i(p)$ the tangent space at p of the leaf of the i -th canonical foliation through $p \in M$.

Theorem 14 *Let $M = M_1 \times M_2$ be a complete semi-Riemannian direct product with M_i weakly irreducible. Suppose that M admits another decomposition as a direct product $M = L_1 \times L_2$ (with $L_1 \neq M_i$), and $M_i(p) \cap L_j(p)$ zero or non degenerate. Then $M = \mathbb{R}^2$ with the Euclidean or Minkowski metric.*

Proof: Suppose that $\dim M_1 = k$, and the signature of M_i is ν_i , such that the signature of M is $\nu = \nu_1 + \nu_2$. Let $i : O_{\nu_1}(k) \rightarrow O_\nu(m)$ and $j : O_{\nu_2}(m-k) \rightarrow O_\nu(m)$ be the natural immersions $i(c) = \begin{pmatrix} c & 0 \\ 0 & I_{m-k} \end{pmatrix}$, $j(d) = \begin{pmatrix} I_k & 0 \\ 0 & d \end{pmatrix}$. We call $G = i(O_{\nu_1}(k))j(O_{\nu_2}(m-k))$. It is clear that if M is a direct product $M_1 \times M_2$, its holonomy group H is reducible to a subgroup of G , that is, $H = H_1 H_2$ with $H_1 \subset i(O_{\nu_1}(k))$ and $H_2 \subset j(O_{\nu_2}(m-k))$. Let $\pi : OM \rightarrow M$ be the orthonormal frame bundle. Call

$$E = \{r \in OM / r : \mathbb{R}^m \rightarrow T_{\pi r} M \text{ carries an adapted basis of } \mathbb{R}^k \times \mathbb{R}^{m-k} \text{ to an adapted basis of } M_1 \times M_2\}.$$

With respect to the decomposition $M = L_1 \times L_2$, fixed an element $r \in E$, there exists another decomposition of \mathbb{R}^m as a direct product $S_1 \times S_2$ such that r carries an adapted basis of $S_1 \times S_2$ to an adapted basis of $L_1 \times L_2$.

Both tuples of foliations in M are invariant by parallel transport, that is, the subspaces \mathbb{R}^k , \mathbb{R}^{m-k} , S_1 and S_2 of \mathbb{R}^m are invariant by the holonomy group H .

Given $h \in H_1$, we can write $h = \begin{pmatrix} c & 0 \\ 0 & I \end{pmatrix}$ with $c \in O_{\nu_1}(k)$, and if we call (x_1, x_2) the components of $x = \pi(r) \in M$ in $M_1 \times M_2$ and (x'_1, x'_2) its components in $L_1 \times L_2$, we have the following two ways in which we can write the composition $r \circ h$

$$\begin{array}{ccc} \mathbb{R}^k \times \mathbb{R}^{m-k} & \xrightarrow{h} & \mathbb{R}^k \times \mathbb{R}^{m-k} & \xrightarrow{r} & T_{x_1} M_1 \times T_{x_2} M_2 \\ S_1 \times S_2 & \xrightarrow{h} & S_1 \times S_2 & \xrightarrow{r} & T_{x'_1} L_1 \times T_{x'_2} L_2. \end{array}$$

Given $(u, 0) \in S_1 \times \{0\}$, we have $h(u, 0) \in S_1 \times \{0\}$ because S_1 is invariant by H . On the other hand if we write u with its components in the other decomposition, $u = (u_1, u_2) \in \mathbb{R}^k \times \mathbb{R}^{m-k}$, we have $h(u) = (cu_1, u_2)$ and

$$u - h(u) = (u_1 - cu_1, 0) \in (S_1 \cap \mathbb{R}^k) \times \mathbb{R}^{m-k}.$$

By hypothesis $S_1 \cap \mathbb{R}^k$ is zero or a non degenerated subspace of \mathbb{R}^k invariant by H , but M_1 is weakly irreducible, so it must be zero, thus $H_1 = \{1\}$. A similar argument for H_2 implies that $H = \{1\}$. The associated fibre bundle $OM \times_{O_\nu(m)} (O_\nu(m)/H)$ to OM , which is diffeomorphic to OM itself, has a parallel cross section, this means that M admits a global orthonormal basis formed by parallel

vector fields E_1, \dots, E_m . By completeness, the universal covering \widetilde{M} splits as \mathbb{R}^m with a flat metric. The group of deck transformation preserves the parallel basis, otherwise H would not be trivial, thus M_i is a product of $m_i = \dim M_i$ factors each one being \mathbb{R} or \mathbb{S}^1 , but being M_i weakly irreducible we have $m_i = 1$, therefore $m = \dim M = 2$. The only complete flat surfaces that admits two different structures as a direct product are the Euclidean and the Minkowski plane. ■

Note that in this proof, we do not suppose a priori that L_i must also be weakly irreducible. If we suppose it, then the uniqueness of the decomposition in the Theorem of the de Rham-Wu can be used to give a direct proof, [24, Appendix I].

4 Topologies on the space of all metrics with precompact holonomy

Having made several statements about single metrics with precompact holonomy, let us try to explore the topology of the space of *all* time-oriented metrics on an orientable manifold M that have precompact holonomy, in analogy to the situation in positive curvature. However, it will turn out in the following that much of its topology is hidden behind the not quite accessible topology of the space of Lorentzian metrics. In the light of applications like the Einstein equation considered as variational problem, it is outermost desirable to construct an appropriate topology on the space of Lorentzian metrics. Several topologies on that space and on related spaces have been considered. In a row of articles, Bombelli, Meyer, Noldus and Sorkin, e.g., introduced topologies on the quotient $\text{Lor}(M)/\text{Diff}(M)$ based on a splitting between the conformal and the volume part (for an overview, see [14]), but unfortunately, this topology is not a manifold topology in general. As we are ultimately interested in variational problems, and thus look for a manifold topology on the space $\text{Lor}_+(M)$ of time-oriented Lorentzian metrics, the simplest choice is the subspace topology with respect to a topological vector space topology on the space $\text{Bil}(M)$ of bilinear forms on M . Let

$$K(M) := \{g \in \text{Lor}_+(M) / \overline{\text{Hol}_g} \text{ compact}\}$$

thus, following Theorem 4, $K(M)$ is the set of time-oriented Lorentzian metrics with a parallel timelike vector field.

We define $G(M)$ to be the space of all globally hyperbolic metrics and $C(M)$ to be the set of all causally complete metrics on M .

First of all we want to compare the different possible topologies on $K(M)$ (understood as a subset of $\text{Lor}(M)$). On one hand, if M is noncompact, only a topology at least as fine as the C^0 -fine (Whitney) topology on $\text{Bil}(M)$ ensures that $\text{Lor}(M)$ is an open subset of $\text{Bil}(M)$. On the other hand, as we want to be able to define parallel vector fields, all metrics should at least be C^1 ,

which recommends us (together with the desired completeness of the vector space topology) to choose a topology at least as strong as the C^1 -compact-open topology. First of all, for $L(M)$ being the set of time-oriented Lorentzian metrics with a parallel causal vector field, we observe that we can control the closure of $K(M)$ in terms of $L(M)$:

Theorem 15 (Closure of $K(M)$) *Let M be a manifold.*

1. *In any topology finer or equal to the C^1 -compact-open topology, the closure of the set $K(M)$ is contained in $L(M)$.*
2. *If M is diffeomorphic to $\mathbb{R} \times S$ for some manifold S (to ensure $G(M) \neq \emptyset$), then for $E := L(M) \cap G(M) \cap C(M)$ and $F := K(M) \cap G(M) \cap C(M)$, there is some $e \in L(M) \cap C(M) \cap G(M) \setminus K(M)$ and a curve $g : [0, 1] \rightarrow E$ that is smooth w.r.t. every C^k -compact open topology and with $g([0, 1]) \subset F$.*

Proof: Obviously $K(M) \subset L(M)$, so for the first assertion it is enough to see that $L(M)$ is closed. Take any $g \in \text{Lor}(M) \setminus L(M)$. For every causal vector $v \in T_p M$ there exists a closed loop c_v at p such that the g -parallel transport along c_v does not fix v , that is $v \neq P_{c_v}^g(v)$. It is easy to see that still $v \neq P_{c_v}^h(v)$ for h in an open neighborhood of g , where now v may or may not be an h -causal vector. Associated to v we can take a tuple (W_v, V_v) consisting of open neighborhoods of g and v respectively, small enough such that $u \neq P_{c_v}^h(u)$ for every $h \in W_v$ and $u \in V_v$. The set L_g of g -causal vectors in $T_p M$ itself is not compact, however, for every auxiliary scalar product in $T_p M$ and associated norm $|\cdot|$, we can consider its unit sphere $S_p^k M := \{v \in T_p M \mid |v| = 1\}$, so $L_g \cap S_p^k M$ is compact, and therefore covered by a finite number of open sets V_{v_1}, \dots, V_{v_k} . Take an open set $W \subset \cap_{i=1}^k W_{v_i}$ such that $g \in W$ and V_{v_1}, \dots, V_{v_k} still cover $L_h \cap S_p^k M$ for every $h \in W$. If $h \in W$ and $v \in L_h$, there exists i such that $\frac{v}{|v|} \in V_{v_i}$ so $P_{c_{v_i}}^h(v) \neq v$ because $h \in W \subset W_{v_i}$. This shows that $W \subset \text{Lor}(M) \setminus L(M)$.

Now for the second part, assume $(M, g) := (\mathbb{R}_t \times S, \alpha \otimes dt + dt \otimes \alpha + \bar{g})$ for a complete metric \bar{g} on S and a \bar{g} -bounded one-form α on S . Furthermore assume that there is a point $x \in S$ with sectional curvature $k_x^S > 0$. This can be done with an arbitrarily small perturbation of a given metric in the C^k -compact open topology.

Define

$$g_r = -(1-t)dt^2 + r(dt \otimes \alpha + \alpha \otimes dt) + \bar{g}$$

for $r \in [0, 1]$ which is a continuous curve in $\text{Lor}(M)$. One finds that t is a Cauchy time function for all r . In fact, it is easy to see that any future vector v has positive scalar product with $\text{grad}_{g_r}(t)$.

Let $c : \mathbb{R} \rightarrow M$ be a causal curve. Now, if $t \circ c$ is bounded, it has a limit t_0 due to its monotonicity.

Now we parametrize c according to t , that is, $c(t) = (t, \bar{c})$, on a bounded interval $[0, b)$. The Cauchy-Schwarz inequality implies that $|\alpha(\bar{c})'| \leq |\alpha| |\bar{c}'|$, the norm always being the one defined by \bar{g} .

Then, for $r = 1$, using the causal character of c , we get $|\bar{c}'| \leq 2|\alpha|$.

In case of $r < 1$ we can solve the corresponding quadratic inequality for $|\bar{c}'|$ and get as a condition necessary for c causal

$$|\bar{c}'| \leq r|\alpha| + \sqrt{r^2|\alpha|^2 + (1-r)}.$$

Thus, by completeness of \bar{g} , also the S -coordinate along c has a limit at b , thus t is Cauchy, so $g_r \in G(M)$. Moreover, as $\text{grad}_{g_r}(t)$ is g_r -parallel, in particular it is g_r -Killing, we have $g_r(\text{grad}_{g_r}(t), c')$ is constant along any geodesic. Thus g_r is causally geodesically complete. This (and the fact that $\text{grad}_{g_r}(t)$ is timelike for every $r \in [0, 1)$ and lightlike for $r = 1$), shows that $g([0, 1)) \subset F$ and $g_1 \in E$.

Suppose now that $g_1 \in K(M)$, that is, there exists a timelike g_1 -parallel vector field $Z \in \mathfrak{X}(M)$, in particular it is linearly independent to $\frac{\partial}{\partial t}$ at any point. So there are g_1 -degenerate planes π in $T_q M$ for any point $q \in M$ such that its null sectional curvature is zero, but this is not possible at points $p = (t, x) \in M$ for any $t \in \mathbb{R}$ because by hypothesis $k_x^S > 0$, see [8, Theorem 6.3 and Lemma 5.2]. Contradiction. \blacksquare

Now let us consider more closely the fine topologies. We want to argue in the following that they are *not* appropriate to consider spaces of metrics of precompact holonomy. The C^0 -fine topology for continuous sections of a bundle $\pi : E \rightarrow M$ has as a neighborhood basis of a section f the family of sets $W_U := \{\gamma \in \Gamma^0(\pi) / g(M) \subset U\}$ where U is an open neighborhood of $f \subset E$. If π is a vector bundle such that the fibers are locally convex metric vector spaces with an arbitrary translational-invariant metric¹ then we can describe the topology in a different manner: Let P be the space of smooth positive functions on M , then, for $p \in P$, which could be called a *profile function*, we set

$$U_p := \{f \in \Gamma^0(\pi) / d(f(x), 0_x) < p(x)\}$$

where 0_x is the zero in $\pi^{-1}(x)$. Then $\{f + U_p\}$ is a neighborhood basis for f as well. The equivalence of these two descriptions is easy to see, the arbitrariness of the auxiliary metric is compensated by the flexible choice of the profile function. The C^k -fine topology is defined by applying the same to the map $d^k \gamma$ as a section of the bundle $S^k E \rightarrow S^k M$ where, for a manifold N , $S^k N$ is the bundle of unit vectors in $T^k N$ for an arbitrary auxiliary Riemannian metrics. For more details cf. [2] and the references therein.

The following theorem should be well-known to the experts, however we could not find any reference in the literature and thus include a proof here:

Theorem 16 *Let $\pi : E \rightarrow M$ be a metric vector bundle with locally convex fibers over a finite-dimensional manifold. Let a, b be two k -times continuously differentiable sections of π . Then a and b are in the same path connected component of $\Gamma^k(\pi)$ if and only if $\text{supp}(a - b)$ is compact.*

¹Keep in mind that here we use the word 'metric' not in the sense of bilinear form but in the sense of distance on a metric space.

Proof: As everything is translationally invariant, w.l.o.g. we can assume $b = 0$, the zero section. Assume the opposite of the statement of the theorem, that is, there is a noncompactly supported section a in the same path connected component as 0. By assumption, there is a C^0 curve $c : [0, 1] \rightarrow \Gamma^k(\pi)$ from 0 to a . Choose $p_n \in \text{supp}(a)$, $p_n \rightarrow \infty$ (a sequence leaving every compact set) and define $d_n := d(a(p_n), 0) > 0$. Let $(C_n)_{n \in \mathbb{N}}$ be a compact exhaustion with $p_n \in C_{n+1} \setminus C_n$. And consider an open neighborhood W_U of 0 as above with $U \cap \pi^{-1}(M \setminus C_n) \subset B_{d_n/n}$, for all n . As $c([0, 1])$ is compact, it has a finite covering by sets of the form $U_i := c(t_i) + U$, say U_1, \dots, U_m . Then iterative application of the triangle inequality implies that $d(a(p_i), 0) \leq m \cdot d_i/i < d_i$ for $i > m$, contradiction. ■

Now, the first corollary of the previous theorem is that $G(M)$ alone has uncountably many path connected components each of which is intersected nontrivially by $K(M)$. This holds even if we mod out the action of the diffeomorphism group on the space of metrics as it leaves the topology of the Cauchy hypersurfaces unchanged.

Corollary 17 *Within each path connected component of $G(M)$ in $\text{Lor}(M)$ equipped with the C^0 -fine topology, the topology of the Cauchy surface does not vary. Consequently, for M diffeomorphic to \mathbb{R}^n with $n \geq 4$, the set $G(M)/\text{Diff}(M)$ has uncountably many path connected components. Each of the components contains elements of $K(M)$.*

Proof: For the first assertion, single out two metrics $g_1, g_2 \in G(M)$ in the same path connected component, then apply the previous theorem to $\text{Lor}(M)$ (equipped with any auxiliary Riemannian metric on the fibers) to obtain that $g_1 = g_2$ outside of a compact set C . Now, any Cauchy surface in $M \setminus C$ is a Cauchy surface for either metric. A recent result of Chernov-Nemirovski ([6], Remark 2.3) states that for an open contractible manifold C of dimension $n - 1$, the product $\mathbb{R} \times C$ is diffeomorphic to \mathbb{R}^n . Now equip C with a complete metric g and consider the standard static manifold over (C, g) . It is obviously diffeomorphic to \mathbb{R}^n . The Cauchy surfaces, however, are diffeomorphic to C . As we know (see [22] and the references therein) that for $n - 1 \geq 3$, there are uncountably many pairwise non-diffeomorphic contractible open manifolds (the Whitehead manifold being an example for $n - 1 = 3$), the statement follows. ■

At this point, the reader probably is tempted to allow for an additional compact factor N to \mathbb{R}^n and then to repeat the proof above. However, this is not possible as the proof would yield $C_1 \times N \cong C_2 \times N$ which could be true even for C_1 not homotopy equivalent to C_2 , for an example with $N = \mathbb{S}^1$ see [5]. However, we think that it should be possible to use the argument above for any noncompact Cauchy surface, by replacing Whitehead's manifold suitably.

It is well-known (see e.g. Corollary 7.32 and 7.37 in [2]) that causal completeness and causal incompleteness are C^1 -fine-stable properties, i.e., given a globally hyperbolic causally complete resp. causally incomplete metric g , there

is a C^1 -fine open neighborhood U of g such that all metrics $h \in U$ are causally complete resp. causally incomplete. Using connectedness arguments we get easily that each connected component of a globally hyperbolic metric g_0 either consists entirely of causally complete or consists entirely of causally incomplete metrics. Now it becomes clear that the C^0 -fine topology is already too fine for our purposes, as it isolates geometrically different metrics from each other. Namely, if we focus on one of the uncountably many path connected components, the result is only one $\text{Diff}(M)$ -orbit:

Corollary 18 1. *If $g_0 \in K(M) \cap G(M)$ is timelike complete, then any timelike complete metric in the path connected component of g_0 in the C^0 -fine topology is isometric to g_0 .*

2. *If $g_0 \in K(M) \cap G(M)$ is timelike complete, then any metric in the path connected component of g_0 in the C^1 -fine topology is isometric to g_0 .*

Proof: Let g_1 be another element of $K(M) \cap G(M)$ path connected to g_0 . Both g_0 and g_1 admit global decompositions $I_0 : (M, g_0) \rightarrow \mathbb{R} \times (S, h_0)$ and $I_1 : (M, g_1) \rightarrow \mathbb{R} \times (S, h_1)$ (taking into account that the Cauchy surfaces of g_0 are diffeomorphic to those of g_1 following Corollary 17) with corresponding parallel vector fields P^0 resp. P^1 and time functions t_0 resp. t_1 . As the metrics coincide outside of a compact set K following Theorem 16, the vector field P^0 is g_0 -parallel and g_1 -parallel on $(M \setminus K, g_1)$. We can see the above decomposition of (M, g_0) as $\mathbb{R} \times t_0^{-1}(a)$ for some $a \in \mathbb{R}$, such that $K \subset I^-(t_0^{-1}(a))$. We now construct a g_1 -parallel transport extension Q^0 of $P^0|_{I^+(t_0^{-1}(a))}$ on the whole M along the integral curves of P^1 . Observe that the integral curves of P^1 form a foliation in M , in particular every point in $I^-(t_0^{-1}(a))$ is on exactly one integral curve of P^1 . Take a Cauchy surface for g_1 in the complementary of K , and observe that there are local basis $\{P^1, E_2, \dots, E_n\}$ of vector fields, such that E_i are vector fields in the Cauchy surface such that $[P^1, E_i] = 0$. Using that the curvature of the mixed planes $\text{span}\{P^1, E_i\}$ is zero, it is easy to see that Q^0 is g_1 -parallel. This shows that (M, g_1) is also isometric to $\mathbb{R} \times t_0^{-1}(a)$.

The second assertion follows from the first part and from the observation above that the whole path connected component of g_0 consists of causally complete metrics. ■

It remains as an interesting question to examine the topology of subsets of globally hyperbolic metrics with holonomy of certain kinds for other manifold structures on $\text{Lor}(M)$, possibly not coming from a vector space topology on $\text{Bil}(M)$.

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