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Conjugate points along null geodesics on Lorentzian manifolds with symmetries

M. Gutiérrez[1], F. J. Palomo[1] and A. Romero[2]

[1]*Departamento de Álgebra, Geometría y Topología,*
Universidad de Málaga, Spain

[2]*Departamento de Geometría y Topología,*
Universidad de Granada, Spain

e-mails: mgl@agt.cie.uma.es, fjpalomo@eresmas.com, aromero@ugr.es

Abstract

Several integral inequalities, involving the Ricci tensor, for a compact Lorentzian manifold which admits a timelike conformal vector field are given. These inequalities relate conjugate points along null geodesics to global geometric properties. As a consequence, some classification results on certain compact Lorentzian manifolds without conjugate points along its null geodesics are shown.

Key words: *Lorentzian manifolds, timelike conformal vector fields, null geodesics, conjugate points.*
MSC 2000: *53C50, 53C22.*

1 Introduction

On a Lorentzian manifold (M, g), tangent vectors are classified into *timelike, null,* or *spacelike*, and so a (smooth) curve on M is said to be timelike, null, or spacelike if its tangent vectors are always timelike, null, or spacelike, respectively. Lorentzian Geometry is the mathematical theory used in General Relativity. In fact, a timelike curve corresponds to the path of an observer moving at less than the speed of light. Null curves correspond to moving at the speed of light, and spacelike curves to moving faster than light. Although Relativity predicts that physical particles cannot move faster than light, spacelike curves are of undeniable geometric interest. If a timelike or null curve is supposed to be a geodesic, then it represents the path of a "free falling" particle or the path of a lightlike particle, respectively.

It is commonly argued in General Relativity that the curvature tensor codifies the gravity, and the relative position of neighbours of a free falling particle γ, is given by the *Jacobi fields* on γ. In this way, if the gravity attracts then it will cause conjugate points while if gravity does not attract then it will prevent them. From a geometric point of view, a conjugate point $\gamma(a)$ of $p = \gamma(0)$ along a geodesic γ can be interpreted as an "almost-meeting point" of geodesics starting from p with initial velocities near $\gamma'(0)$. These neighbouring geodesics may, but need not, actually pass through the point $\gamma(a)$. The existence of conjugate points on timelike geodesics gives an effect rather like the "twin paradox" and for null geodesics, may be related to the phenomenon of gravitational lensing [16]. Moreover, the study of conjugate points along causal geodesics plays an important role to state Singularity theorems and to study Causality Theory, see [26] and [15], for instance.

We will be interested here in compact Lorentzian manifolds. All these manifolds are acausal, i.e. they possess closed timelike curves [26, Lemma 14.10], while a physically admissible spacetime is generally assumed to be free of closed causal curves. Yurtsever introduced [36] a new notion of causality for general acausal Lorentzian manifolds. Recall that remarkable examples of spacetimes M are often merely a connected open set in a larger spacetime $\widetilde{M}$, and M can be causal even though $\widetilde{M}$ is not [27]. Moreover, it has been pointed out [25] that general properties of compact spacetimes should be identified in order to show they are incompatible with astronomical observations, since the objections to chronology violations are based more on philosophy than physics. On the other hand, following [36], it is reasonable to expect that Lorentzian theory on a compact manifold will provide valuable information on the topology of the underlying manifold, complementary to the one obtained through the study of Riemannian theory.

It is well-known that not every compact manifold, M, can be endowed with a Lorentzian metric. This holds if and only if the Euler number of M vanishes [26, Prop. 5.57]. Moreover, for Lorentzian metrics, compactness does not imply geodesic completeness, in fact the *Clifton-Pohl torus* is an example of this situation [26, Ex. 7.16], [30]. Of course, this is not an isolated example. A general procedure to construct many incomplete Lorentzian metrics on the torus was explained in [29]. On the other hand, geodesic completeness in the Lorentzian setting can be separated into spacelike, null and timelike completeness. These conditions are shown to be independent [26, Ex. 5.43], [30] and references therein. See also the comment at the end of this paper.

A way to get completeness in the compact Lorentzian case is to impose some geometric extra condition on the manifold [30]. In fact, it is relevant for this work that the existence of a timelike conformal vector field on a compact Lorentzian manifold yields to its geodesic completeness [31].

A compact Riemannian manifold must be complete as an easy consequence of the classical Hopf-Rinow theorem. Another argument to obtain the same conclusion is [4, p. 32]: take the unit tangent bundle UB of an $n-$dimensional Riemannian manifold (B, g_B). It is a fiber bundle over B with fiber type the $(n-1)-$dimensional unit sphere $\mathbb{S}^{n-1}$. If B is assumed to be compact, then UB is also compact and therefore the geodesic flow is automatically complete as it is a flow of a vector field on a compact manifold. This argument fails in the Lorentzian setting because there is no reasonable fiber bundle over a Lorentzian manifold (M, g) with compact fiber and having a tangent vector for every direction in T_pM for all $p \in M$. In fact, if we put $U^+M = \{v \in TM : g(v, v) = +1\}$ and $U^-M = \{v \in TM : g(v, v) = -1\}$, then both fiber bundles have not compact fibers and neither of them has a vector for every direction of TM. In order to avoid this difficulty, we will restrict our attention to null tangent directions on a Lorentzian manifold which admits a timelike conformal vector field. Thus the key tool in our work is the *null congruence associated with a timelike conformal vector field* K which can be seen as the manifold of all null tangent directions of (M, g), and it is denoted by C_KM. This null congruence is a fiber bundle over M with compact fiber type $\mathbb{S}^{n-2}$. As we will point out (see section 4) the geodesic flow preserves C_KM. So, a reasoning as in the previous Riemannian case gives the null completeness of M, already known from [31].

The main aim of this note is to introduce, following [11] and [12], two *new integral inequalities* on compact Lorentzian manifolds which admit a timelike conformal vector field. These integral inequalities relate conjugate points along null geodesics with global properties of the Lorentzian manifold. In particular, we have found an "splitting" result (Corollary 5.4) and several classification results on certain compact Lorentzian manifolds without conjugate points on their null geodesics.

The study of conformal vector fields on pseudo-Riemannian geometry is a topic of special importance, see for instance [24]. In particular, the study of timelike conformal vector fields on Lorentzian Geometry has been developed mainly under assumptions of interest in physics and, from a mathematical viewpoint, the case of timelike Killing vector fields has appeared to be a useful tool to get classification theorems in some area of Lorentzian Geometry [34]. Moreover, standard Lorentzian space forms (those with a time-orientable double Lorentzian covering admitting a timelike Killing vector field) have been studied in [23]. Recall furthermore, that the existence of a timelike Killing vector field has been also used to classify compact Lorentzian space forms [18] and more generally to study and classify compact Einstein Lorentzian manifolds [32], [33]. Finally, note that the existence of a timelike conformal vector field has been also used to study compact spacelike hypersurfaces of constant mean curvature (see [1], [2] and references therein).

2 Motivations from the Riemannian case

In 1948, E. Hopf published a remarkable theorem where he showed that *the total scalar curvature of a closed surface without conjugate points is nonpositive and vanishes only if the surface is flat.* Therefore, thanks to the Gauss-Bonnet formula, a Riemannian torus without conjugate points is flat [17]. Hopf's result was, ten years after, extended by L.W. Green [8] to compact Riemannian manifolds as follows: *if* (B, g_B) *has no conjugate points then*

$$\int_B S d\mu_{g_B} \leq 0, \tag{1}$$

and it vanishes only if the metric is flat, (here S and $d\mu_{g_B}$ denote, respectively, the scalar curvature of (B, g_B) and the canonical measure associated to g_B). More recently, F. Guimaraes, generalized in [10] the Green theorem to complete Riemannian manifolds under the assumption that the Ricci curvature on the unit tangent bundle has an integrable positive or negative part.

On the other hand, in [9] L. W. Green solved the famous Blaschke conjecture in dimension two. A useful tool to prove it was the so-called Berger inequality which asserts that $\mathrm{area}(B, g_B) \geq \frac{2a^2}{\pi}\chi(B)$, being $\chi(B)$ the Euler number of B, for a $2-$dimensional compact Riemannian manifold without conjugate points before a fixed distance a in the parameter of any (unit) geodesic. Moreover, the equality holds only if (B, g_B) has constant sectional curvature $\frac{\pi^2}{a^2}$. This inequality was later generalized to higher dimensions by Berger and independently by Green [4, Prop. 5.64] as follows:

$$\mathrm{Vol}(B, g_B) \geq \frac{a^2}{\pi^2 n(n-1)} \int_B S d\mu_{g_B}, \tag{2}$$

where (B, g_B) is a compact Riemannian manifold of dimension n without conjugate points before a fixed distance a in the parameter of any geodesic. Moreover, the equality holds if and only if (B, g_B) has constant sectional curvature $\frac{\pi^2}{a^2}$. Note that Berger's inequality is a direct consequence of (2), via the Gauss-Bonnet formula.

Inspired from these results, we want to study Lorentzian manifolds in which no null geodesic contains a pair of mutually conjugate points before a fixed value of a special affine parameter. In fact, assuming the existence of a timelike conformal vector field, we will distinguish an affine parameter for each null geodesic, and we deduce an integral inequality in the same philosophy to (2). We will explote several geometric consequences and show that the Hopf-Green and the Berger-Green inequalities above can be deduced as a particular case of our approach.

3 Setup

Let (M, g) be an $n(\geq 2)-$dimensional Lorentzian manifold; that is a (connected) smooth manifold M endowed with a non-degenerate metric g with signature $(-, +, ..., +)$. We denote TM for the tangent fibre bundle of M, $\pi : TM \longrightarrow M$ for the natural projection and for every $v \in TM$, we write $(\ )_v : T_{\pi(v)}M \longrightarrow T_vT_{\pi(v)}M$ for the natural identification. We shall write ∇ for its Levi-Civita connection, R for its Riemannian curvature tensor (our convention on the curvature tensor is $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$), Ric for its Ricci tensor, S for its scalar curvature and $d\mu_g$ for the canonical measure associated with g.

As usual, the causal character of a tangent vector $v \in T_pM$ is timelike (resp. null, spacelike) if $g(v, v) < 0$ (resp. $g(v, v) = 0$ and $v \neq 0$, $g(v, v) > 0$ or $v = 0$). If $v \in T_pM$ then, γ_v will denote the unique geodesic such that $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. It is well-known that the causal character of $\gamma'(t)$, for any geodesic γ of (M, g), does not depend on the parameter t. In particular, a null geodesic γ of (M, g) is a geodesic such that $\gamma'(t)$ is a null vector. A vector field $K \in \mathfrak{X}(M)$ is said to be timelike if K_x is timelike for all $x \in M$.

Recall that the cotangent bundle T^*M, of an arbitrary manifold M, carries a natural symplectic structure given by $d\alpha$ where α is the 1-form defined by $\alpha(\xi) = -q(\xi)\,[p_*(\xi)]$ for all $\xi \in TT^*M$, being $q : TT^*M \longrightarrow T^*M$, and $p : T^*M \longrightarrow M$ the natural projections. If (M, g) is a Lorentzian manifold, then g determines a vector bundle isomorphism from TM to T^*M, $\flat : TM \longrightarrow T^*M$, by putting $v \mapsto g(v,\ )$. If we call α_g the pull-back of α by $\flat$, then $\alpha_g(X) = -g(v, \pi_*(X))$, where $X \in T_v(TM)$. The geodesic vector field, Z_g, is the vector field on TM defined by $i_{Z_g} d\alpha_g = dE$, where E is given by $E(v) = \frac{1}{2}g(v, v)$. The flow $\{\Phi_t\}$ of Z_g, $\Phi_t(v) = \gamma_v'(t)$, is called the *geodesic flow* of (M, g).

We point out that the *Sasaki metric* $\widehat{g}$ induced from the Lorentzian metric g may be introduced on TM in a similar way to the Riemannian case. Note that now $(TM, \widehat{g})$ is a semi-Riemannian manifold with index two, and

$\pi : (TM, \widehat{g}) \longrightarrow (M, g)$ is a semi-Riemannian submersion.

As it has been said above, we are here interested in the study of conjugate points along null geodesics (or null conjugate points). Many authors have thought about this topic, we will quote here several significant results:

1. A conjugate point along a null geodesic γ is an almost-meeting point of nearby null geodesics, i.e. the geodesic variation of γ could be chosen such that all its longitudinal geodesics are null geodesics [26, p. 291]. This fact is not obvious from continuity arguments.

2. An essential problem in Lorentzian Geometry is to determine if a given pair of points can be joined by a timelike curve (Causality Theory). This is related to conjugate points along null geodesics as follows: if $\gamma : [0, a] \longrightarrow M$ is a null geodesic with $\gamma(0)$ and $\gamma(a)$ conjugate points along γ, then for every $\varepsilon > 0$ there is a timelike curve from $\gamma(0)$ to $\gamma(a + \varepsilon)$ "arbitrarily near" to γ [26, Prop. 10.48].

3. Another important question is to find incomplete geodesics (Singularity Theory). The *Hawking-Penrose conjugancy* theorem for null geodesics connects Singularity Theory to null conjugate points, and it can be stated as follows: if $\gamma_v : I \longrightarrow M$ is a null geodesic, $\mathrm{Ric}(\gamma_v', \gamma_v') \geq 0$, and $R(\ ,\gamma_v'(t))\gamma_v'(t)$, contemplated as a linear operator of the quotient space $\mathrm{Span}\{\gamma_v'(t)\}^{\perp} / \mathrm{Span}\{\gamma_v'(t)\}$, is not the zero operator for some $t \in I$, then γ_v is incomplete or has a pair of conjugate points (see for instance [3, Prop. 12.17]).

4. No null geodesic γ in any two-dimensional Lorentzian manifold has conjugate points [3, Lemma 10.45].

5. There is no conjugate point along null geodesics in any Lorentzian manifold of constant sectional curvature [26, Ex. 10-11]. Nevertheless, it is easy to construct examples of Lorentzian manifolds without conjugate points along their null geodesics and which do not have constant sectional curvature.

Of course, sectional curvature can be defined on Lorentzian manifolds, but it plays a different role than in the Riemannian case. In fact, it cannot be stated for null planes (degenerate planes in the induced metric). Now, we close this section by pointing out two of these differences.

1. There is no compact Lorentzian manifold of constant sectional curvature $c > 0$. In fact, for $n = 2$, it can be directly deduced from the Gauss-Bonnet formula for Lorentzian metrics. For $n \geq 3$, this is a consequence

of a classical result by Calabi and Markus [5] which implies the inexistence of compact quotients of *de Sitter space,* and a more recent result of completeness by Klinger [20].

2. In 1979, Kulkarni showed that if a (connected) $n(\geq 3)-$dimensional Lorentzian manifold (M, g) has sectional curvature bounded from above or bounded from below, then (M, g) has constant sectional curvature [22] (see also [26, Prop. 8.28]).

We end this section by recalling the well-known notion of *null sectional curvature* [13], [14], [3, Definition A.6] to be used later. If v is a null vector and σ a null plane containing it, the null sectional curvature with respect to v of the plane σ is defined to be $\mathcal{K}_v(\sigma) = g(R(u,v)v,u)/g(u,u)$, where $u(\neq 0)$ is any vector in σ independent with v (and therefore spacelike). This curvature does not depend on the choice of the non-zero spacelike vector u, but it does quadratically on v. An $n(\geq 3)-$dimensional Lorentzian manifold has constant sectional curvature if and only if has null sectional curvature everywhere zero [26, Prop. 8.28].

4 Null Congruence

From now on (M, g) will be a (time orientable) Lorentzian manifold with dimension $n \geq 3$, and K a timelike vector field on M. Recall [14], [21] that the *null congruence associated to* K is defined by

$$C_K M = \left\{ v \in TM : g(v,v) = 0,\ g(v, K_{\pi(v)}) = 1 \right\}. \qquad (3)$$

The null congruence has the following nice properties: for each null tangent vector v, there exists a unique $t \in \mathbb{R}$ such that $tv \in C_K M$, and the map $v \longmapsto [v]$ is a diffeomorphism from $C_K M$ to the manifold of the null directions of M, $\mathcal{N} = \{[v] \in PM : g(v,v) = 0\}$ (here we denote by PM the projective fiber bundle associated to TM).

The null congruence $C_K M$ is an orientable embedded submanifold of TM with dimension $2(n-1)$. Moreover $(C_K M, \pi, M)$ is a fiber bundle with fibre type $\mathbb{S}^{n-2}$, and so $C_K M$ will be compact if M is compact. If $\widehat{g}$ also represents the induced metric on $C_K M$ from the Sasaki one of TM, then $(C_K M, \widehat{g})$ is a Lorentzian manifold and the restriction of π to $C_K M$ is a semi-Riemannian submersion with spacelike fibers (see [11] for details). We shall write $d\mu_{\widehat{g}}$ for the canonical measure on $C_K M$ induced from $\widehat{g}$.

This manifold is our main tool in the study of null conjugate points. From a geometric point of view, the null congruence has been systematically studied in [11], where the reader can find the following results:

1. The volume element of $(C_KM, \widehat{g})$ is given, up to a constant multiple, by the $2(n-1)-$form $\beta_g \wedge \alpha_g \wedge (d\alpha_g)^{n-2}$, where $\beta_g(X) = -g(K_{\pi v}, \pi_*(X))$ for every $X \in T_v C_K M$.

2. The fiber $\pi^{-1}(x) = (C_KM)_x$ is isometric to $\big(\mathbb{S}^{n-2}, -g(K_x, K_x)^{-1} g_{can}\big)$, where $\big(\mathbb{S}^{n-2}, g_{can}\big)$ is the canonical unit Riemannian sphere. Therefore $(C_KM)_x$ is isometric to a Riemannian sphere of radius $[-g(K_x, K_x)]^{-1/2}$. Note that there is a remarkable difference with the unit tangent bundle of a Riemannian manifold, where all the fibers are isometric. We will write $h = [-g(K,K)]^{-1/2}$ and so $g(U,U) = -1$ with $U = hK$.

3. C_KM is invariant by the geodesic flow if and only if the vector field K is assumed to be conformal, i.e. $\mathfrak{L}_K g = \rho g$ for some $\rho \in C^\infty(M)$. Moreover, in such a case div $(Z_g \mid_{C_KM}) = 0$, where div denotes the divergence operator of $(C_KM, \widehat{g})$. So, if (M,g) is compact and K is conformal (therefore (M,g) is geodesically complete [31]) then, we have:

$$\int_{C_KM} (f \circ \Phi_t) d\mu_{\widehat{g}} = \int_{C_KM} f d\mu_{\widehat{g}} \tag{4}$$

   for every $f \in C^0(C_KM)$ and $t \in \mathbb{R}$.

4. If K is conformal, then every null geodesic γ_v of (M,g) with $v \in C_KM$, gives rise to the null geodesic γ'_v of $(C_KM, \widehat{g})$. Furthermore, each null geodesic β of (M,g) may be reparametrized to obtain a null geodesic α which satisfies $\alpha'(t) \in C_KM$. In fact, $g(\beta', K_\beta) = a \in \mathbb{R}$, $a \neq 0$. Thus, if we put $\alpha(t) = \beta(\frac{t}{a})$ we achieve $g(\alpha', K_\alpha) = 1$.

If a null congruence associated with a timelike vector field K (conformal or not) has been fixed, then we may choose, for every null plane σ, the unique null vector $v \in C_KM \cap \sigma$. Therefore, the null sectional curvature can be thought as a function on null planes. Along this note we always use such convention, and we will call it the $K-$*normalized null sectional curvature*.

5 The integral inequalities

We recall that if $\gamma_v : [0,a] \longrightarrow M$ is a null geodesic such that there are no conjugate points of $\gamma_v(0)$ in $[0,a)$, then the *Hessian form* $H^{\perp}_{\gamma_v}$ is positive semidefinite, i.e.

$$H^{\perp}_{\gamma_v}(V,V) := \int_0^a \left[g\Big(\frac{\nabla V}{dt}, \frac{\nabla V}{dt}\Big) - g(R(V,\gamma'_v)\gamma'_v, V) \right] dt \geq 0, \tag{5}$$

for every vector field V along γ_v such that $V(0) = 0$, $V(a) = 0$ and $g(\gamma_v', V) = 0$, [26, pp. 290-1]. Moreover, a standard argument permits us to show, that if $H^{\perp}_{\gamma_v}(V,V) = 0$ then $\frac{\nabla^2 V}{dt^2} + R(V,\gamma_v')\gamma_v' = f\gamma_v'$ where f is a smooth function. Observe that, in contrast to the well-known Riemannian case, here the condition $H^{\perp}_{\gamma_v}(V,V) = 0$ does not imply, in general, that V is a Jacobi vector field.

From now on, we will assume that M is (($n \geq 3$)-dimensional) compact and (M,g) admits a timelike conformal vector field K.

We are ready to introduce the integral inequalities that involve null conjugate points. The proof of the following result makes use of (4) and (5). It can be found in [11, Th. 3.2]. We put $\widehat{\mathrm{Ric}}(v) = \mathrm{Ric}(v,v)$ for all $v \in C_K(M)$.

Theorem 5.1 *If there exists $a \in (0,+\infty)$ such that every null geodesic $\gamma_v : [0,a] \longrightarrow M$, with $v \in C_K M$, has no conjugate point of $\gamma_v(0)$ in $[0,a)$, then*

$$\mathrm{Vol}(C_K M, \widehat{g}) \geq \frac{a^2}{\pi^2(n-2)} \int_{C_K M} \widehat{\mathrm{Ric}}\; d\mu_{\widehat{g}}. \tag{6}$$

Moreover, equality holds if and only if (M,g) has constant $K-$normalized null sectional curvature $\frac{\pi^2}{a^2}$.

Notice that the equality holds in (6) if and only if (M,g) has $U-$normalized null sectional curvature $-\frac{\pi^2}{a^2}g(K,K)$. That is, the $U-$normalized null sectional curvature of (M,g) is an everywhere non-zero point function. An equivalent condition to this [11, Remark 3.4] was studied by H. Karcher, [19]. He showed, when $n \geq 4$, the following theorem:

> *Let U be a unit timelike vector field on an $n(\geq 4)-$ dimensional Lorentzian manifold (M,g). The $U-$normalized null sectional curvature is a everywhere non-zero point function if and only if:*
> *1. The distribution $U^{\perp}$ is integrable.*
> *2. The integral manifolds of $U^{\perp}$ are totally umbilic and have constant sectional curvature.*
> *3. (M,g) is locally conformal to a flat Lorentzian space.*

We would like to point out that conclusion 1 above does not remain true if it is assumed $\dim M = 3$ [11, Remark 4.4].

On the other hand, as it has been proved in [11]:

$$\mathrm{Vol}(C_K M, \widehat{g}) = \mathrm{Vol}(\mathbb{S}^{n-2}, g_{can}) \int_M h^{n-2} d\mu_g \tag{7}$$

and

$$\int_{C_K M} \widehat{\mathrm{Ric}}\, d\mu_{\widehat{g}} = \frac{\mathrm{Vol}(\mathbb{S}^{n-2}, g_{can})}{n-1} \int_M \left[n\mathrm{Ric}(U,U) + S \right] h^n d\mu_g. \qquad (8)$$

Therefore, we can rewrite Theorem 5.1 as follows:

Theorem 5.2 *If there exists $a \in (0, +\infty)$ such that every null geodesic $\gamma_v : [0,a] \longrightarrow M$, with $v \in C_K M$, has no conjugate point of $\gamma_v(0)$ in $[0,a)$, then*

$$\int_M h^{n-2} d\mu_g \geq \frac{a^2}{\pi^2 (n-1)(n-2)} \int_M \left[n\mathrm{Ric}(U,U) + S \right] h^n d\mu_g. \qquad (9)$$

Moreover, equality holds if and only if (M,g) has $U-$normalized null sectional curvature $-\frac{\pi^2}{a^2} g(K,K)$.

If any null geodesic does not contain a pair of mutually conjugate points, then (9) is valid for all positive a. Since $\int_M h^{n-2} d\mu_g < +\infty$, we achieve (10). The equality case in (10) requires a more intricate argument [12].

Theorem 5.3 *If there are no conjugate points along the null geodesics, then*

$$\int_M \left[n\mathrm{Ric}(U,U) + S \right] h^n d\mu_g \leq 0. \qquad (10)$$

Moreover, equality holds if and only if (M,g) has constant sectional curvature $c \leq 0$.

This result allows us to show the existence of null conjugate points without using the Jacobi equation [11, Cor. 4.3]. Moreover, it can be thought as an integral condition to obtain constant sectional curvature from the absence of null conjugate points (compare with [7]).

If K is assumed, more restrictibly, to be *Killing*, then we have proved in [11, Lemma 3.12] that $\int_M \mathrm{Ric}(U,U) h^n d\mu_g \geq 0$, and equality holds if and only if U is parallel (and hence, h is a constant function). This fact permits us to improve, in that case, Theorems 5.2 and 5.3 to get:

Corollary 5.4 *If K is Killing and there is $a \in (0, +\infty)$ such that every null geodesic $\gamma_v : [0,a] \longrightarrow M$, $v \in C_K M$, has no conjugate point of $\gamma_v(0)$ in $[0,a)$, then*

$$\int_M h^{n-2} d\mu_g \geq \frac{a^2}{\pi^2 (n-1)(n-2)} \int_M S h^n d\mu_g. \qquad (11)$$

Moreover equality holds if and only if $g(K,K)$ is constant and the universal covering of (M,g) is globally isometric to the semi-Riemannian product $\left(\mathbb{R} \times \mathbb{S}^{n-1}(\frac{ah}{\pi}), -dt^2 + g_{can} \right)$.

Corollary 5.5 *If K is Killing and there are no conjugate points along the null geodesics, then*

$$\int_M S h^n d\mu_g \leq 0. \tag{12}$$

The equality holds if and only if (M, g) is flat. Moreover, in this case, U is parallel, the first Betti number of M is not zero and the Levi-Civita connection of g is Riemannian.

Remark 5.6 Kamishima proved that if a compact Lorentzian manifold admits a timelike Killing vector field and has constant sectional curvature c, then it is complete and $c \leq 0$. Moreover, it is affinely diffeomorphic to a Riemannian manifold with non-zero first Betti number if $c = 0$ [18, Th. A (1)]. Last result was extended in [32, Th. 3.2] to a compact Ricci-flat Lorentzian manifold which admits a timelike Killing vector field, and later, in [33, Cor. 3.9], to the conformal case. Since a Lorentzian manifold of constant sectional curvature has no conjugate point along its null geodesics, Corollary 5.5 is a proper extension (in other direction to [33, Cor. 3.9]) of Kamishima's theorem.

Note that in case that the equality holds in Corollary 5.5, from [35, Cor. 3.4.6], there is a (finite) Lorentzian covering of (M, g) by a flat Lorentzian torus [12].

Corollaries 5.4 and 5.5 contain, as a very particular case, the classical Berger-Green [9] and Hopf-Green [8] inequalities in Riemannian geometry, respectively.

Corollary 5.7 *Let (B, g_B) be a compact Riemannian manifold of dimension $n \geq 2$ and scalar curvature S. Suppose that no unit geodesic $\gamma : [0, a] \longrightarrow B$ has a conjugate point of $\gamma(0)$ in $[0, a)$, then*

$$\mathrm{Vol}(B, g_B) \geq \frac{a^2}{\pi^2 n(n-1)} \int_B S d\mu_{g_B}. \tag{13}$$

Moreover, equality holds if and only if (B, g_B) has constant sectional curvature $\frac{\pi^2}{a^2}$.

Proof. If we take $(M, g) = (\mathbb{S}^1 \times B, -g_{can} + g_B)$ and the unit timelike Killing vector field K as the lift to $\mathbb{S}^1 \times B$ of the vector field $z \longmapsto iz$ on $\mathbb{S}^1 \subset \mathbb{C}$, then it is easy to see that every null geodesic γ_v of (M, g), with $v \in C_K M$, has no conjugate point in $[0, a)$. Therefore Corollary 5.4 gives

$$\mathrm{Vol}(M, g) \geq \frac{a^2}{\pi^2 n(n-1)} \int_M S_g d\mu_g, \tag{14}$$

where S_g denotes the scalar curvature of (M, g), and equality holds if and only if (M, g) has $K-$normalized null sectional curvature π^2/a^2. Taking into account that $\mathrm{Vol}(M, g) = 2\pi\mathrm{Vol}(B, g_B)$ and $S_g(z, x) = S(x)$ for $(z, x) \in M$, we obtain (13) from (14). Finally note that (M, g) has $K-$normalized null sectional curvature π^2/a^2 if and only if (B, g_B) has constant sectional curvature π^2/a^2. ■

Corollary 5.8 *Let (B, g_B) be a compact Riemannian manifold of dimension $n \geq 2$ and scalar curvature S, if (B, g_B) has no conjugate point then*

$$\int_B S d\mu_{g_B} \leq 0. \tag{15}$$

Moreover, equality holds if and only if (B, g_B) is flat.

Proof. It is easy to see that no null geodesic γ_v of (M, g) has conjugate points. Therefore Corollary 5.5 gives

$$\int_M S_g d\mu_g \leq 0, \tag{16}$$

and equality holds if and only if (M, g) is flat. Then we get (15) from (16). Finally (M, g) is flat if and only if (B, g_B) is flat. ■

Note that, in the Lorentzian case, only null geodesics are involved in the assumptions of the integral inequalities, whereas in the Riemannian setting, all geodesics are implicated. Nevertheless, Theorem 5.2 and Corollary 5.4 give us, in particular, information about the *volume* of the Lorentzian manifold (M, g) only from the study of its null conjugate points. In fact, compactness of M implies that there exists an upper bound $b > 0$ for the function h, and so we have $\int_M h^{n-2} d\mu_g \leq b^{n-2}\mathrm{Vol}(M, g)$.

Another relationships of null geodesics to global properties of Lorentzian manifolds have been pointed out in previous works. For instance, in [28] it was conjectured that a null complete compact Lorentzian manifold must be complete. Although this conjecture has deserved the attention of some geometers [6], it remains now as an open problem.

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