A New Approach to the Study of Conjugate Points along Null Geodesics on Certain Compact Lorentzian Manifolds

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Abstract

An integral inequality on a compact Lorentzian manifold admitting a timelike conformal vector field is shown under some assumption on its conjugate points along null geodesics. The inequality relates the behaviour of these conjugate points to global geometrical results. As an application, several properties of the null geodesics of a natural Lorentzian metric on each odd dimensional sphere are obtained.

1 Introduction

In [5], [6] the authors have introduced a new integral inequality on a remarkable family of compact Lorentzian manifolds. It reproves a classical result of M. Berger and L.W. Green in Riemannian geometry [4, Theors. 4.2, 5.3], and, in a suitable way, extends it to the Lorentzian setting. The main aim of this note is to show the use of that integral inequality to the study of conjugate points along null geodesics on Lorentzian odd dimensional spheres. In fact, each odd dimensional sphere may be endowed with a natural Lorentzian metric (section 3). Our method permits us to study null conjugate points without using the Jacobi equation and related techniques. Moreover, as far as we know, there are not many examples of compact Lorentzian manifolds where the behaviour of their null geodesics, null conjugate points and null conjugate loci have been described.

Compact Lorentzian manifolds have been historically neglected because of both physical and mathematical reasons. Recall that they have closed timelike curves, and therefore they are acausal (in particular, they cannot be isometrically immersed in a Lorentz-Minkowski space of any dimension) and not physically admissible. On the other hand, a compact Lorentzian manifold may be geodesically incomplete (this fact is well known) and the elliptic model of Lorentzian space form is not compact (contrary to the Riemannian case). However, it has been recently argued [20] that the study of field theory on compact spacetimes could be interesting for Physics and it could give valuable information about the underlying manifold, complementary to the one obtained from the Riemannian theory. From a mathematical point of view, the lack of completeness in the compact case gave rise to the obtention of extra conditions which joint to compactness would imply completeness of the Lorentzian manifold. For instance, in [10] it has been proved that every compact Lorentzian manifold with constant sectional curvature is geodesically complete (the flat case was previously shown in [2]); in [16] that every compact Lorentzian manifold which admits a timelike conformal vector field is geodesically complete (see also [14] for a wide information on completeness of Lorentzian manifolds). Physicists are familiarized with the study of conformal vector fields, in fact the assumption of their existence on spacetime is a way to impose some symmetry useful, for instance, to study the Einstein equations (see, for example [3]). Finally, recall the outstanding role of timelike conformal vector fields in the introduction of Bochner's technique in Lorentzian manifolds [17], [18], [13].

The content of this note is organized as follows. Section 2 is first devoted to recall the notion and main properties of the null congruence associated to a timelike conformal vector field on a Lorentzian manifold. In the compact case, an integral inequality is shown, Theorem 2.1, and, using a well known result of H. Karcher, it is analyzed when the equality holds. Moreover, we also show that Theorem 2.1 provides information on the manifold from the nonexistence of null conjugate points.

Finally, in section 3 we consider a natural Lorentzian metric g on each sphere $\mathbb{S}^{2n+1}$ (it was called *canonical* in [20]) which is introduced from three different procedures. It is shown that g has a large group of isometries, an isotropic property for null tangent directions and that it is homogeneous, Proposition 3.2. Its null geodesics are studied, showing that no null geodesic is closed. Null conjugate points and null conjugate loci are analyzed, Proposition 3.3. In fact, it is shown that all past (or future) null geodesics starting from a point p meet at the second conjugate point of p, and that the null conjugate locus at every point is an imbedded $(2n-1)-$dimensional sphere $\mathbb{S}^{2n-1}$.

2 Preliminaries

Let (M,g) be an $n(\geq 2)-$dimensional Lorentzian manifold; that is a (connected) smooth manifold M endowed with a non-degenerate metric g with index 1, i.e. with signature $(-,+,...,+)$. As usual, T_pM denotes the tangent space at $p \in M$, TM the tangent bundle of M, and $\pi : TM \longrightarrow M$ the natural projection. We shall write ∇ for the Levi-Civita connection of g, R for the Riemannian curvature tensor (our convention on the curvature tensor is $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z)$, Ric for the Ricci tensor, $\widetilde{\mathrm{Ric}}$ for the corresponding quadratic form, S for the scalar curvature and $d\mu_g$ for the canonical measure induced from g.

The causal character of a tangent vector $v \in T_pM$ is *timelike* (resp. *null*, *spacelike*) if $g(v,v) < 0$ (resp. $g(v,v) = 0$ and $v \neq 0$, $g(v,v) > 0$ or $v = 0$). If $v \in T_pM$ then, γ_v will denote the unique geodesic such that $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. It is well-known that the causal character of the velocities $\gamma'(t)$, for any geodesic γ of (M,g), does not depend on the parameter t. In particular, a null geodesic γ of (M,g) is a geodesic such that $\gamma'(t)$ is a null vector. A vector field $K \in \mathfrak{X}(M)$ is said to be timelike if K_p is timelike for all $p \in M$. A timelike or null tangent vector $v \in T_pM$ is said to be future (resp. past) with respect to K if $g(v,K_p) < 0$ (resp. $g(v,K_p) > 0$). We will write $U = hK$ where $h = [-g(K,K)^{-\frac{1}{2}}]$ and so $g(U,U) = -1$ holds on all M.

Let $\widehat{g}$ be the *Sasaki metric* on TM induced from the Lorentzian metric g. We point out that it may be introduced in a similar way to the Riemannian case. But now $\widehat{g}$ is semi-Riemannian with index 2, and the fact that the natural projection $\pi : (TM,\widehat{g}) \longrightarrow (M,g)$ is a semi-Riemannian submersion remains true.

Throughout the remainder of this paper, (M,g) will denote a Lorentzian manifold with dimension $n \geq 3$, time oriented by a timelike vector field K. Recall [8], [12] that the *null congruence associated to* K is defined as follows:

$$C_KM = \left\{ v \in TM \,:\, g(v,v) = 0,\, g(v,K_{\pi(v)}) = 1 \right\}. \tag{1}$$

This subset of TM has the following nice properties [5], [6]:

(i) For each null tangent vector v, there exists a unique $t \in \mathbb{R}$ such that $tv \in C_KM$, and the map $v \longmapsto [v]$ is a diffeomorphism from C_KM to the manifold $\mathcal{N} = \{[v] \in PM : g(v,v) = 0\}$ of the null directions of M (here PM denotes the projective fiber bundle associated to TM).

(ii) It is an orientable imbedded submanifold of TM with dimension $2(n-1)$. Moreover (C_KM,π,M) is a fiber bundle with fibre type $\mathbb{S}^{n-2}$, and so C_KM will be compact if M is assumed to be compact.

(iii) The induced metric on C_KM from the Sasaki metric of TM, which we agree also to represent by $\widehat{g}$, is Lorentzian. Moreover, the restriction of π to C_KM is a semi-Riemannian submersion with spacelike fibers.

Sectional curvature of a Lorentzian metric can be defined for nondegenerate tangent planes but it cannot be stated for null planes (i.e. degenerate planes). If v is a null tangent vector and σ a null plane containing it, the *null sectional curvature* with respect to v of the plane σ is defined to be $\mathcal{K}_v(\sigma) = g(R(u,v)v,u)/g(u,u)$, where $\{u,v\}$ is a basis of σ [7], [8], [1, Def. A.6]. Note that $\mathcal{K}_v(\sigma)$ does not depend on the choice of the non-zero spacelike vector u, but it does quadratically on v.

From now on let us suppose that a null congruence associated with a timelike vector field K has been fixed. Then, we may choice, for every null plane σ, the unique null vector $v \in C_KM \cap \sigma$, thus the null sectional curvature can be thought as a function on null tangent planes. In this note we always use such convention, and we will call it the $K-$*normalized null sectional curvature*.

Until now, no extra hypothesis on the timelike vector field K has been assumed. Recall that a vector field X is called *conformal* (resp. *Killing*) if each of its (local) fluxes consists of (local) conformal (resp. isometric) transformations. It is well known that X is conformal if and only if the Lie derivative of g with respect to X satisfies $\mathcal{L}_X g = \rho g$, where $\rho : M \to \mathbb{R}$ (Killing when $\rho = 0$). If K is assumed to be conformal, then every null geodesic γ_v of (M,g) with $v \in C_KM$, provides us with the null geodesic γ_v' of $(C_KM, \widehat{g})$. Furthermore, each null geodesic β of (M,g) may be reparametrized to obtain a null geodesic α which satisfies $\alpha'(t) \in C_KM$. In fact, consider the real number $a = g(\beta', K_\beta)$, which satisfies $a \neq 0$. If we put $\alpha(t) = \beta(\frac{t}{a})$, then $g(\alpha', K_\alpha) = 1$ holds for all t. Null geodesics will be considered to be parametrized by this $K-$*affine parameter*.

We will next assume that (M,g) is a compact Lorentzian manifold and K a timelike conformal vector field. Recall that in this case (M,g) is geodesically complete [16]. The following integral inequality is the key tool to relate null conjugate points to global geometric properties:

Theorem 2.1 [5, Theor. 3.5] *Let (M,g) be a compact Lorentzian manifold which admits a timelike conformal vector field K. If there exists $a \in (0,+\infty)$ such that every null geodesic $\gamma_v : [0,a] \longrightarrow M$, with $v \in C_KM$, has no conjugate point of $\gamma_v(0)$ in $[0,a)$, then*

$$\int_M h^{n-2} d\mu_g \geq \frac{a^2}{\pi^2(n-1)(n-2)} \int_M \left[n\widetilde{\mathrm{Ric}}(U) + S \right] h^n d\mu_g. \qquad (2)$$

Moreover, equality holds if and only if (M,g) has $U-$normalized null sectional curvature $\frac{\pi^2}{a^2h^2}$.

Observe that if equality holds in (2) then the $U-$normalized null sectional curvature of (M, g) is an everywhere non-zero point function. On the other hand, it was proven by H. Karcher, [9] the following result:

Let U be a unit timelike vector field on an $n(\geq 4)-$ dimensional Lorentzian manifold (M, g). The $U-$normalized null sectional curvature is an everywhere non-zero point function if and only if the following conditions hold:

1. The distribution $U^{\perp}$ is integrable.

2. The integral manifolds of $U^{\perp}$ are totally umbilic and have constant sectional curvature.

3. (M, g) is locally conformal to a flat Lorentzian manifold.

Combining Theorem 2.1 and Karcher's result we can give a characterization of the equality in (2) in terms of the distribution $U^{\perp}(= K^{\perp})$ and the locally conformal flatness of (M, g).

Moreover, Theorem 2.1 provides also information of (M, g) from the nonexistence of null conjugate points. In fact, if it is assumed that every null geodesic does not contain a pair of mutually conjugate points, then (2) is valid for any positive real number a. Therefore it must happen

$$\int_M \Big[n\widetilde{\mathrm{Ric}}(U) + S \Big] h^n d\mu_g \leq 0. \tag{3}$$

In the next section we will use the integral inequality (2) to get a bound of a for a relevant family of compact Lorentzian manifolds which admit a unit timelike Killing vector field.

3 Lorentzian Odd Dimensional Spheres

We consider $\mathbb{R}^{2n+2}$ identified to $\mathbb{C}^{n+1}$ as usual: $(x_1, ..., x_{2n+2}) = (z_1, ..., z_{n+1})$, with $z_j = x_j + i x_{n+1+j}$. So that, the unit sphere of $\mathbb{R}^{2n+2}$ is written

$$\mathbb{S}^{2n+1} = \Big\{ z = (z_1, ..., z_{n+1}) \in \mathbb{C}^{n+1} : \sum_{j=1}^{n+1} |z_j|^2 = 1 \Big\}.$$

Let $U \in \mathfrak{X}(\mathbb{S}^{2n+1})$ be given by $U_z = iz$ at any $z \in \mathbb{S}^{2n+1}$. For the canonical Riemannian metric g_{can} of $\mathbb{S}^{2n+1}$, U is Killing and satisfies $g_{can}(U, U) = 1$. Therefore, $\nabla_U U = 0$, where ∇ is the Levi-Civita connection of g_{can}; that is, the integral curves of U are geodesics of g_{can}.

Let ω be the $1-$form metrically equivalent to U with respect to g_{can}. A Lorentzian metric on $\mathbb{S}^{2n+1}$ can be defined by

$$g \,=\, g_{can} \,-\, 2\,\omega \otimes \omega. \tag{4}$$

This construction of g from g_{can} is standard, but the Lorentzian metric g deserves of making stand out among all the Lorentzian metrics of $\mathbb{S}^{2n+1}$. In fact, it has previously considered [20]. It is not difficult to show that the Levi-Civita connection $\widetilde{\nabla}$ of g satisfies:

$$\widetilde{\nabla}_X Y = \nabla_X Y - 2\,\omega(X)\nabla_Y U - 2\,\omega(Y)\nabla_X U, \tag{5}$$

where $X, Y \in \mathfrak{X}(\mathbb{S}^{2n+1})$. Moreover, the vector field U satisfies $g(U,U) = -1$, it is Killing for g and $\widetilde{\nabla}_U U = 0$; so that, its integral curves are unit timelike geodesics of g. On the other hand, observe that the inclusion $\mathbb{S}^{2n+1} \hookrightarrow \mathbb{S}^{2m+1}$, $n < m$, $(z_1, .., z_{n+1}) \mapsto (z_1, .., z_{n+1}, 0, .., 0)$ is a totally geodesic Lorentzian submanifold, when both spheres are endowed with the corresponding Lorentzian metrics (4).

Recall now the classical Hopf fibration $\Pi : (\mathbb{S}^{2n+1}, g_{can}) \to (\mathbb{C}P^n, g_{FS})$, $z \mapsto [z]$, where $\mathbb{C}P^n$ is the complex projective space endowed with its Fubini-Study Kähler metric g_{FS} of constant holomorphic sectional curvature 4 [11, p. 273]. Recall that Π permits to consider $\mathbb{S}^{2n+1}$ as a principal fiber bundle over $\mathbb{C}P^n$ with structural group $\mathbb{S}^1$. Moreover, Π is a Riemannian submersion with totally geodesic fibres. If the Riemannian metric g_{can} is replaced by the Lorentzian metric g, then Π becomes a semi-Riemannian submersion from $(\mathbb{S}^{2n+1}, g)$ to $(\mathbb{C}P^n, g_{FS})$ with timelike totally geodesics fibres. Let us remark that g may be considered as a particular case of a Kaluza-Klein metric. In fact, if we put $\mathfrak{s}^1 = i\mathbb{R}$ for the Lie algebra of $\mathbb{S}^1$ then $i\omega$ is a connection form on $\mathbb{S}^{2n+1}$, and $g \,=\, \Pi^*(g_{FS}) \,-\, \omega \otimes \omega$, [15].

As a third description of the Lorentzian metric g, note that it can be characterized from the properties:

$$g_{|\mathcal{V}} = -g_{can|\mathcal{V}}, \quad g_{|\mathcal{H}} = g_{can|\mathcal{H}}, \quad g(\mathcal{V}, \mathcal{H}) = 0, \tag{6}$$

where $\mathcal{V}$ and $\mathcal{H}$ are respectively the vertical and the horizontal distributions for the canonical connection of the Hopf fibration.

Now recall that if U is a unit timelike vector field on a Lorentzian manifold (M, g), and $p \in M$, (M, g) is said to be *spatially isotropic* with respect to U at p if for every two unit vectors $u_1, u_2 \in U_p^{\perp}$ there exists an isometry $\phi : M \longrightarrow M$ such that $\phi(p) = p$, $d\phi_p(U_p) = U_p$ and $d\phi_p(u_1) = u_2$. (M, g) is said to be *spatially isotropic* with respect to U if it is *spatially isotropic* with respect to U at every point, [19, p. 47]. The following results are easy to show:

Lemma 3.1 *Let (M, g) be a Lorentzian manifold which admits a unit timelike vector field U. Then (M, g) is spatially isotropic with respect to U if and only if for every $p \in M$ and for every $u, v \in (C_U M)_p$ there exists an isometry $\phi : M \longrightarrow M$ such that $\phi(p) = p$, $d\phi_p(U_p) = U_p$ and $d\phi_p(u) = v$.*

Proposition 3.2 [5, Prop. 4.2] $(\mathbb{S}^{2n+1}, g)$ *is spatially isotropic with respect to U and the unitary group $U(n+1)$ acts transitively by $g-$isometries on $\mathbb{S}^{2n+1}$.*

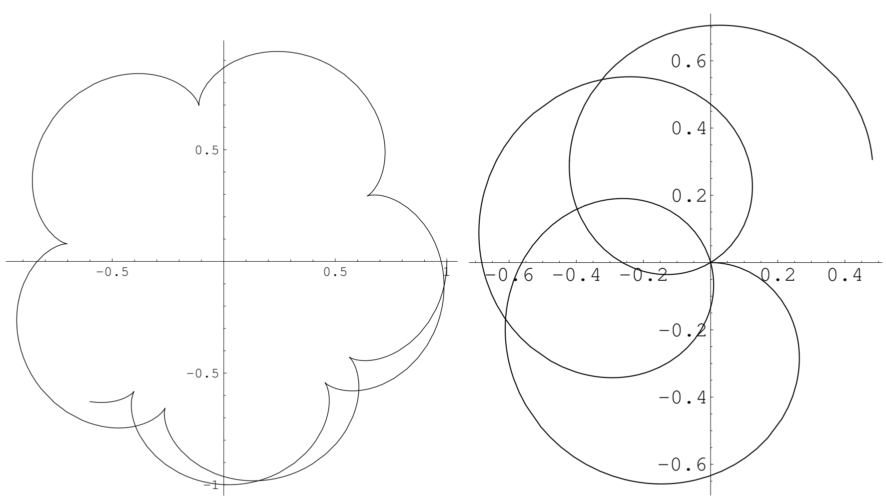
Now note that in order to analyze the behaviour of the null geodesics of the Lorentzian odd dimensional spheres, it suffices to consider the ones starting from the specific point $p = (1, ..., 0) \in \mathbb{S}^{2n+1}$. Observe that $v \in (C_U \mathbb{S}^{2n+1})_p$ if and only if $v = (-i, v_2, ..., v_{n+1})$ with $\sum_{j=2}^{n+1} \mid v_j \mid^2 = 1$.

If we agree to represent $\gamma_v(t) = (\Theta_1^v(t), ..., \Theta_{n+1}^v(t))$, with $\Theta_k^v : \mathbb{R} \to \mathbb{C}$, $1 \leq k \leq n+1$, then we get:

$$\Theta_1^v(t) = \frac{2-\sqrt{2}}{4}\, e^{(-2-\sqrt{2})it} + \frac{2+\sqrt{2}}{4}\, e^{(-2+\sqrt{2})it}$$

$$\Theta_j^v(t) = \frac{\sqrt{2} i v_j}{4} \left[e^{(-2-\sqrt{2})it} - e^{(-2+\sqrt{2})it} \right],\ j \geq 2,$$

The following figures show each kind of components of a lightlike geodesic.



From the previous equations the following facts directly follow:

(1) *There is no closed null geodesic in* $(\mathbb{S}^{2n+1}, g)$,

(2) *For every $v, u \in (C_U \mathbb{S}^{2n+1})_p$, $v \neq u$, $\gamma_v(t) = \gamma_u(t)$ holds if and only if $t = \frac{k\pi}{\sqrt{2}}$ for some $k \in \mathbb{Z}$.*

Now we pay attention to curvature properties of $(\mathbb{S}^{2n+1}, g)$. Its scalar curvature S can be computed to obtain $S = 2n(2n+3)$. On the other hand, we get $\widetilde{\mathrm{Ric}}(U) = 2n$ and the $U-$normalized null sectional curvature of $(\mathbb{S}^{2n+1}, g)$ is a point function if and only if $n = 1$, with $\mathcal{K}_v(v^{\perp}) = 8$ for any $v \in C_U\mathbb{S}^3$, (see [5] for details). So, it should be pointed out that the first conclusion in Karcher's theorem does not remain true if it is assumed $\dim M = 3$, because of non integrability of the distribution $U^{\perp}$.

We end this note with an application of our integral inequality (2) to the study of the behaviour of conjugate points along null geodesics in Lorentzian odd dimensional spheres.

Proposition 3.3 [5, Prop. 4.4] *For every null geodesic γ_v of $(\mathbb{S}^{2n+1}, g)$ with $v \in C_U\mathbb{S}^{2n+1}$, the points $\gamma_v(0)$ and $\gamma_v(\frac{\pi}{2\sqrt{2}})$ are conjugate and there is no conjugate point to $\gamma_v(0)$ on $[0, \frac{\pi}{2\sqrt{2}})$. Moreover the past null conjugate locus of each point $p \in \mathbb{S}^{2n+1}$ is a $(2n-1)-$dimensional imbedded sphere.*

Observe that previous result may be dualized to analyze the future null conjugate locus.

Remark 3.4 A conjugate point $\gamma(a)$ of $\gamma(0) = p$ along a null geodesic γ can be interpreted as an "almost-meeting point" of null geodesics starting from p. In our case, the first conjugate point along any null geodesic is exactly at the middle of the path to the first "meeting point" which is the second null conjugate point. Thus, null geodesics of $\mathbb{S}^{2n+1}$ have an "Auf wiedersehensflächen" type property as in Riemannian case, but in contrast to that, in the Lorentzian setting the first "meeting point" is not the first conjugate point.

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