Additional Topics

3.A Universal Coefficients for Homology

The main goal in this section is an algebraic formula for computing homology with arbitrary coefficients in terms of homology with $\mathbb{Z}$ coefficients. The theory parallels rather closely the universal coefficient theorem for cohomology in §3.1.

The first step is to formulate the definition of homology with coefficients in terms of tensor products. The chain group $C_n(X;G)$ as defined in §2.2 consists of the finite formal sums $\sum_i g_i\sigma_i$ with $g_i \in G$ and $\sigma_i : \Delta^n \to X$. This means that $C_n(X;G)$ is a direct sum of copies of G, with one copy for each singular n-simplex in X. More generally, the relative chain group $C_n(X,A;G) = C_n(X;G)/C_n(A;G)$ is also a direct sum of copies of G, one for each singular n-simplex in X not contained in A. From the basic properties of tensor products listed in the discussion of the Künneth formula in §3.2 it follows that $C_n(X,A;G)$ is naturally isomorphic to $C_n(X,A)\otimes G$, via the correspondence $\sum_i g_i\sigma_i \mapsto \sum_i \sigma_i \otimes g_i$. Under this isomorphism the boundary map $C_n(X,A;G)\to C_{n-1}(X,A;G)$ becomes the map $\partial\otimes 1\!\!1 : C_n(X,A)\otimes G \to C_{n-1}(X,A)\otimes G$ where $\partial : C_n(X,A)\to C_{n-1}(X,A)$ is the usual boundary map for $\mathbb{Z}$ coefficients. Thus we have the following algebraic problem:

> Given a chain complex $\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$ of free abelian groups C_n, is it possible to compute the homology groups $H_n(C;G)$ of the associated chain complex $\cdots \longrightarrow C_n\otimes G \xrightarrow{\partial_n\otimes 1\!\!1} C_{n-1}\otimes G \longrightarrow \cdots$ just in terms of G and the homology groups $H_n(C)$ of the original complex?

To approach this problem, the idea will be to compare the chain complex C with two simpler subcomplexes, the subcomplexes consisting of the cycles and the boundaries in C, and see what happens upon tensoring all three complexes with G.

Let $Z_n = \operatorname{Ker}\partial_n \subset C_n$ and $B_n = \operatorname{Im}\partial_{n+1} \subset C_n$. The restrictions of ∂_n to these two subgroups are zero, so they can be regarded as subcomplexes Z and B of C with trivial boundary maps. Thus we have a short exact sequence of chain complexes consisting of the commutative diagrams

$$\text{(i)}\qquad \begin{array}{ccccccccc}
0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{\partial_n} & B_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_{n-1}} & & \\
0 & \longrightarrow & Z_{n-1} & \longrightarrow & C_{n-1} & \xrightarrow{\partial_{n-1}} & B_{n-2} & \longrightarrow & 0
\end{array}$$

The rows in this diagram split since each B_n is free, being a subgroup of the free group C_n. Thus $C_n \approx Z_n \oplus B_{n-1}$, but the chain complex C is not the direct sum of the chain complexes Z and B since the latter have trivial boundary maps but the boundary maps in C may be nontrivial. Now tensor with G to get a commutative diagram

(ii)
$$\begin{tikzcd}
0 \arrow[r] & Z_n \otimes G \arrow[r] \arrow[d, "\partial_n \otimes 1\!\!1"] & C_n \otimes G \arrow[r, "\partial_n \otimes 1\!\!1"] \arrow[d, "\partial_n \otimes 1\!\!1"] & B_{n-1} \otimes G \arrow[r] \arrow[d, "\partial_{n-1} \otimes 1\!\!1"] & 0 \\
0 \arrow[r] & Z_{n-1} \otimes G \arrow[r] & C_{n-1} \otimes G \arrow[r, "\partial_{n-1} \otimes 1\!\!1"] & B_{n-2} \otimes G \arrow[r] & 0
\end{tikzcd}$$

The rows are exact since the rows in (i) split and tensor products satisfy $(A \oplus B) \otimes G \approx A \otimes G \oplus B \otimes G$, so the rows in (ii) are split exact sequences too. Thus we have a short exact sequence of chain complexes $0 \to Z \otimes G \to C \otimes G \to B \otimes G \to 0$. Since the boundary maps are trivial in $Z \otimes G$ and $B \otimes G$, the associated long exact sequence of homology groups has the form

(iii) $$\cdots \longrightarrow B_n \otimes G \longrightarrow Z_n \otimes G \longrightarrow H_n(C;G) \longrightarrow B_{n-1} \otimes G \longrightarrow Z_{n-1} \otimes G \longrightarrow \cdots$$

The 'boundary' maps $B_n \otimes G \to Z_n \otimes G$ in this sequence are simply the maps $i_n \otimes 1\!\!1$ where $i_n : B_n \to Z_n$ is the inclusion. This is evident from the definition of the boundary map in a long exact sequence of homology groups: In diagram (ii) one takes an element of $B_{n-1} \otimes G$, pulls it back via $(\partial_n \otimes 1\!\!1)^{-1}$ to $C_n \otimes G$, then applies $\partial_n \otimes 1\!\!1$ to get into $C_{n-1} \otimes G$, then pulls back to $Z_{n-1} \otimes G$.

The long exact sequence (iii) can be broken up into short exact sequences

(iv) $$0 \longrightarrow \mathrm{Coker}(i_n \otimes 1\!\!1) \longrightarrow H_n(C;G) \longrightarrow \mathrm{Ker}(i_{n-1} \otimes 1\!\!1) \longrightarrow 0$$

where $\mathrm{Coker}(i_n \otimes 1\!\!1) = (Z_n \otimes G)/\mathrm{Im}(i_n \otimes 1\!\!1)$. The next lemma shows this cokernel is just $H_n(C) \otimes G$.

Lemma 3A.1. *If the sequence of abelian groups* $A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$ *is exact, then so is* $A \otimes G \xrightarrow{i \otimes 1\!\!1} B \otimes G \xrightarrow{j \otimes 1\!\!1} C \otimes G \longrightarrow 0$.

Proof: Certainly the compositions of two successive maps in the latter sequence are zero. Also, $j \otimes 1\!\!1$ is clearly surjective since j is. To check exactness at $B \otimes G$ it suffices to show that the map $B \otimes G/\mathrm{Im}(i \otimes 1\!\!1) \to C \otimes G$ induced by $j \otimes 1\!\!1$ is an isomorphism, which we do by constructing its inverse. Define a map $\varphi : C \times G \to B \otimes G/\mathrm{Im}(i \otimes 1\!\!1)$ by $\varphi(c,g) = b \otimes g$ where $j(b) = c$. This φ is well-defined since if $j(b) = j(b') = c$ then $b - b = i(a)$ for some $a \in A$ by exactness, so $b \otimes g - b' \otimes g = (b - b') \otimes g = i(a) \otimes g \in \mathrm{Im}(i \otimes 1\!\!1)$. Since φ is a homomorphism in each variable separately, it induces a homomorphism $C \otimes G \to B \otimes G/\mathrm{Im}(i \otimes 1\!\!1)$. This is clearly an inverse to the map $B \otimes G/\mathrm{Im}(i \otimes 1\!\!1) \to C \otimes G$. $\square$

It remains to understand $\mathrm{Ker}(i_{n-1} \otimes 1\!\!1)$, or equivalently $\mathrm{Ker}(i_n \otimes 1\!\!1)$. The situation is that tensoring the short exact sequence

(v) $$0 \longrightarrow B_n \xrightarrow{i_n} Z_n \longrightarrow H_n(C) \longrightarrow 0$$

with G produces a sequence which becomes exact only by insertion of the extra term $\mathrm{Ker}(i_n \otimes 1\!\!1)$:

(vi) $$0 \longrightarrow \mathrm{Ker}(i_n \otimes 1\!\!1) \longrightarrow B_n \otimes G \xrightarrow{i_n \otimes 1\!\!1} Z_n \otimes G \longrightarrow H_n(C) \otimes G \longrightarrow 0$$

What we will show is that $\mathrm{Ker}(i_n \otimes 1\!\!1)$ does not really depend on B_n and Z_n but only on their quotient $H_n(C)$, and of course G.

The sequence (v) is a free resolution of $H_n(C)$, where as in §3.1 a free resolution of an abelian group H is an exact sequence

$$\cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \longrightarrow 0$$

with each F_n free. Tensoring a free resolution of this form with a fixed group G produces a chain complex

$$\cdots \longrightarrow F_1 \otimes G \xrightarrow{f_1 \otimes 1\!\!1} F_0 \otimes G \xrightarrow{f_0 \otimes 1\!\!1} H \otimes G \longrightarrow 0$$

By the preceding lemma this is exact at $F_0 \otimes G$ and $H \otimes G$, but to the left of these two terms it may not be exact. For the moment let us write $H_n(F \otimes G)$ for the homology group $\mathrm{Ker}(f_n \otimes 1\!\!1)/\,\mathrm{Im}(f_{n+1} \otimes 1\!\!1)$.

Lemma 3A.2. *For any two free resolutions F and F' of H there are canonical isomorphisms $H_n(F \otimes G) \approx H_n(F' \otimes G)$ for all n.*

Proof: We will use Lemma 3.1(a). In the situation described there we have two free resolutions F and F' with a chain map between them. If we tensor the two free resolutions with G we obtain chain complexes $F \otimes G$ and $F' \otimes G$ with the maps $\alpha_n \otimes 1\!\!1$ forming a chain map between them. Passing to homology, this chain map induces homomorphisms $\alpha_* : H_n(F \otimes G) \to H_n(F' \otimes G)$ which are independent of the choice of α_n's since if α_n and α'_n are chain homotopic via a chain homotopy λ_n then $\alpha_n \otimes 1\!\!1$ and $\alpha'_n \otimes 1\!\!1$ are chain homotopic via $\lambda_n \otimes 1\!\!1$.

For a composition $H \xrightarrow{\alpha} H' \xrightarrow{\beta} H''$ with free resolutions F, F', and F'' of these three groups also given, the induced homomorphisms satisfy $(\beta\alpha)_* = \beta_*\alpha_*$ since we can choose for the chain map $F \to F''$ the composition of chain maps $F \to F' \to F''$. In particular, if we take α to be an isomorphism, with β its inverse and $F'' = F$, then $\beta_*\alpha_* = (\beta\alpha)_* = 1\!\!1_* = 1\!\!1$, and similarly with β and α reversed. So α_* is an isomorphism if α is an isomorphism. Specializing further, taking α to be the identity but with two different free resolutions F and F', we get a canonical isomorphism $1\!\!1_* : H_n(F \otimes G) \to H_n(F' \otimes G)$. □

The group $H_n(F \otimes G)$, which depends only on H and G, is denoted $\mathrm{Tor}_n(H, G)$. Since a free resolution $0 \to F_1 \to F_0 \to H \to 0$ always exists, as noted in §3.1, it follows that $\mathrm{Tor}_n(H, G) = 0$ for $n > 1$. Usually $\mathrm{Tor}_1(H, G)$ is written simply as $\mathrm{Tor}(H, G)$. As we shall see later, $\mathrm{Tor}(H, G)$ provides a measure of the common torsion of H and G, hence the name 'Tor.'

Is there a group $\mathrm{Tor}_0(H, G)$? With the definition given above it would be zero since Lemma 3A.1 implies that $F_1 \otimes G \to F_0 \otimes G \to H \otimes G \to 0$ is exact. It is probably better to modify the definition of $H_n(F \otimes G)$ to be the homology groups of the sequence

$\cdots \to F_1 \otimes G \to F_0 \otimes G \to 0$, omitting the term $H \otimes G$ which can be regarded as a kind of augmentation. With this revised definition, Lemma 3A.1 then gives an isomorphism $\mathrm{Tor}_0(H,G) \approx H \otimes G$.

We should remark that $\mathrm{Tor}(H,G)$ is a functor of both G and H: Homomorphisms $\alpha : H \to H'$ and $\beta : G \to G'$ induce homomorphisms $\alpha_* : \mathrm{Tor}(H,G) \to \mathrm{Tor}(H',G)$ and $\beta_* : \mathrm{Tor}(H,G) \to \mathrm{Tor}(H,G')$, satisfying $(\alpha\alpha')_* = \alpha_*\alpha'_*$, $(\beta\beta')_* = \beta_*\beta'_*$, and $\mathbb{1}_* = \mathbb{1}$. The induced map α_* was constructed in the proof of Lemma 3A.2, while for β the construction of β_* is obvious.

Before going into calculations of $\mathrm{Tor}(H,G)$ let us finish analyzing the earlier exact sequence (iv). Recall that we have a chain complex C of free abelian groups, with homology groups denoted $H_n(C)$, and tensoring C with G gives another complex $C \otimes G$ whose homology groups are denoted $H_n(C;G)$. The following result is known as the **universal coefficient theorem for homology** since it describes homology with arbitrary coefficients in terms of homology with the 'universal' coefficient group $\mathbb{Z}$.

Theorem 3A.3. *If C is a chain complex of free abelian groups, then there are natural short exact sequences*

$$0 \longrightarrow H_n(C) \otimes G \longrightarrow H_n(C;G) \longrightarrow \mathrm{Tor}(H_{n-1}(C),G) \longrightarrow 0$$

for all n and all G, and these sequences split, though not naturally.

Naturality means that a chain map $C \to C'$ induces a map between the corresponding short exact sequences, with commuting squares.

Proof: The exact sequence in question is (iv) since we have shown that we can identify $\mathrm{Coker}(i_n \otimes \mathbb{1})$ with $H_n(C) \otimes G$ and $\mathrm{Ker}\, i_{n-1}$ with $\mathrm{Tor}(H_{n-1}(C),G)$. Verifying the naturality of this sequence is a mental exercise in definition-checking, left to the reader.

The splitting is obtained as follows. We observed earlier that the short exact sequence $0 \to Z_n \to C_n \to B_{n-1} \to 0$ splits, so there is a projection $p : C_n \to Z_n$ restricting to the identity on Z_n. The map p gives an extension of the quotient map $Z_n \to H_n(C)$ to a homomorphism $C_n \to H_n(C)$. Letting n vary, we then have a chain map $C \to H(C)$ where the groups $H_n(C)$ are regarded as a chain complex with trivial boundary maps, so the chain map condition is automatic. Now tensor with G to get a chain map $C \otimes G \to H(C) \otimes G$. Taking homology groups, we then have induced homomorphisms $H_n(C;G) \to H_n(C) \otimes G$ since the boundary maps in the chain complex $H(C) \otimes G$ are trivial. The homomorphisms $H_n(C;G) \to H_n(C) \otimes G$ give the desired splitting since at the level of chains they are the identity on cycles in C, by the definition of p. ◻

Corollary 3A.4. *For each pair of spaces (X,A) there are split exact sequences*

$$0 \longrightarrow H_n(X,A) \otimes G \longrightarrow H_n(X,A;G) \longrightarrow \mathrm{Tor}(H_{n-1}(X,A),G) \longrightarrow 0$$

for all n, and these sequences are natural with respect to maps $(X,A) \to (Y,B)$. ◻

The splitting is not natural, for if it were, a map $X \to Y$ that induced trivial maps $H_n(X) \to H_n(Y)$ and $H_{n-1}(X) \to H_{n-1}(Y)$ would have to induce the trivial map

$H_n(X;G) \to H_n(Y;G)$ for all G, but in Example 2.51 we saw an instance where this fails, namely the quotient map $M(\mathbb{Z}_m, n) \to S^{n+1}$ with $G = \mathbb{Z}_m$.

The basic tools for computing Tor are given by:

Proposition 3A.5.

(1) $\mathrm{Tor}(A,B) \approx \mathrm{Tor}(B,A)$.

(2) $\mathrm{Tor}(\bigoplus_i A_i, B) \approx \bigoplus_i \mathrm{Tor}(A_i, B)$.

(3) $\mathrm{Tor}(A,B) = 0$ *if* A *or* B *is free, or more generally torsionfree.*

(4) $\mathrm{Tor}(A,B) \approx \mathrm{Tor}(T(A),B)$ *where* $T(A)$ *is the torsion subgroup of* A.

(5) $\mathrm{Tor}(\mathbb{Z}_n, A) \approx \mathrm{Ker}(A \xrightarrow{n} A)$.

(6) *For each short exact sequence* $0 \to B \to C \to D \to 0$ *there is a naturally associated exact sequence*

$$0 \to \mathrm{Tor}(A,B) \to \mathrm{Tor}(A,C) \to \mathrm{Tor}(A,D) \to A \otimes B \to A \otimes C \to A \otimes D \to 0$$

Proof: Statement (2) is easy since one can choose as a free resolution of $\bigoplus_i A_i$ the direct sum of free resolutions of the A_i's. Also easy is (5), which comes from tensoring the free resolution $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}_n \to 0$ with A.

For (3), if A is free, it has a free resolution with $F_n = 0$ for $n \ge 1$, so $\mathrm{Tor}(A,B) = 0$ for all B. On the other hand, if B is free, then tensoring a free resolution of A with B preserves exactness, since tensoring a sequence with a direct sum of $\mathbb{Z}$'s produces just a direct sum of copies of the given sequence. So $\mathrm{Tor}(A,B) = 0$ in this case too. The generalization to torsionfree A or B will be given below.

For (6), choose a free resolution $0 \to F_1 \to F_0 \to A \to 0$ and tensor with the given short exact sequence to get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F_1 \otimes B \arrow[r] \arrow[d] & F_1 \otimes C \arrow[r] \arrow[d] & F_1 \otimes D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_0 \otimes B \arrow[r] & F_0 \otimes C \arrow[r] & F_0 \otimes D \arrow[r] & 0
\end{tikzcd}$$

The rows are exact since tensoring with a free group preserves exactness. Extending the three columns by zeros above and below, we then have a short exact sequence of chain complexes whose associated long exact sequence of homology groups is the desired six-term exact sequence.

To prove (1) we apply (6) to a free resolution $0 \to F_1 \to F_0 \to B \to 0$. Since $\mathrm{Tor}(A, F_1)$ and $\mathrm{Tor}(A, F_0)$ vanish by the part of (3) which we have proved, the six-term sequence in (6) reduces to the first row of the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}(A,B) \arrow[r] & A \otimes F_1 \arrow[r] \arrow[d, "\approx"] & A \otimes F_0 \arrow[r] \arrow[d, "\approx"] & A \otimes B \arrow[r] \arrow[d, "\approx"] & 0 \\
0 \arrow[r] & \mathrm{Tor}(B,A) \arrow[r] & F_1 \otimes A \arrow[r] & F_0 \otimes A \arrow[r] & B \otimes A \arrow[r] & 0
\end{tikzcd}$$

The second row comes from the definition of $\mathrm{Tor}(B,A)$. The vertical isomorphisms come from the natural commutativity of tensor product. Since the squares commute, there is induced a map $\mathrm{Tor}(A,B) \to \mathrm{Tor}(B,A)$, which is an isomorphism by the five-lemma.

Now we can prove the statement (3) in the torsionfree case. For a free resolution $0 \longrightarrow F_1 \xrightarrow{\varphi} F_0 \longrightarrow A \longrightarrow 0$ we wish to show that $\varphi \otimes 1\!\!1 : F_1 \otimes B \to F_0 \otimes B$ is injective if B is torsionfree. Suppose $\sum_i x_i \otimes b_i$ lies in the kernel of $\varphi \otimes 1\!\!1$. This means that $\sum_i \varphi(x_i) \otimes b_i$ can be reduced to 0 by a finite number of applications of the defining relations for tensor products. Only a finite number of elements of B are involved in this process. These lie in a finitely generated subgroup $B_0 \subset B$, so $\sum_i x_i \otimes b_i$ lies in the kernel of $\varphi \otimes 1\!\!1 : F_1 \otimes B_0 \to F_0 \otimes B_0$. This kernel is zero since $\mathrm{Tor}(A, B_0) = 0$, as B_0 is finitely generated and torsionfree, hence free.

Finally, we can obtain statement (4) by applying (6) to the short exact sequence $0 \to T(A) \to A \to A/T(A) \to 0$ since $A/T(A)$ is torsionfree. $\square$

In particular, (5) gives $\mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) \approx \mathbb{Z}_q$ where q is the greatest common divisor of m and n. Thus $\mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}_n)$ is isomorphic to $\mathbb{Z}_m \otimes \mathbb{Z}_n$, though somewhat by accident. Combining this isomorphism with (2) and (3) we see that for finitely generated A and B, $\mathrm{Tor}(A, B)$ is isomorphic to the tensor product of the torsion subgroups of A and B, or roughly speaking, the common torsion of A and B. This is one reason for the 'Tor' designation, further justification being (3) and (4).

Homology calculations are often simplified by taking coefficients in a field, usually $\mathbb{Q}$ or $\mathbb{Z}_p$ for p prime. In general this gives less information than taking $\mathbb{Z}$ coefficients, but still some of the essential features are retained, as the following result indicates:

Corollary 3A.6. **(a)** *$H_n(X;\mathbb{Q}) \approx H_n(X;\mathbb{Z}) \otimes \mathbb{Q}$, so when $H_n(X;\mathbb{Z})$ is finitely generated, the dimension of $H_n(X;\mathbb{Q})$ as a vector space over $\mathbb{Q}$ equals the rank of $H_n(X;\mathbb{Z})$.*

(b) *If $H_n(X;\mathbb{Z})$ and $H_{n-1}(X;\mathbb{Z})$ are finitely generated, then for p prime, $H_n(X;\mathbb{Z}_p)$ consists of*

(i) *a $\mathbb{Z}_p$ summand for each $\mathbb{Z}$ summand of $H_n(X;\mathbb{Z})$,*

(ii) *a $\mathbb{Z}_p$ summand for each $\mathbb{Z}_{p^k}$ summand in $H_n(X;\mathbb{Z})$, $k \geq 1$,*

(iii) *a $\mathbb{Z}_p$ summand for each $\mathbb{Z}_{p^k}$ summand in $H_{n-1}(X;\mathbb{Z})$, $k \geq 1$.* $\square$

Even in the case of nonfinitely generated homology groups, field coefficients still give good qualitative information:

Corollary 3A.7. **(a)** *$\widetilde{H}_n(X;\mathbb{Z}) = 0$ for all n iff $\widetilde{H}_n(X;\mathbb{Q}) = 0$ and $\widetilde{H}_n(X;\mathbb{Z}_p) = 0$ for all n and all primes p.*

(b) *A map $f : X \to Y$ induces isomorphisms on homology with $\mathbb{Z}$ coefficients iff it induces isomorphisms on homology with $\mathbb{Q}$ and $\mathbb{Z}_p$ coefficients for all primes p.*

Proof: Statement (b) follows from (a) by passing to the mapping cone of f. The universal coefficient theorem gives the 'only if' half of (a). For the 'if' implication it suffices to show that if an abelian group A is such that $A \otimes \mathbb{Q} = 0$ and $\mathrm{Tor}(A, \mathbb{Z}_p) = 0$

for all primes p, then $A = 0$. For the short exact sequences $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}_p \to 0$ and $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, the six-term exact sequences in (6) of the proposition become

$$0 \longrightarrow \mathrm{Tor}(A, \mathbb{Z}_p) \longrightarrow A \xrightarrow{p} A \longrightarrow A \otimes \mathbb{Z}_p \longrightarrow 0$$
$$0 \longrightarrow \mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z}) \longrightarrow A \longrightarrow A \otimes \mathbb{Q} \longrightarrow A \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

If $\mathrm{Tor}(A, \mathbb{Z}_p) = 0$ for all p, then exactness of the first sequence implies that $A \xrightarrow{p} A$ is injective for all p, so A is torsionfree. Then $\mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z}) = 0$ by (3) or (4) of the proposition, so the second sequence implies that $A \to A \otimes \mathbb{Q}$ is injective, hence $A = 0$ if $A \otimes \mathbb{Q} = 0$. $\square$

The algebra by means of which the Tor functor is derived from tensor products has a very natural generalization in which abelian groups are replaced by modules over a fixed ring R with identity, using the definition of tensor product of R-modules given in §3.2. Free resolutions of R-modules are defined in the same way as for abelian groups, using free R-modules, which are direct sums of copies of R. Lemmas 3A.1 and 3A.2 carry over to this context without change, and so one has functors $\mathrm{Tor}_n^R(A, B)$. However, it need not be true that $\mathrm{Tor}_n^R(A, B) = 0$ for $n > 1$. The reason this was true when $R = \mathbb{Z}$ was that subgroups of free groups are free, but submodules of free R-modules need not be free in general. If R is a principal ideal domain, submodules of free R-modules are free, so in this case the rest of the algebra, in particular the universal coefficient theorem, goes through without change. When R is a field F, every module is free and $\mathrm{Tor}_n^F(A, B) = 0$ for $n > 0$ via the free resolution $0 \to A \to A \to 0$. Thus $H_n(C \otimes_F G) \approx H_n(C) \otimes_F G$ if F is a field.

Exercises

1. Use the universal coefficient theorem to show that if $H_*(X; \mathbb{Z})$ is finitely generated, so the Euler characteristic $\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X; \mathbb{Z})$ is defined, then for any coefficient field F we have $\chi(X) = \sum_n (-1)^n \dim H_n(X; F)$.

2. Show that $\mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z})$ is isomorphic to the torsion subgroup of A. Deduce that A is torsionfree iff $\mathrm{Tor}(A, B) = 0$ for all B.

3. Show that if $\widetilde{H}^n(X; \mathbb{Q})$ and $\widetilde{H}^n(X; \mathbb{Z}_p)$ are zero for all n and all primes p, then $\widetilde{H}_n(X; \mathbb{Z}) = 0$ for all n, and hence $\widetilde{H}^n(X; G) = 0$ for all G and n.

4. Show that $\otimes$ and Tor commute with direct limits: $(\varinjlim A_\alpha) \otimes B = \varinjlim (A_\alpha \otimes B)$ and $\mathrm{Tor}(\varinjlim A_\alpha, B) = \varinjlim \mathrm{Tor}(A_\alpha, B)$.

5. From the fact that $\mathrm{Tor}(A, B) = 0$ if A is free, deduce that $\mathrm{Tor}(A, B) = 0$ if A is torsionfree by applying the previous problem to the directed system of finitely generated subgroups A_α of A.

6. Show that $\mathrm{Tor}(A, B)$ is always a torsion group, and that $\mathrm{Tor}(A, B)$ contains an element of order n iff both A and B contain elements of order n.

3.B The General Künneth Formula

Künneth formulas describe the homology or cohomology of a product space in terms of the homology or cohomology of the factors. In nice cases these formulas take the form $H_*(X\times Y;R) \approx H_*(X;R)\otimes H_*(Y;R)$ or $H^*(X\times Y;R) \approx H^*(X;R)\otimes H^*(Y;R)$ for a coefficient ring R. For the case of cohomology, such a formula was given in Theorem 3.16, with hypotheses of finite generation and freeness on the cohomology of one factor. To obtain a completely general formula without these hypotheses it turns out that homology is more natural than cohomology, and the main aim in this section is to derive the general Künneth formula for homology. The new feature of the general case is that an extra Tor term is needed to describe the full homology of a product.

The Cross Product in Homology

A major component of the Künneth formula is a **cross product** map

$$H_i(X;R)\times H_j(Y;R) \xrightarrow{\times} H_{i+j}(X\times Y;R)$$

There are two ways to define this. One is a direct definition for singular homology, involving explicit simplicial formulas. More enlightening, however, is the definition in terms of cellular homology. This necessitates assuming X and Y are CW complexes, but this hypothesis can later be removed by the technique of CW approximation in §4.1. We shall focus therefore on the cellular definition, leaving the simplicial definition to later in this section for those who are curious to see how it goes.

The key ingredient in the definition of the cellular cross product will be the fact that the cellular boundary map satisfies $d(e^i\times e^j) = de^i\times e^j + (-1)^i e^i\times de^j$. Implicit in the right side of this formula is the convention of treating the symbol $\times$ as a bilinear operation on cellular chains. With this convention we can then say more generally that $d(a\times b) = da\times b + (-1)^i a\times db$ whenever a is a cellular i-chain and b is a cellular j-chain. From this formula it is obvious that the cross product of two cycles is a cycle. Also, the product of a boundary and a cycle is a boundary since $da\times b = d(a\times b)$ if $db = 0$, and similarly $a\times db = (-1)^i d(a\times b)$ if $da = 0$. Hence there is an induced homomorphism $H_i(X;R)\times H_j(Y;R)\to H_{i+j}(X\times Y;R)$, which is by definition the cross product in cellular homology. Since it is bilinear, it could also be viewed as a homomorphism $H_i(X;R)\otimes_R H_j(Y;R)\to H_{i+j}(X\times Y;R)$. In either form, this cross product turns out to be independent of the cell structures on X and Y.

Our task then is to express the boundary maps in the cellular chain complex $C_*(X\times Y)$ for $X\times Y$ in terms of the boundary maps in the cellular chain complexes $C_*(X)$ and $C_*(Y)$. For simplicity we consider homology with $\mathbb{Z}$ coefficients here, but the same formula for arbitrary coefficients follows immediately from this special case. With $\mathbb{Z}$ coefficients, the cellular chain group $C_i(X)$ is free with basis the i-cells of X, but there is a sign ambiguity for the basis element corresponding to each cell e^i, namely the choice of a generator for the $\mathbb{Z}$ summand of $H_i(X^i, X^{i-1})$ corresponding to e^i. Only when $i = 0$ is this choice canonical. We refer to these choices as 'choosing orientations for the cells.' A choice of such orientations allows cellular i-chains to be written unambiguously as linear combinations of i-cells.

The formula $d(e^i \times e^j) = de^i \times e^j + (-1)^i e^i \times de^j$ is not completely canonical since it contains the sign $(-1)^i$ but not $(-1)^j$. Evidently there is some distinction being made between the two factors of $e^i \times e^j$. Since the signs arise from orientations, we need to make explicit how an orientation of cells e^i and e^j determines an orientation of $e^i \times e^j$. Via characteristic maps, orientations can be obtained from orientations of the domain disks of the characteristic maps. It will be convenient to choose these domains to be cubes since the product of two cubes is again a cube. Thus for a cell e^i_α we take a characteristic map $\Phi_\alpha : I^i \to X$ where I^i is the product of i intervals $[0,1]$. An orientation of I^i is a generator of $H_i(I^i, \partial I^i)$, and the image of this generator under $\Phi_{\alpha *}$ gives an orientation of e^i_α. We can identify $H_i(I^i, \partial I^i)$ with $H_i(I^i, I^i - \{x\})$ for any point x in the interior of I^i, and then an orientation is determined by a linear embedding $\Delta^i \to I^i$ with x chosen in the interior of the image of this embedding. The embedding is determined by its sequence of vertices $v_0, \cdots, v_i$. The vectors $v_1 - v_0, \cdots, v_i - v_0$ are linearly independent in I^i, thought of as the unit cube in $\mathbb{R}^i$, so an orientation in our sense is equivalent to an orientation in the sense of linear algebra, that is, an equivalence class of ordered bases, two ordered bases being equivalent if they differ by a linear transformation of positive determinant. (An ordered basis can be continuously deformed to an orthonormal basis, by the Gram–Schmidt process, and two orthonormal bases are related either by a rotation or a rotation followed by a reflection, according to the sign of the determinant of the transformation taking one to the other.)

With this in mind, we adopt the convention that an orientation of $I^i \times I^j = I^{i+j}$ is obtained by choosing an ordered basis consisting of an ordered basis for I^i followed by an ordered basis for I^j. Notice that reversing the orientation for either I^i or I^j then reverses the orientation for I^{i+j}, so all that really matters is the order of the two factors of $I^i \times I^j$.

Proposition 3B.1. *The boundary map in the cellular chain complex $C_*(X \times Y)$ is determined by the boundary maps in the cellular chain complexes $C_*(X)$ and $C_*(Y)$ via the formula $d(e^i \times e^j) = de^i \times e^j + (-1)^i e^i \times de^j$.*

Proof: Let us first consider the special case of the cube I^n. We give I the CW structure with two vertices and one edge, so the i^{th} copy of I has a 1-cell e_i and 0-cells 0_i and 1_i, with $de_i = 1_i - 0_i$. The n-cell in the product I^n is $e_1 \times \cdots \times e_n$, and we claim that the boundary of this cell is given by the formula

$$(*) \qquad d(e_1 \times \cdots \times e_n) = \sum_i (-1)^{i+1} e_1 \times \cdots \times de_i \times \cdots \times e_n$$

This formula is correct modulo the signs of the individual terms $e_1\times\cdots\times 0_i\times\cdots\times e_n$ and $e_1\times\cdots\times 1_i\times\cdots\times e_n$ since these are exactly the $(n-1)$-cells in the boundary sphere ∂I^n of I^n. To obtain the signs in $(*)$, note that switching the two ends of an I factor of I^n produces a reflection of ∂I^n, as does a transposition of two adjacent I factors. Since reflections have degree -1, this implies that $(*)$ is correct up to an overall sign. This final sign can be determined by looking at any term, say the term $0_1\times e_2\times\cdots\times e_n$, which has a minus sign in $(*)$. To check that this is right, consider the n-simplex $[v_0,\cdots,v_n]$ with v_0 at the origin and v_k the unit vector along the k^{th} coordinate axis for $k>0$. This simplex defines the 'positive' orientation of I^n as described earlier, and in the usual formula for its boundary the face $[v_0,v_2,\cdots,v_n]$, which defines the positive orientation for the face $0_1\times e_2\times\cdots\times e_n$ of I^n, has a minus sign.

If we write $I^n=I^i\times I^j$ with $i+j=n$ and we set $e^i=e_1\times\cdots\times e_i$ and $e^j=e_{i+1}\times\cdots\times e_n$, then the formula $(*)$ becomes $d(e^i\times e^j)=de^i\times e^j+(-1)^ie^i\times de^j$. We will use naturality to reduce the general case of the boundary formula to this special case. When dealing with cellular homology, the maps $f:X\to Y$ that induce chain maps $f_*:C_*(X)\to C_*(Y)$ of the cellular chain complexes are the *cellular maps*, taking X^n to Y^n for all n, hence (X^n,X^{n-1}) to (Y^n,Y^{n-1}). The naturality statement we want is then:

Lemma 3B.2. *For cellular maps $f:X\to Z$ and $g:Y\to W$, the cellular chain maps $f_*:C_*(X)\to C_*(Z)$, $g_*:C_*(Y)\to C_*(W)$, and $(f\times g)_*:C_*(X\times Y)\to C_*(Z\times W)$ are related by the formula $(f\times g)_*=f_*\times g_*$.*

Proof: The relation $(f\times g)_*=f_*\times g_*$ means that if $f_*(e^i_\alpha)=\sum_\gamma m_{\alpha\gamma}e^i_\gamma$ and if $g_*(e^j_\beta)=\sum_\delta n_{\beta\delta}e^j_\delta$, then $(f\times g)_*(e^i_\alpha\times e^j_\beta)=\sum_{\gamma\delta}m_{\alpha\gamma}n_{\beta\delta}(e^i_\gamma\times e^j_\delta)$. The coefficient $m_{\alpha\gamma}$ is the degree of the composition $f_{\alpha\gamma}:S^i\to X^i/X^{i-1}\to Z^i/Z^{i-1}\to S^i$ where the first and third maps are induced by characteristic maps for the cells e^i_α and e^i_γ, and the middle map is induced by the cellular map f. With the natural choices of basepoints in these quotient spaces, $f_{\alpha\gamma}$ is basepoint-preserving. The $n_{\beta\delta}$'s are obtained similarly from maps $g_{\beta\delta}:S^j\to S^j$. For $f\times g$, the map $(f\times g)_{\alpha\beta,\gamma\delta}:S^{i+j}\to S^{i+j}$ whose degree is the coefficient of $e^i_\gamma\times e^j_\delta$ in $(f\times g)_*(e^i_\alpha\times e^j_\beta)$ is obtained from the product map $f_{\alpha\gamma}\times g_{\beta\delta}:S^i\times S^j\to S^i\times S^j$ by collapsing the $(i+j-1)$-skeleton of $S^i\times S^j$ to a point. In other words, $(f\times g)_{\alpha\beta,\gamma\delta}$ is the smash product map $f_{\alpha\gamma}\wedge g_{\beta\delta}$. What we need to show is the formula $\deg(f\wedge g)=\deg(f)\deg(g)$ for basepoint-preserving maps $f:S^i\to S^i$ and $g:S^j\to S^j$.

Since $f\wedge g$ is the composition of $f\wedge \mathbb{1}$ and $\mathbb{1}\wedge g$, it suffices to show that $\deg(f\wedge\mathbb{1})=\deg(f)$ and $\deg(\mathbb{1}\wedge g)=\deg(g)$. We do this by relating smash products to suspension. The smash product $X\wedge S^1$ can be viewed as $X\times I/(X\times\partial I\cup\{x_0\}\times I)$, so it is the reduced suspension ΣX, the quotient of the ordinary suspension SX obtained by collapsing the segment $\{x_0\}\times I$ to a point. If X is a CW complex with x_0 a 0-cell, the quotient map $SX \to X \wedge S^1$ induces an isomorphism on homology since it collapses a contractible subcomplex to a point. Taking $X = S^i$, we have the commutative diagram at the right, and from the induced commutative diagram of homology groups H_{i+1} we deduce that Sf and $f \wedge 1\!\!1$ have the same degree. Since suspension preserves degree by Proposition 2.33, we conclude that $\deg(f \wedge 1\!\!1) = \deg(f)$. The $1\!\!1$ in this formula is the identity map on S^1, and by iteration we obtain the same result for $1\!\!1$ the identity map on S^j since S^j is the smash product of j copies of S^1. This implies also that $\deg(1\!\!1 \wedge g) = \deg(g)$ since a permutation of coordinates in S^{i+j} does not affect the degree of maps $S^{i+j} \to S^{i+j}$. $\square$

$$\begin{array}{ccc} S(S^i) & \xrightarrow{Sf} & S(S^i) \\ \downarrow & & \downarrow \\ S^i \wedge S^1 & \xrightarrow{f \wedge 1\!\!1} & S^i \wedge S^1 \end{array}$$

Now to finish the proof of the proposition, let $\Phi : I^i \to X^i$ and $\Psi : I^j \to Y^j$ be characteristic maps of cells $e^i_\alpha \subset X$ and $e^j_\beta \subset Y$. The restriction of Φ to ∂I^i is the attaching map of e^i_α. We may perform a preliminary homotopy of this attaching map $\partial I^i \to X^{i-1}$ to make it cellular. There is no need to appeal to the cellular approximation theorem to do this since a direct argument is easy: First deform the attaching map so that it sends all but one face of I^i to a point, which is possible since the union of these faces is contractible, then do a further deformation so that the image point of this union of faces is a 0-cell. A homotopy of the attaching map $\partial I^i \to X^{i-1}$ does not affect the cellular boundary de^i_α, since de^i_α is determined by the induced map $H_{i-1}(\partial I^i) \to H_{i-1}(X^{i-1}) \to H_{i-1}(X^{i-1}, X^{i-2})$. So we may assume Φ is cellular, and likewise Ψ, hence also $\Phi \times \Psi$. The map of cellular chain complexes induced by a cellular map between CW complexes is a chain map, commuting with the cellular boundary maps.

If e^i is the i-cell of I^i and e^j the j-cell of I^j, then $\Phi_*(e^i) = e^i_\alpha$, $\Psi_*(e^j) = e^j_\beta$, and $(\Phi \times \Psi)_*(e^i \times e^j) = e^i_\alpha \times e^j_\beta$, hence

$$\begin{aligned} d(e^i_\alpha \times e^j_\beta) &= d\big((\Phi\times\Psi)_*(e^i \times e^j)\big) \\ &= (\Phi\times\Psi)_* d(e^i \times e^j) && \text{since } (\Phi\times\Psi)_* \text{ is a chain map} \\ &= (\Phi\times\Psi)_*(de^i \times e^j + (-1)^i e^i \times de^j) && \text{by the special case} \\ &= \Phi_*(de^i)\times\Psi_*(e^j) + (-1)^i \Phi_*(e^i)\times\Psi_*(de^j) && \text{by the lemma} \\ &= d\Phi_*(e^i)\times\Psi_*(e^j) + (-1)^i \Phi_*(e^i)\times d\Psi_*(e^j) && \text{since } \Phi_* \text{ and } \Psi_* \text{ are chain maps} \\ &= de^i_\alpha \times e^j_\beta + (-1)^i e^i_\alpha \times de^j_\beta \end{aligned}$$

which completes the proof of the proposition. $\square$

Example 3B.3. Consider $X \times S^k$ where we give S^k its usual CW structure with two cells. The boundary formula in $C_*(X \times S^k)$ takes the form $d(a \times b) = da \times b$ since $d = 0$ in $C_*(S^k)$. So the chain complex $C_*(X \times S^k)$ is just the direct sum of two copies of the chain complex $C_*(X)$, one of the copies having its dimension shifted upward by k. Hence $H_n(X\times S^k;\mathbb{Z}) \approx H_n(X;\mathbb{Z})\oplus H_{n-k}(X;\mathbb{Z})$ for all i. In particular, we see that all the homology classes in $X\times S^k$ are cross products of homology classes in X and S^k.

Example 3B.4. More subtle things can happen when X and Y both have torsion in their homology. To take the simplest case, let X be S^1 with a cell e^2 attached by a map $S^1\to S^1$ of degree m, so $H_1(X;\mathbb{Z}) \approx \mathbb{Z}_m$ and $H_i(X;\mathbb{Z}) = 0$ for $i > 1$. Similarly, let Y be obtained from S^1 by attaching a 2-cell by a map of degree n. Thus X and Y each have CW structures with three cells and so $X\times Y$ has nine cells. These are indicated by the dots in the diagram at the right, with X in the horizontal direction and Y in the vertical direction. The arrows denote the nonzero cellular boundary maps. For example the two arrows leaving the dot in the upper right corner indicate that $\partial(e^2\times e^2) = m(e^1\times e^2) + n(e^2\times e^1)$. Obviously

$$
\begin{tikzcd}
e^2 & \bullet \arrow[d, "n"] & \bullet \arrow[d, "-n"] & \bullet \arrow[l, "m"'] \arrow[d, "n"] \\
e^1 & \bullet & \bullet & \bullet \arrow[l, "m"'] \\
e^0 & \bullet & \bullet & \bullet \arrow[l, "m"'] \\
 & e^0 & e^1 & e^2
\end{tikzcd}
$$

$H_1(X\times Y;\mathbb{Z})$ is $\mathbb{Z}_m\oplus\mathbb{Z}_n$. In dimension 2, $\operatorname{Ker}\partial$ is generated by $e^1\times e^1$, and the image of the boundary map from dimension 3 consists of the multiples $(\ell m - kn)(e^1\times e^1)$. These form a cyclic group generated by $q(e^1\times e^1)$ where q is the greatest common divisor of m and n, so $H_2(X\times Y;\mathbb{Z}) \approx \mathbb{Z}_q$. In dimension 3 the cycles are the multiples of $(m/q)(e^1\times e^2) + (n/q)(e^2\times e^1)$, and the smallest such multiple that is a boundary is $q[(m/q)(e^1\times e^2) + (n/q)(e^2\times e^1)] = m(e^1\times e^2) + n(e^2\times e^1)$, so $H_3(X\times Y;\mathbb{Z}) \approx \mathbb{Z}_q$. Since X and Y have no homology above dimension 1, this 3-dimensional homology of $X\times Y$ cannot be realized by cross products. As the general theory will show, $H_2(X\times Y;\mathbb{Z})$ is $H_1(X;\mathbb{Z})\otimes H_1(Y;\mathbb{Z})$ and $H_3(X\times Y;\mathbb{Z})$ is $\operatorname{Tor}(H_1(X;\mathbb{Z}),H_1(Y;\mathbb{Z}))$.

This example generalizes easily to higher dimensions, with $X = S^i\cup e^{i+1}$ and $Y = S^j\cup e^{j+1}$, the attaching maps having degrees m and n, respectively. Essentially the same calculation shows that $X\times Y$ has both H_{i+j} and H_{i+j+1} isomorphic to $\mathbb{Z}_q$.

We should say a few words about why the cross product is independent of CW structures. For this we will need a fact proved in the next chapter in Theorem 4.8, that every map between CW complexes is homotopic to a cellular map. As we mentioned earlier, a cellular map induces a chain map between cellular chain complexes. It is easy to see from the equivalence between cellular and singular homology that the map on cellular homology induced by a cellular map is the same as the map induced on singular homology. Now suppose we have cellular maps $f:X\to Z$ and $g:Y\to W$. Then Lemma 3B.2 implies that we have a commutative diagram

$$
\begin{tikzcd}
H_i(X;\mathbb{Z}) \times H_j(Y;\mathbb{Z}) \arrow[r, "\times"] \arrow[d, "f_*\times g_*"] & H_{i+j}(X\times Y;\mathbb{Z}) \arrow[d, "(f\times g)_*"] \\
H_i(Z;\mathbb{Z}) \times H_j(W;\mathbb{Z}) \arrow[r, "\times"] & H_{i+j}(Z\times W;\mathbb{Z})
\end{tikzcd}
$$

Now take Z and W to be the same spaces as X and Y but with different CW structures, and let f and g be cellular maps homotopic to the identity. The vertical maps in the diagram are then the identity, and commutativity of the diagram says that the cross products defined using the different CW structures coincide.

Cross product is obviously bilinear, or in other words, distributive. It is not hard to check that it is also associative. What about commutativity? If $T: X\times Y \to Y\times X$ is transposition of the factors, then we can ask whether $T_*(a\times b)$ equals $b\times a$. The only effect transposing the factors has on the definition of cross product is in the convention for orienting a product $I^i\times I^j$ by taking an ordered basis in the first factor followed by an ordered basis in the second factor. Switching the two factors can be achieved by moving each of the i coordinates of I^i past each of the coordinates of I^j. This is a total of ij transpositions of adjacent coordinates, each realizable by a reflection, so a sign of $(-1)^{ij}$ is introduced. Thus the correct formula is $T_*(a\times b) = (-1)^{ij} b\times a$ for $a\in H_i(X)$ and $b \in H_j(Y)$.

The Algebraic Künneth Formula

By adding together the various cross products we obtain a map

$$\textstyle\bigoplus_i \big(H_i(X;\mathbb{Z})\otimes H_{n-i}(Y;\mathbb{Z})\big) \longrightarrow H_n(X\times Y;\mathbb{Z})$$

and it is natural to ask whether this is an isomorphism. Example 3B.4 above shows that this is not always the case, though it is true in Example 3B.3. Our main goal in what follows is to show that the map is always injective, and that its cokernel is $\bigoplus_i \mathrm{Tor}\big(H_i(X;\mathbb{Z}), H_{n-i-1}(Y;\mathbb{Z})\big)$. More generally, we consider other coefficients besides $\mathbb{Z}$ and show in particular that with field coefficients the map is an isomorphism.

For CW complexes X and Y, the relationship between the cellular chain complexes $C_*(X)$, $C_*(Y)$, and $C_*(X\times Y)$ can be expressed nicely in terms of tensor products. Since the n-cells of $X\times Y$ are the products of i-cells of X with $(n-i)$-cells of Y, we have $C_n(X\times Y) \approx \bigoplus_i \big(C_i(X)\otimes C_{n-i}(Y)\big)$, with $e^i\times e^j$ corresponding to $e^i\otimes e^j$. Under this identification the boundary formula of Proposition 3B.1 becomes $d(e^i\otimes e^j) = de^i\otimes e^j + (-1)^i e^i\otimes de^j$. Our task now is purely algebraic, to compute the homology of the chain complex $C_*(X\times Y)$ from the homology of $C_*(X)$ and $C_*(Y)$.

Suppose we are given chain complexes C and C' of abelian groups C_n and C'_n, or more generally R-modules over a commutative ring R. The **tensor product chain complex** $C\otimes_R C'$ is then defined by $(C\otimes_R C')_n = \bigoplus_i (C_i\otimes_R C'_{n-i})$, with boundary maps given by $\partial(c\otimes c') = \partial c\otimes c' + (-1)^i c\otimes \partial c'$ for $c\in C_i$ and $c' \in C'_{n-i}$. The sign $(-1)^i$ guarantees that $\partial^2 = 0$ in $C\otimes_R C'$, since

$$\begin{aligned}\partial^2(c\otimes c') &= \partial\big(\partial c\otimes c' + (-1)^i c\otimes\partial c'\big)\\ &= \partial^2 c\otimes c' + (-1)^{i-1}\partial c\otimes\partial c' + (-1)^i\partial c\otimes \partial c' + (-1)^i c\otimes\partial^2 c' = 0\end{aligned}$$

From the boundary formula $\partial(c\otimes c') = \partial c\otimes c' + (-1)^i c\otimes\partial c'$ it follows that the tensor product of cycles is a cycle, and the tensor product of a cycle and a boundary, in either order, is a boundary, just as for the cross product defined earlier. So there is induced a natural map on homology groups $H_i(C)\otimes_R H_{n-i}(C') \to H_n(C\otimes_R C')$. Summing over i

then gives a map $\bigoplus_i (H_i(C)\otimes_R H_{n-i}(C')) \to H_n(C\otimes_R C')$. This figures in the following algebraic version of the Künneth formula:

Theorem 3B.5. *If R is a principal ideal domain and the R-modules C_i are free, then for each n there is a natural short exact sequence*

$$0\to\bigoplus_i (H_i(C)\otimes_R H_{n-i}(C'))\to H_n(C\otimes_R C')\to\bigoplus_i (\mathrm{Tor}_R(H_i(C),H_{n-i-1}(C'))\to 0$$

and this sequence splits.

This is a generalization of the universal coefficient theorem for homology, which is the case that C' consists of just the coefficient group G in dimension zero. The proof will also be a natural generalization of the proof of the universal coefficient theorem.

Proof: First we do the special case that the boundary maps in C are all zero, so $H_i(C) = C_i$. In this case $\partial(c\otimes c') = (-1)^i c\otimes\partial c'$ and the chain complex $C\otimes_R C'$ is simply the direct sum of the complexes $C_i\otimes_R C'$, each of which is a direct sum of copies of C' since C_i is free. Hence $H_n(C_i\otimes_R C') \approx C_i\otimes_R H_{n-i}(C') = H_i(C)\otimes_R H_{n-i}(C')$. Summing over i yields an isomorphism $H_n(C\otimes_R C')\approx\bigoplus_i (H_i(C)\otimes_R H_{n-i}(C'))$, which is the statement of the theorem since there are no Tor terms, $H_i(C) = C_i$ being free.

In the general case, let $Z_i\subset C_i$ and $B_i\subset C_i$ denote kernel and image of the boundary homomorphisms for C. These give subchain complexes Z and B of C with trivial boundary maps. We have a short exact sequence of chain complexes $0\to Z\to C\to B\to 0$ made up of the short exact sequences $0\to Z_i\to C_i\xrightarrow{\partial} B_{i-1}\to 0$ each of which splits since B_{i-1} is free, being a submodule of C_{i-1} which is free by assumption. Because of the splitting, when we tensor $0\to Z\to C\to B\to 0$ with C' we obtain another short exact sequence of chain complexes, and hence a long exact sequence in homology

$$\cdots\longrightarrow H_n(Z\otimes_R C')\longrightarrow H_n(C\otimes_R C')\longrightarrow H_{n-1}(B\otimes_R C')\longrightarrow H_{n-1}(Z\otimes_R C')\longrightarrow\cdots$$

where we have $H_{n-1}(B\otimes_R C')$ instead of the expected $H_n(B\otimes_R C')$ since $\partial : C\to B$ decreases dimension by one. Checking definitions, one sees that the 'boundary' map $H_{n-1}(B\otimes_R C')\to H_{n-1}(Z\otimes_R C')$ in the preceding long exact sequence is just the map induced by the natural map $B\otimes_R C'\to Z\otimes_R C'$ coming from the inclusion $B\subset Z$.

Since Z and B are chain complexes with trivial boundary maps, the special case at the beginning of the proof converts the preceding exact sequence into

$$\cdots\xrightarrow{i_n}\bigoplus_i (Z_i\otimes_R H_{n-i}(C'))\longrightarrow H_n(C\otimes_R C')\longrightarrow\bigoplus_i (B_i\otimes_R H_{n-i-1}(C'))\xrightarrow{i_{n-1}}$$
$$\bigoplus_i (Z_i\otimes_R H_{n-i-1}(C'))\longrightarrow\cdots$$

So we have short exact sequences

$$0\longrightarrow \mathrm{Coker}\, i_n\longrightarrow H_n(C\otimes_R C')\longrightarrow \mathrm{Ker}\, i_{n-1}\longrightarrow 0$$

where $\operatorname{Coker} i_n = \bigoplus_i (Z_i \otimes_R H_{n-i}(C'))/\operatorname{Im} i_n$, and this equals $\bigoplus_i (H_i(C) \otimes_R H_{n-i}(C'))$ by Lemma 3A.1. It remains to identify $\operatorname{Ker} i_{n-1}$ with $\bigoplus_i \operatorname{Tor}_R(H_i(C), H_{n-i}(C'))$.

By the definition of Tor, tensoring the free resolution $0 \to B_i \to Z_i \to H_i(C) \to 0$ with $H_{n-i}(C')$ yields an exact sequence

$$0 \longrightarrow \operatorname{Tor}_R(H_i(C), H_{n-i}(C')) \longrightarrow B_i \otimes_R H_{n-i}(C') \longrightarrow Z_i \otimes_R H_{n-i}(C') \longrightarrow H_i(C) \otimes_R H_{n-i}(C') \longrightarrow 0$$

Hence, summing over i, $\operatorname{Ker} i_n = \bigoplus_i \operatorname{Tor}_R(H_i(C), H_{n-i}(C'))$.

Naturality should be obvious, and we leave it for the reader to fill in the details.

We will show that the short exact sequence in the statement of the theorem splits assuming that both C and C' are free. This suffices for our applications. For the extra argument needed to show splitting when C' is not free, see the exposition in [Hilton & Stammbach 1970].

The splitting is via a homomorphism $H_n(C \otimes_R C') \to \bigoplus_i (H_i(C) \otimes_R H_{n-i}(C'))$ constructed in the following way. As already noted, the sequence $0 \to Z_i \to C_i \to B_{i-1} \to 0$ splits, so the quotient maps $Z_i \to H_i(C)$ extend to homomorphisms $C_i \to H_i(C)$. Similarly we obtain $C'_j \to H_j(C')$ if C' is free. Viewing the sequences of homology groups $H_i(C)$ and $H_j(C')$ as chain complexes $H(C)$ and $H(C')$ with trivial boundary maps, we thus have chain maps $C \to H(C)$ and $C' \to H(C')$, whose tensor product is a chain map $C \otimes_R C' \to H(C) \otimes_R H(C')$. The induced map on homology for this last chain map is the desired splitting map since the chain complex $H(C) \otimes_R H(C')$ equals its own homology, the boundary maps being trivial. $\square$

The Topological Künneth Formula

Now we can apply the preceding algebra to obtain the topological statement we are looking for:

Theorem 3B.6. *If X and Y are CW complexes and R is a principal ideal domain, then there are natural short exact sequences*

$$0 \longrightarrow \bigoplus_i (H_i(X;R) \otimes_R H_{n-i}(Y;R)) \longrightarrow H_n(X \times Y;R) \longrightarrow \bigoplus_i \operatorname{Tor}_R(H_i(X;R), H_{n-i-1}(Y;R)) \longrightarrow 0$$

and these sequences split.

Naturality means that maps $X \to X'$ and $Y \to Y'$ induce a map from the short exact sequence for $X \times Y$ to the corresponding short exact sequence for $X' \times Y'$, with commuting squares. The splitting is not natural, however, as an exercise at the end of this section demonstrates.

Proof: When dealing with products of CW complexes there is always the bothersome fact that the compactly generated CW topology may not be the same as the product topology. However, in the present context this is not a real problem. Since the two topologies have the same compact sets, they have the same singular simplices and hence the same singular homology groups.

Let $C = C_*(X;R)$ and $C' = C_*(Y;R)$, the cellular chain complexes with coefficients in R. Then $C \otimes_R C' = C_*(X \times Y;R)$ by Proposition 3B.1, so the algebraic Künneth formula gives the desired short exact sequences. Their naturality follows from naturality in the algebraic Künneth formula, since we can homotope arbitrary maps $X \to X'$ and $Y \to Y'$ to be cellular by Theorem 4.8, assuring that they induce chain maps of cellular chain complexes. $\square$

With field coefficients the Künneth formula simplifies because the Tor terms are always zero over a field:

Corollary 3B.7. *If F is a field and X and Y are CW complexes, then the cross product map $h : \bigoplus_i (H_i(X;F) \otimes_F H_{n-i}(Y;F)) \longrightarrow H_n(X \times Y;F)$ is an isomorphism for all n.* $\square$

There is also a relative version of the Künneth formula for CW pairs (X,A) and (Y,B). This is a split short exact sequence

$$0 \longrightarrow \bigoplus_i (H_i(X,A;R) \otimes_R H_{n-i}(Y,B;R)) \longrightarrow H_n(X \times Y, A \times Y \cup X \times B;R) \longrightarrow \bigoplus_i \mathrm{Tor}_R(H_i(X,A;R), H_{n-i-1}(Y,B;R)) \longrightarrow 0$$

for R a principal ideal domain. This too follows from the algebraic Künneth formula since the isomorphism of cellular chain complexes $C_*(X \times Y) \approx C_*(X) \otimes C_*(Y)$ passes down to a quotient isomorphism

$$C_*(X \times Y)/C_*(A \times Y \cup X \times B) \approx C_*(X)/C_*(A) \otimes C_*(Y)/C_*(B)$$

since bases for these three relative cellular chain complexes correspond bijectively with the cells of $(X - A) \times (Y - B)$, $X - A$, and $Y - B$, respectively.

As a special case, suppose A and B are basepoints $x_0 \in X$ and $y_0 \in Y$. Then the subcomplex $A \times Y \cup X \times B$ can be identified with the wedge sum $X \vee Y$ and the quotient $X \times Y / X \vee Y$ is the smash product $X \wedge Y$. Thus we have a reduced Künneth formula

$$0 \longrightarrow \bigoplus_i (\widetilde{H}_i(X;R) \otimes_R \widetilde{H}_{n-i}(Y;R)) \longrightarrow \widetilde{H}_n(X \wedge Y;R) \longrightarrow \bigoplus_i \mathrm{Tor}_R(\widetilde{H}_i(X;R), \widetilde{H}_{n-i-1}(Y;R)) \longrightarrow 0$$

If we take $Y = S^k$ for example, then $X \wedge S^k$ is the k-fold reduced suspension of X, and we obtain isomorphisms $\widetilde{H}_n(X;\mathbb{Z}) \approx \widetilde{H}_{n+k}(X \wedge S^k;\mathbb{Z})$. More generally, by taking Y to be a Moore space $M(G,k)$ and then applying the universal coefficient theorem we obtain:

Corollary 3B.2. *There are natural isomorphisms $\widetilde{H}_n(X;G) \approx \widetilde{H}_{n+k}(X \wedge M(G,k);\mathbb{Z})$ for all CW complexes X and abelian groups G.* $\square$

This says that homology with arbitrary coefficients is obtainable from homology with $\mathbb{Z}$ coefficients by a geometric construction as well as by the algebra of tensor products. For general homology theories this formula can be used as a definition of homology with coefficients.

The Künneth formula and the universal coefficient theorem can be combined to give a more concise formula $H_n(X\times Y;G) \approx \bigoplus_i H_i(X;H_{n-i}(Y;G))$, at least when $G=\mathbb{Z}$. In fact, with a little more algebra one can show that this formula is valid for arbitrary coefficient groups G; see [Hilton & Wylie 1967], p. 227, or [Spanier 1966], p. 235. However the naturality of this isomorphism is problematic since it uses the splittings in the Künneth formulas and universal coefficient theorems.

One might wonder about a cohomology version of the Künneth formula. Taking coefficients in a field F and using the natural isomorphism $\mathrm{Hom}(A\otimes B,C) \approx \mathrm{Hom}(A,\mathrm{Hom}(B,C))$, the Künneth formula for homology and the universal coefficient theorem give isomorphisms

$$\begin{aligned} H^n(X\times Y;F) &\approx \mathrm{Hom}_F(H_n(X\times Y;F),F) \approx \textstyle\bigoplus_i \mathrm{Hom}_F(H_i(X;F)\otimes H_{n-i}(Y;F),F) \\ &\approx \textstyle\bigoplus_i \mathrm{Hom}_F(H_i(X;F),\mathrm{Hom}_F(H_{n-i}(Y;F),F)) \\ &\approx \textstyle\bigoplus_i \mathrm{Hom}_F(H_i(X;F),H^{n-i}(Y;F)) \\ &\approx \textstyle\bigoplus_i H^i(X;H^{n-i}(Y;F)) \end{aligned}$$

More generally, there are isomorphisms $H^n(X\times Y;G) \approx \bigoplus_i H^i(X;H^{n-i}(Y;G))$ for any coefficient group G; see [Hilton & Wylie 1967], p. 227. However, in practice it usually suffices to apply the Künneth formula for homology and the universal coefficient theorem for cohomology separately. Also, Theorem 3.16 shows that with stronger hypotheses one can draw stronger conclusions using cup products.

The Simplicial Cross Product

Let us sketch how the cross product $H_m(X;R)\otimes H_n(Y;R)\to H_{m+n}(X\times Y;R)$ can be defined directly in terms of singular homology. What one wants is a cross product at the level of singular chains, $C_m(X;R)\otimes C_n(Y;R)\to C_{m+n}(X\times Y;R)$. If we are given singular simplices $f:\Delta^m\to X$ and $g:\Delta^n\to Y$, then we have the product map $f\times g:\Delta^m\times\Delta^n\to X\times Y$, and the idea is to subdivide $\Delta^m\times\Delta^n$ into simplices of dimension $m+n$ and then take the sum of the restrictions of $f\times g$ to these simplices, with appropriate signs.

In the special cases that m or n is 1 we have already seen how to subdivide $\Delta^m\times\Delta^n$ into simplices when we constructed prism operators in §2.1. The generalization to $\Delta^m\times\Delta^n$ is not completely obvious, however. Label the vertices of Δ^m as $v_0,v_1,\cdots,v_m$ and the vertices of Δ^n as $w_0,w_1,\cdots,w_n$. Think of the pairs (i,j) with $0\le i\le m$ and $0\le j\le n$ as the vertices of an $m\times n$ rectangular grid in $\mathbb{R}^2$. Let σ be a path formed by a sequence of $m+n$ horizontal and vertical edges in this grid starting at $(0,0)$ and ending at (m,n), always moving either to the right or upward. To such a path σ we associate a linear map $\ell_\sigma:\Delta^{m+n}\to\Delta^m\times\Delta^n$ sending the k^{th} vertex of Δ^{m+n} to (v_{i_k},w_{j_k}) where (i_k,j_k) is the k^{th} vertex of the edgepath σ. Then we define a simplicial cross product

$$C_m(X;R)\otimes C_n(Y;R)\xrightarrow{\ \times\ } C_{m+n}(X\times Y;R)$$

by the formula

$$f\times g=\sum_\sigma(-1)^{|\sigma|}(f\times g)\ell_\sigma$$

where $|\sigma|$ is the number of squares in the grid lying below the path σ. Note that the symbol '$\times$' means different things on the two sides of the equation. From this definition it is a calculation to show that $\partial(f\times g)=\partial f\times g+(-1)^m f\times\partial g$. This implies that the cross product of two cycles is a cycle, and the cross product of a cycle and a boundary is a boundary, so there is an induced cross product in singular homology.

One can see that the images of the maps ℓ_σ give a simplicial structure on $\Delta^m\times\Delta^n$ in the following way. We can view Δ^m as the subspace of $\mathbb{R}^m$ defined by the inequalities $0\le x_1\le\cdots\le x_m\le 1$, with the vertex v_i as the point having coordinates $m-i$ zeros followed by i ones. Similarly we have $\Delta^n\subset\mathbb{R}^n$ with coordinates $0\le y_1\le\cdots\le y_n\le 1$. The product $\Delta^m\times\Delta^n$ then consists of $(m+n)$-tuples $(x_1,\cdots,x_m,y_1,\cdots,y_n)$ satisfying both sets of inequalities. The combined inequalities $0\le x_1\le\cdots\le x_m\le y_1\le\cdots\le y_n\le 1$ define a simplex Δ^{m+n} in $\Delta^m\times\Delta^n$, and every other point of $\Delta^m\times\Delta^n$ satisfies a similar set of inequalities obtained from $0\le x_1\le\cdots\le x_m\le y_1\le\cdots\le y_n\le 1$ by a permutation of the variables 'shuffling' the y_j's into the x_i's. Each such shuffle corresponds to an edgepath σ consisting of a rightward edge for each x_i and an upward edge for each y_j in the shuffled sequence. Thus we have $\Delta^m\times\Delta^n$ expressed as the union of simplices Δ^{m+n}_σ indexed by the edgepaths σ. One can check that these simplices fit together nicely to form a Δ-complex structure on $\Delta^m\times\Delta^n$, which is also a simplicial complex structure. See [Eilenberg & Steenrod 1952], p. 68. In fact this construction is sufficiently natural to make the product of any two Δ-complexes into a Δ-complex.

The Cohomology Cross Product

In §3.2 we defined a cross product

$$H^k(X;R)\times H^\ell(Y;R)\xrightarrow{\ \times\ } H^{k+\ell}(X\times Y;R)$$

in terms of the cup product. Let us now describe the alternative approach in which this cross product is defined directly via cellular cohomology, and then cup product is defined in terms of this cross product.

The cellular definition of cohomology cross product is very much like the definition in homology. Given CW complexes X and Y, define a cross product of cellular cochains $\varphi\in C^k(X;R)$ and $\psi\in C^\ell(Y;R)$ by setting

$$(\varphi\times\psi)(e^k_\alpha\times e^\ell_\beta)=\varphi(e^k_\alpha)\psi(e^\ell_\beta)$$

and letting $\varphi\times\psi$ take the value 0 on $(k+\ell)$-cells of $X\times Y$ which are not the product of a k-cell of X with an ℓ-cell of Y. Another way of saying this is to use the convention that a cellular cochain in $C^k(X;R)$ takes the value 0 on cells of dimension different from k, and then we can let $(\varphi\times\psi)(e^m_\alpha\times e^n_\beta)=\varphi(e^m_\alpha)\psi(e^n_\beta)$ for all m and n.

The cellular coboundary formula $\delta(\varphi\times\psi)=\delta\varphi\times\psi+(-1)^k\varphi\times\delta\psi$ for cellular cochains $\varphi\in C^k(X;R)$ and $\psi\in C^\ell(Y;R)$ follows easily from the corresponding boundary formula in Proposition 3B.1, namely

$$\begin{aligned}\delta(\varphi\times\psi)(e^m_\alpha\times e^n_\beta) &= (\varphi\times\psi)\big(\partial(e^m_\alpha\times e^n_\beta)\big)\\ &= (\varphi\times\psi)(\partial e^m_\alpha\times e^n_\beta+(-1)^m e^m_\alpha\times\partial e^n_\beta)\\ &= \delta\varphi(e^m_\alpha)\psi(e^n_\beta)+(-1)^m\varphi(e^m_\alpha)\delta\psi(e^n_\beta)\\ &= (\delta\varphi\times\psi+(-1)^k\varphi\times\delta\psi)(e^m_\alpha\times e^n_\beta)\end{aligned}$$

where the coefficient $(-1)^m$ in the next-to-last line can be replaced by $(-1)^k$ since $\varphi(e^m_\alpha)=0$ unless $k=m$. From the formula $\delta(\varphi\times\psi)=\delta\varphi\times\psi+(-1)^k\varphi\times\delta\psi$ it follows just as for homology and for cup product that there is an induced cross product in cellular cohomology.

To show this agrees with the earlier definition, we can first reduce to the case that X has trivial $(k-1)$-skeleton and Y has trivial $(\ell-1)$-skeleton via the commutative diagram

$$\begin{array}{ccc} H^k(X/X^{k-1};R)\times H^\ell(Y/Y^{\ell-1};R) & \xrightarrow{\ \times\ } & H^{k+\ell}(X/X^{k-1}\times Y/Y^{\ell-1};R)\\ \downarrow & & \downarrow\\ H^k(X;R)\times H^\ell(Y;R) & \xrightarrow{\ \times\ } & H^{k+\ell}(X\times Y;R)\end{array}$$

The left-hand vertical map is surjective, so by commutativity, if the two definitions of cross product agree in the upper row, they agree in the lower row. Next, assuming X^{k-1} and $Y^{\ell-1}$ are trivial, consider the commutative diagram

$$\begin{array}{ccc} H^k(X;R)\times H^\ell(Y;R) & \xrightarrow{\ \times\ } & H^{k+\ell}(X\times Y;R)\\ \downarrow & & \downarrow\\ H^k(X^k;R)\times H^\ell(Y^\ell;R) & \xrightarrow{\ \times\ } & H^{k+\ell}(X^k\times Y^\ell;R)\end{array}$$

The vertical maps here are injective, $X^k\times Y^\ell$ being the $(k+\ell)$-skeleton of $X\times Y$, so it suffices to see that the two definitions agree in the lower row. We have $X^k=\bigvee_\alpha S^k_\alpha$ and $Y^\ell=\bigvee_\beta S^\ell_\beta$, so by restriction to these wedge summands the question is reduced finally to the case of a product $S^k_\alpha\times S^\ell_\beta$. In this case, taking $R=\mathbb{Z}$, we showed in Theorem 3.16 that the cross product in question is the map $\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ sending $(1,1)$ to ±1, with the original definition of cross product. The same is obviously true using the cellular cross product. So for $R=\mathbb{Z}$ the two cross products agree up to sign, and it follows that this is also true for arbitrary R. We leave it to the reader to sort out the matter of signs.

To relate cross product to cup product we use the diagonal map $\Delta: X\to X\times X$, $x\mapsto(x,x)$. If we are given a definition of cross product, we can define cup product as the composition

$$H^k(X;R)\times H^\ell(X;R)\xrightarrow{\ \times\ }H^{k+\ell}(X\times X;R)\xrightarrow{\ \Delta^*\ }H^{k+\ell}(X;R)$$

This agrees with the original definition of cup product since we have $\Delta^*(a\times b) = \Delta^*(p_1^*(a)\smile p_2^*(b)) = \Delta^*(p_1^*(a))\smile\Delta^*(p_2^*(b)) = a\smile b$, as both compositions $p_1\Delta$ and $p_2\Delta$ are the identity map of X.

Unfortunately, the definition of cellular cross product cannot be combined with Δ to give a definition of cup product at the level of cellular cochains. This is because Δ is not a cellular map, so it does not induce a map of cellular cochains. It is possible to homotope Δ to a cellular map by Theorem 4.8, but this involves arbitrary choices. For example, the diagonal of a square can be pushed across either adjacent triangle. In particular cases one might hope to understand the geometry well enough to compute an explicit cellular approximation to the diagonal map, but usually other techniques for computing cup products are preferable.

The cohomology cross product satisfies the same commutativity relation as for homology, namely $T^*(a\times b) = (-1)^{k\ell}b\times a$ for $T:X\times Y\to Y\times X$ the transposition map, $a\in H^k(Y;R)$, and $b\in H^\ell(X;R)$. The proof is the same as for homology. Taking $X=Y$ and noting that $\Delta T=\Delta$, we obtain a new proof of the commutativity property of cup product.

Exercises

1. Compute the groups $H_i(\mathbb{R}\mathrm{P}^m\times\mathbb{R}\mathrm{P}^n;G)$ and $H^i(\mathbb{R}\mathrm{P}^m\times\mathbb{R}\mathrm{P}^n;G)$ for $G=\mathbb{Z}$ and $\mathbb{Z}_2$ via the cellular chain and cochain complexes. [See Example 3B.4.]

2. Let C and C' be chain complexes, and let I be the chain complex consisting of $\mathbb{Z}$ in dimension 1 and $\mathbb{Z}\times\mathbb{Z}$ in dimension 0, with the boundary map taking a generator e in dimension 1 to the difference v_1-v_0 of generators v_i of the two $\mathbb{Z}$'s in dimension 0. Show that a chain map $f:I\otimes C\to C'$ is precisely the same as a chain homotopy between the two chain maps $f_i:C\to C'$, $c\mapsto f(v_i\otimes c)$, $i=0,1$. [The chain homotopy is $h(c)=f(e\otimes c)$.]

3. Show that the splitting in the topological Künneth formula cannot be natural by considering the map $f\times 1\!\!1:M(\mathbb{Z}_m,n)\times M(\mathbb{Z}_m,n)\to S^{n+1}\times M(\mathbb{Z}_m,n)$ where f collapses the n-skeleton of $M(\mathbb{Z}_m,n)=S^n\cup e^{n+1}$ to a point.

4. Show that the cross product of fundamental classes for closed R-orientable manifolds M and N is a fundamental class for $M\times N$.

5. Show that **slant products**

$$H_n(X\times Y;R)\times H^j(Y;R)\longrightarrow H_{n-j}(Y;R),\quad (e^i\times e^j,\varphi)\mapsto \varphi(e^j)e^i$$
$$H^n(X\times Y;R)\times H_j(Y;R)\longrightarrow H^{n-j}(Y;R),\quad (\varphi,e^j)\mapsto (e^i\mapsto\varphi(e^i\times e^j))$$

can be defined via the indicated cellular formulas. [These 'products' are in some ways more like division than multiplication, and this is reflected in the common notation a/b for them, or $a\backslash b$ when the order of the factors is reversed. The first of the two slant products is related to cap product in the same way that the cohomology cross product is related to cup product.]

3.C H-Spaces and Hopf Algebras

Of the three axioms for a group, it would seem that the least subtle is the existence of an identity element. However, we shall see in this section that when topology is added to the picture, the identity axiom becomes much more potent. To give a name to the objects we will be considering, define a space X to be an **H–space**, 'H' standing for 'Hopf,' if there is a continuous multiplication map $\mu : X \times X \to X$ and an 'identity' element $e \in X$ such that the two maps $X \to X$ given by $x \mapsto \mu(x,e)$ and $x \mapsto \mu(e,x)$ are homotopic to the identity through maps $(X,e) \to (X,e)$. In particular, this implies that $\mu(e,e) = e$.

In terms of generality, this definition represents something of a middle ground. One could weaken the definition by dropping the condition that the homotopies preserve the basepoint e, or one could strengthen it by requiring that e be a strict identity, without any homotopies. An exercise at the end of the section is to show the three possible definitions are equivalent if X is a CW complex. An advantage of allowing homotopies in the definition is that a space homotopy equivalent in the basepointed sense to an H–space is again an H–space. Imposing basepoint conditions is fairly standard in homotopy theory, and is usually not a serious restriction.

The most classical examples of H–spaces are **topological groups**, spaces X with a group structure such that both the multiplication map $X \times X \to X$ and the inversion map $X \to X$, $x \mapsto x^{-1}$, are continuous. For example, the group $GL_n(\mathbb{R})$ of invertible $n \times n$ matrices with real entries is a topological group when topologized as a subspace of the n^2-dimensional vector space $M_n(\mathbb{R})$ of all $n \times n$ matrices over $\mathbb{R}$. It is an open subspace since the invertible matrices are those with nonzero determinant, and the determinant function $M_n(\mathbb{R}) \to \mathbb{R}$ is continuous. Matrix multiplication is certainly continuous, being defined by simple algebraic formulas, and it is not hard to see that matrix inversion is also continuous if one thinks for example of the classical adjoint formula for the inverse matrix.

Likewise $GL_n(\mathbb{C})$ is a topological group, as is the quaternionic analog $GL_n(\mathbb{H})$, though in the latter case one needs a somewhat different justification since determinants of quaternionic matrices do not have the good properties one would like. Since these groups GL_n over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ are open subsets of Euclidean spaces, they are examples of *Lie groups*, which can be defined as topological groups which are also manifolds. The GL_n groups are noncompact, being open subsets of Euclidean spaces, but they have the homotopy types of compact Lie groups called $O(n)$, $U(n)$, and $Sp(n)$, as we shall see in §3.D.

Among the simplest H–spaces from a topological viewpoint are the unit spheres S^1 in $\mathbb{C}$, S^3 in the quaternions $\mathbb{H}$, and S^7 in the octonions $\mathbb{O}$. These are H–spaces since the multiplications in these division algebras are continuous, being defined by polynomial formulas, and are norm-preserving, $|ab| = |a||b|$, hence restrict to multiplications on the unit spheres, and the identity element of the division algebra lies in the unit sphere in each case. Both S^1 and S^3 are Lie groups since the multiplications in $\mathbb{C}$ and $\mathbb{H}$ are associative and inverses exist since $a\overline{a} = |a|^2 = 1$ if $|a| = 1$. However, S^7 is not a group since multiplication of octonions is not associative. Of course $S^0 = \{\pm 1\}$ is also a topological group, trivially. A famous theorem of J. F. Adams asserts that S^0, S^1, S^3, and S^7 are the only spheres that are H–spaces; see §4.B for a fuller discussion.

Let us describe now some associative H–spaces where inverses fail to exist. Multiplication of polynomials provides an H–space structure on $\mathbb{C}\mathrm{P}^\infty$ in the following way. A nonzero polynomial $a_0 + a_1 z + \cdots + a_n z^n$ with coefficients $a_i \in \mathbb{C}$ corresponds to a point $(a_0, \cdots, a_n, 0, \cdots) \in \mathbb{C}^\infty - \{0\}$. Multiplication of two such polynomials determines a multiplication $\mathbb{C}^\infty - \{0\} \times \mathbb{C}^\infty - \{0\} \to \mathbb{C}^\infty - \{0\}$ which is associative, commutative, and has an identity element $(1, 0, \cdots)$. Since $\mathbb{C}$ is commutative we can factor out by scalar multiplication by nonzero constants and get an induced product $\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$ with the same properties. Thus $\mathbb{C}\mathrm{P}^\infty$ is an associative, commutative H–space with a strict identity. Instead of factoring out by all nonzero scalars, we could factor out only by scalars of the form $\rho e^{2\pi i k/q}$ with ρ an arbitrary positive real, k an arbitrary integer, and q a fixed positive integer. The quotient of $\mathbb{C}^\infty - \{0\}$ under this identification, an infinite-dimensional lens space L^∞ with $\pi_1(L^\infty) \approx \mathbb{Z}_q$, is therefore also an associative, commutative H–space. This includes $\mathbb{R}\mathrm{P}^\infty$ in particular.

The spaces $J(X)$ defined in §3.2 are also H–spaces, with the multiplication given by $(x_1, \cdots, x_m)(y_1, \cdots, y_n) = (x_1, \cdots, x_m, y_1, \cdots, y_n)$, which is associative and has an identity element (e) where e is the basepoint of X. One could describe $J(X)$ as the free associative H–space generated by X. There is also a commutative analog of $J(X)$ called the **infinite symmetric product** $SP(X)$ defined in the following way. Let $SP_n(X)$ be the quotient space of the n-fold product X^n obtained by identifying all n-tuples $(x_1, \cdots, x_n)$ that differ only by a permutation of their coordinates. The inclusion $X^n \hookrightarrow X^{n+1}$, $(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_n, e)$ induces an inclusion $SP_n(X) \hookrightarrow SP_{n+1}$, and $SP(X)$ is defined to be the union of this increasing sequence of $SP_n(X)$'s, with the weak topology. Alternatively, $SP(X)$ is the quotient of $J(X)$ obtained by identifying points that differ only by permutation of coordinates. The H–space structure on $J(X)$ induces an H–space structure on $SP(X)$ which is commutative in addition to being associative and having a strict identity. The spaces $SP(X)$ are studied in more detail in §4.K.

The goal of this section will be to describe the extra structure which the multiplication in an H–space gives to its homology and cohomology. This is of particular interest since many of the most important spaces in algebraic topology turn out to be H–spaces.

Hopf Algebras

Let us look at cohomology first. Choosing a commutative ring R as coefficient ring, we can regard the cohomology ring $H^*(X;R)$ of a space X as an algebra over R rather than merely a ring. Suppose X is an H–space satisfying two conditions:

(1) X is path-connected, hence $H^0(X;R) \approx R$.

(2) $H^n(X;R)$ is a finitely generated free R-module for each n, so the cross product $H^*(X;R) \otimes_R H^*(X;R) \to H^*(X \times X;R)$ is an isomorphism.

The multiplication $\mu : X \times X \to X$ induces a map $\mu^* : H^*(X;R) \to H^*(X \times X;R)$, and when we combine this with the cross product isomorphism in (2) we get a map

$$H^*(X;R) \xrightarrow{\Delta} H^*(X;R) \otimes_R H^*(X;R)$$

which is an algebra homomorphism since both μ^* and the cross product isomorphism are algebra homomorphisms. The key property of Δ turns out to be that for any $\alpha \in H^n(X;R)$, $n > 0$, we have

$$\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha + \sum_{0<i<n} \alpha'_i \otimes \alpha''_{n-i} \quad \text{where } |\alpha'_j| = j = |\alpha''_j|$$

To verify this, let $i : X \to X \times X$ be the inclusion $x \mapsto (x, e)$ for e the identity element of X, and consider the commutative diagram

$$\begin{tikzcd}
H^*(X;R) \arrow[r, "\mu^*"] \arrow[dr, "\Delta"'] & H^*(X \times X;R) \arrow[r, "i^*"] & H^*(X;R) \\
& H^*(X;R) \otimes_R H^*(X;R) \arrow[u, "\times", "\approx"'] \arrow[r, "\mathbb{1} \otimes i^*"] \arrow[ur, "P"] & H^*(X;R) \otimes_R H^*(e;R) \arrow[u, "\times", "\approx"']
\end{tikzcd}$$

The map P is defined by commutativity, and by looking at the lower right triangle we see that $P(\alpha \otimes 1) = \alpha$ and $P(\alpha \otimes \beta) = 0$ if $|\beta| > 0$. The H–space property says that $\mu i \simeq \mathbb{1}$, so $P\Delta = \mathbb{1}$. This implies that the component of $\Delta(\alpha)$ in $H^n(X;R) \otimes_R H^0(X;R)$ is $\alpha \otimes 1$. A similar argument shows the component in $H^0(X;R) \otimes_R H^n(X;R)$ is $1 \otimes \alpha$.

We can summarize this situation by saying that $H^*(X;R)$ is a **Hopf algebra**, that is, a graded algebra $A = \bigoplus_{n \ge 0} A^n$ over a commutative base ring R, satisfying the following two conditions:

(1) There is an identity element $1 \in A^0$ such that the map $R \to A^0$, $r \mapsto r \cdot 1$, is an isomorphism; one says A is *connected*.

(2) There is a **diagonal** or **coproduct** $\Delta : A \to A \otimes A$, a homomorphism of graded algebras satisfying $\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha + \sum_{0<i<n} \alpha'_i \otimes \alpha''_{n-i}$ for $\alpha \in A^n$, $n > 0$, and $\alpha'_j, \alpha''_j \in A^j$.

Here and in what follows we take $\otimes$ to mean $\otimes_R$. The multiplication in $A \otimes A$ is given by the standard formula $(\alpha \otimes \beta)(\gamma \otimes \delta) = (-1)^{jk}(\alpha\gamma \otimes \beta\delta)$ where $\beta \in A^j$ and $\gamma \in A^k$. For a general Hopf algebra the multiplication is not assumed to be either associative or commutative (in the graded sense), though in the example of $H^*(X;R)$ for X an H–space the algebra structure is of course associative and commutative.

Example 3C.1. One of the simplest Hopf algebras is a polynomial ring $R[\alpha]$. The coproduct $\Delta(\alpha)$ must equal $\alpha\otimes 1 + 1\otimes\alpha$ since the only elements of $R[\alpha]$ of lower dimension than α are the elements of R in dimension zero, so the terms α'_i and α''_{n-i} in the coproduct formula $\Delta(\alpha) = \alpha\otimes 1 + 1\otimes\alpha + \sum_{0<i<n}\alpha'_i\otimes\alpha''_{n-i}$ must be zero. The requirement that Δ be an algebra homomorphism then determines Δ completely. To describe Δ explicitly we distinguish two cases. If the dimension of α is even or if $2=0$ in R, then the multiplication in $R[\alpha]\otimes R[\alpha]$ is strictly commutative and $\Delta(\alpha^n) = (\alpha\otimes 1 + 1\otimes\alpha)^n = \sum_i\binom{n}{i}\alpha^i\otimes\alpha^{n-i}$. In the opposite case that α is odd-dimensional, then $\Delta(\alpha^2) = (\alpha\otimes 1 + 1\otimes\alpha)^2 = \alpha^2\otimes 1 + 1\otimes\alpha^2$ since $(\alpha\otimes 1)(1\otimes\alpha) = \alpha\otimes\alpha$ and $(1\otimes\alpha)(\alpha\otimes 1) = -\alpha\otimes\alpha$ if α has odd dimension. Thus if we set $\beta = \alpha^2$, then β is even-dimensional and we have $\Delta(\alpha^{2n}) = \Delta(\beta^n) = (\beta\otimes 1 + 1\otimes\beta)^n = \sum_i\binom{n}{i}\beta^i\otimes\beta^{n-i}$ and $\Delta(\alpha^{2n+1}) = \Delta(\alpha\beta^n) = \Delta(\alpha)\Delta(\beta^n) = \sum_i\binom{n}{i}\alpha\beta^i\otimes\beta^{n-i} + \sum_i\binom{n}{i}\beta^i\otimes\alpha\beta^{n-i}$.

Example 3C.2. The exterior algebra $\Lambda_R[\alpha]$ on an odd-dimensional generator α is a Hopf algebra, with $\Delta(\alpha) = \alpha\otimes 1 + 1\otimes\alpha$. To verify that Δ is an algebra homomorphism we must check that $\Delta(\alpha^2) = \Delta(\alpha)^2$, or in other words, since $\alpha^2 = 0$, we need to see that $\Delta(\alpha)^2 = 0$. As in the preceding example we have $\Delta(\alpha)^2 = (\alpha\otimes 1 + 1\otimes\alpha)^2 = \alpha^2\otimes 1 + 1\otimes\alpha^2$, so $\Delta(\alpha)^2$ is indeed 0. Note that if α were even-dimensional we would instead have $\Delta(\alpha)^2 = \alpha^2\otimes 1 + 2\alpha\otimes\alpha + 1\otimes\alpha^2$, which would be 0 in $\Lambda_R[\alpha]\otimes\Lambda_R[\alpha]$ only if $2=0$ in R.

An element α of a Hopf algebra is called **primitive** if $\Delta(\alpha) = \alpha\otimes 1 + 1\otimes\alpha$. As the preceding examples illustrate, if a Hopf algebra is generated as an algebra by primitive elements, then the coproduct Δ is uniquely determined by the product. This happens in a number of interesting special cases, but certainly not in general, as we shall see.

The existence of the coproduct in a Hopf algebra turns out to restrict the multiplicative structure considerably. Here is an important example illustrating this:

Example 3C.3. Suppose that the truncated polynomial algebra $F[\alpha]/(\alpha^n)$ over a field F is a Hopf algebra. Then α is primitive, just as it is in $F[\alpha]$, so if we assume either that α is even-dimensional or that F has characteristic 2, then the relation $\alpha^n = 0$ yields an equation

$$0 = \Delta(\alpha^n) = \alpha^n\otimes 1 + 1\otimes\alpha^n + \sum_{0<i<n}\binom{n}{i}\alpha^i\otimes\alpha^{n-i} = \sum_{0<i<n}\binom{n}{i}\alpha^i\otimes\alpha^{n-i}$$

which implies that $\binom{n}{i} = 0$ in F for each i in the range $0<i<n$. This is impossible if F has characteristic 0, and if the characteristic of F is $p>0$ then it happens only when n is a power of p. For $p=2$ this was shown in the proof of Theorem 3.20, and the argument given there works just as well for odd primes. Conversely, it is easy to check that if F has characteristic p then $F[\alpha]/(\alpha^{p^i})$ is a Hopf algebra, assuming still that α is even-dimensional if p is odd.

The characteristic 0 case of this result implies that $\mathbb{C}\mathrm{P}^n$ is not an H–space for finite n, in contrast with $\mathbb{C}\mathrm{P}^\infty$ which is an H–space as we saw earlier. Similarly, taking

$F = \mathbb{Z}_2$, we deduce that $\mathbb{R}P^n$ can be an H–space only if $n+1$ is a power of 2. Indeed, $\mathbb{R}P^1 = S^1/\pm 1$, $\mathbb{R}P^3 = S^3/\pm 1$, and $\mathbb{R}P^7 = S^7/\pm 1$ have quotient H–space structures from S^1, S^3 and S^7 since -1 commutes with all elements of S^1, S^3, or S^7. However, these are the only cases when $\mathbb{R}P^n$ is an H–space since, by an exercise at the end of this section, the universal cover of an H–space is an H–space, and S^1, S^3, and S^7 are the only spheres that are H–spaces, by the theorem of Adams mentioned earlier.

It is an easy exercise to check that the tensor product of Hopf algebras is again a Hopf algebra, with the coproduct $\Delta(\alpha \otimes \beta) = \Delta(\alpha) \otimes \Delta(\beta)$. So the preceding examples yield many other Hopf algebras, tensor products of polynomial, truncated polynomial, and exterior algebras on any number of generators. The following theorem of Hopf is a partial converse:

Theorem 3C.4. *If A is a commutative, associative Hopf algebra over a field F of characteristic 0, and A^n is finite-dimensional over F for each n, then A is isomorphic as an algebra to the tensor product of an exterior algebra on odd-dimensional generators and a polynomial algebra on even-dimensional generators.*

There is an analogous theorem of Borel when F is a finite field of characteristic p. In this case A is again isomorphic to a tensor product of single-generator Hopf algebras, of one of the following types:

- $F[\alpha]$, with α even-dimensional if $p \neq 2$.
- $\Lambda_F[\alpha]$ with α odd-dimensional.
- $F[\alpha]/(\alpha^{p^i})$, with α even-dimensional if $p \neq 2$.

For a proof see [Borel 1953] or [Kane 1988].

Proof of 3C.4: Since A^n is finitely generated over F for each n, we may choose algebra generators $x_1, x_2, \cdots$ for A with $x_i \in A^{|x_i|}$ and $|x_i| \le |x_{i+1}|$ for all i. Let A_n be the subalgebra generated by $x_1, \cdots, x_n$. This is a Hopf subalgebra of A, that is, $\Delta(A_n) \subset A_n \otimes A_n$, since $\Delta(x_i)$ involves only x_i and terms of smaller dimension. We may assume x_n does not lie in A_{n-1}. Since A is associative and commutative, there is a natural surjection $A_{n-1} \otimes F[x_n] \to A_n$ if $|x_n|$ is even, or $A_{n-1} \otimes \Lambda_F[x_n] \to A_n$ if $|x_n|$ is odd. By induction on n it will suffice to prove these surjections are injective. Thus in the two cases we must rule out nontrivial relations $\sum_i \alpha_i x_n^i = 0$ and $\alpha_0 + \alpha_1 x_n = 0$, respectively, with coefficients $\alpha_i \in A_{n-1}$.

Let I be the ideal in A_n generated by x_n^2 and the positive-dimensional elements of A_{n-1}, so I consists of the polynomials $\sum_i \alpha_i x_n^i$ with coefficients $\alpha_i \in A_{n-1}$, the first two coefficients α_0 and α_1 having trivial components in A^0. Note that $x_n \notin I$ since elements of I having dimension $|x_n|$ must lie in A_{n-1}. Consider the composition

$$A_n \xrightarrow{\Delta} A_n \otimes A_n \xrightarrow{q} A_n \otimes (A_n/I)$$

with q the natural quotient map. By the definition of I, this composition $q\Delta$ sends $\alpha \in A_{n-1}$ to $\alpha \otimes 1$ and x_n to $x_n \otimes 1 + 1 \otimes \overline{x}_n$ where $\overline{x}_n$ is the image of x_n in A_n/I.

In case $|x_n|$ is even, applying $q\Delta$ to a nontrivial relation $\sum_i \alpha_i x_n^i = 0$ gives

$$0 = \sum_i (\alpha_i \otimes 1)(x_n \otimes 1 + 1 \otimes \overline{x}_n)^i = \left(\sum_i \alpha_i x_n^i\right) \otimes 1 + \sum_i i\alpha_i x_n^{i-1} \otimes \overline{x}_n$$

Since $\sum_i \alpha_i x_n^i = 0$, this implies that $\sum_i i\alpha_i x_n^{i-1} \otimes \overline{x}_n$ is zero in the tensor product $A_n \otimes (A_n/I)$, hence $\sum_i i\alpha_i x_n^{i-1} = 0$ since $x_n \notin I$ implies $\overline{x}_n \neq 0$. The relation $\sum_i i\alpha_i x_n^{i-1} = 0$ has lower degree than the original relation, and is not the trivial relation since F has characteristic 0, $\alpha_i \neq 0$ implying $i\alpha_i \neq 0$ if $i > 0$. Since we could assume the original relation had minimum degree, we have reached a contradiction.

The case $|x_n|$ odd is similar. Applying $q\Delta$ to a relation $\alpha_0 + \alpha_1 x_n = 0$ gives $0 = \alpha_0 \otimes 1 + (\alpha_1 \otimes 1)(x_n \otimes 1 + 1 \otimes \overline{x}_n) = (\alpha_0 + \alpha_1 x_n) \otimes 1 + \alpha_1 \otimes \overline{x}_n$. Since $\alpha_0 + \alpha_1 x_n = 0$, we get $\alpha_1 \otimes \overline{x}_n = 0$, which implies $\alpha_1 = 0$ and hence $\alpha_0 = 0$. $\square$

The structure of Hopf algebras over $\mathbb{Z}$ is much more complicated than over a field. Here is an example that is still fairly simple.

Example 3C.5: Divided Polynomial Algebras. We showed in Proposition 3.22 that the H-space $J(S^n)$ for n even has $H^*(J(S^n);\mathbb{Z})$ a divided polynomial algebra, the algebra $\Gamma_{\mathbb{Z}}[\alpha]$ with additive generators α_i in dimension $2i$ and multiplication given by $\alpha_1^k = k!\alpha_k$, hence $\alpha_i\alpha_j = \binom{i+j}{i}\alpha_{i+j}$. The coproduct in $\Gamma_{\mathbb{Z}}[\alpha]$ is uniquely determined by the multiplicative structure since $\Delta(\alpha_1^k) = (\alpha_1 \otimes 1 + 1 \otimes \alpha_1)^k = \sum_i \binom{k}{i}\alpha_1^i \otimes \alpha_1^{k-i}$, which implies that $\Delta(\alpha_1^k/k!) = \sum_i (\alpha_1^i/i!) \otimes (\alpha_1^{k-i}/(k-i)!)$, that is, $\Delta(\alpha_k) = \sum_i \alpha_i \otimes \alpha_{k-i}$. So in this case the coproduct has a simpler description than the product.

It is interesting to see what happens to the divided polynomial algebra $\Gamma_{\mathbb{Z}}[\alpha]$ when we change to field coefficients. Clearly $\Gamma_{\mathbb{Q}}[\alpha]$ is the same as $\mathbb{Q}[\alpha]$. In contrast with this, $\Gamma_{\mathbb{Z}_p}[\alpha]$, with multiplication defined by $\alpha_i\alpha_j = \binom{i+j}{i}\alpha_{i+j}$, happens to be isomorphic as an algebra to the infinite tensor product $\bigotimes_{i\geq 0} \mathbb{Z}_p[\alpha_{p^i}]/(\alpha_{p^i}^p)$, as we will show in a moment. However, as Hopf algebras these two objects are different since α_{p^i} is primitive in $\bigotimes_{i\geq 0} \mathbb{Z}_p[\alpha_{p^i}]/(\alpha_{p^i}^p)$ but not in $\Gamma_{\mathbb{Z}_p}[\alpha]$ where the coproduct is given by $\Delta(\alpha_k) = \sum_i \alpha_i \otimes \alpha_{k-i}$.

Now let us show that there is an algebra isomorphism

$$\Gamma_{\mathbb{Z}_p}[\alpha] \approx \bigotimes_{i\geq 0} \mathbb{Z}_p[\alpha_{p^i}]/(\alpha_{p^i}^p)$$

Since $\Gamma_{\mathbb{Z}_p}[\alpha] = \Gamma_{\mathbb{Z}}[\alpha] \otimes \mathbb{Z}_p$, this is equivalent to:

$(*)$ The element $\alpha_1^{n_0}\alpha_p^{n_1}\cdots\alpha_{p^k}^{n_k}$ in $\Gamma_{\mathbb{Z}}[\alpha]$ is divisible by p iff $n_i \geq p$ for some i.

The product $\alpha_1^{n_0}\alpha_p^{n_1}\cdots\alpha_{p^k}^{n_k}$ equals $m\alpha_n$ for $n = n_0 + n_1 p + \cdots + n_k p^k$ and some integer m. The question is whether p divides m. We will show:

$(**)$ $\alpha_n\alpha_{p^k}$ is divisible by p iff $n_k = p-1$, assuming that $n_i < p$ for each i.

This implies $(*)$ by an inductive argument in which we build up the product in $(*)$ by repeated multiplication on the right by terms α_{p^i}.

To prove $(**)$ we recall that $\alpha_n\alpha_{p^k} = \binom{n+p^k}{n}\alpha_{n+p^k}$. The mod p value of this binomial coefficient can be computed using Lemma 3C.6 below. Assuming that $n_i < p$ for each i and that $n_k+1 < p$, the p-adic representations of $n+p^k$ and n differ only in the coefficient of p^k, so mod p we have $\binom{n+p^k}{n} = \binom{n_k+1}{n_k} = n_k+1$. This conclusion also holds if $n_k+1 = p$, when the p-adic representations of $n+p^k$ and n differ also in the coefficient of p^{k+1}. The statement $(**)$ then follows.

Lemma 3C.6. *If p is a prime, then $\binom{n}{k} \equiv \prod_i \binom{n_i}{k_i}$ mod p where $n = \sum_i n_i p^i$ and $k = \sum_i k_i p^i$ with $0 \le n_i < p$ and $0 \le k_i < p$ are the p-adic representations of n and k.*

Here the convention is that $\binom{n}{k} = 0$ if $n < k$, and $\binom{n}{0} = 1$ for all $n \ge 0$.

Proof: In $\mathbb{Z}_p[x]$ there is an identity $(1+x)^p = 1+x^p$ since p clearly divides $\binom{p}{k} = p!/k!(p-k)!$ for $0 < k < p$. By induction it follows that $(1+x)^{p^i} = 1+x^{p^i}$. Hence if $n = \sum_i n_i p^i$ is the p-adic representation of n then:

$$\begin{aligned}(1+x)^n &= (1+x)^{n_0}(1+x^p)^{n_1}(1+x^{p^2})^{n_2}\cdots \\ &= \Big[1+\tbinom{n_0}{1}x+\tbinom{n_0}{2}x^2+\cdots+\tbinom{n_0}{p-1}x^{p-1}\Big] \\ &\quad\times\Big[1+\tbinom{n_1}{1}x^p+\tbinom{n_1}{2}x^{2p}+\cdots+\tbinom{n_1}{p-1}x^{(p-1)p}\Big] \\ &\qquad\times\Big[1+\tbinom{n_2}{1}x^{p^2}+\tbinom{n_2}{2}x^{2p^2}+\cdots+\tbinom{n_2}{p-1}x^{(p-1)p^2}\Big]\times\cdots\end{aligned}$$

When this is multiplied out, one sees that no terms combine, and the coefficient of x^k is just $\prod_i \binom{n_i}{k_i}$ where $k = \sum_i k_i p^i$ is the p-adic representation of k. □

Pontryagin Product

Another special feature of H–spaces is that their homology groups have a product operation, called the **Pontryagin product**. For an H–space X with multiplication $\mu : X\times X \to X$, this is the composition

$$H_*(X;R)\otimes H_*(X;R) \xrightarrow{\times} H_*(X\times X;R) \xrightarrow{\mu_*} H_*(X;R)$$

where the first map is the cross product defined in §3.B. Thus the Pontryagin product consists of bilinear maps $H_i(X;R)\times H_j(X;R)\to H_{i+j}(X;R)$. Unlike cup product, the Pontryagin product is not in general associative unless the multiplication μ is associative or at least associative up to homotopy, in the sense that the maps $X\times X\times X\to X$, $(x,y,z)\mapsto \mu(x,\mu(y,z))$ and $(x,y,z)\mapsto \mu(\mu(x,y),z)$ are homotopic. Fortunately most H–spaces one meets in practice satisfy this associativity property. Nor is the Pontryagin product generally commutative, even in the graded sense, unless μ is commutative or homotopy-commutative, which is relatively rare for H–spaces. We will give examples shortly where the Pontryagin product is not commutative.

In case X is a CW complex and μ is a cellular map the Pontryagin product can be computed using cellular homology via the cellular chain map

$$C_i(X;R)\times C_j(X;R)\xrightarrow{\times}C_{i+j}(X\times X;R)\xrightarrow{\mu_*}C_{i+j}(X;R)$$

where the cross product map sends generators corresponding to cells e^i and e^j to the generator corresponding to the product cell $e^i\times e^j$, and then μ_* is applied to this product cell.

Example 3C.7. Let us compute the Pontryagin product for $J(S^n)$. Here there is one cell e^{in} for each $i\ge 0$, and μ takes the product cell $e^{in}\times e^{jn}$ homeomorphically onto the cell $e^{(i+j)n}$. This means that $H_*(J(S^n);\mathbb{Z})$ is simply the polynomial ring $\mathbb{Z}[x]$ on an n-dimensional generator x. This holds for n odd as well as for n even, so the Pontryagin product need not satisfy the same general commutativity relation as cup product. In this example the Pontryagin product structure is simpler than the cup product structure, though for some H–spaces it is the other way round. In applications it is often convenient to have the choice of which product structure to use.

This calculation immediately generalizes to $J(X)$ where X is any connected CW complex whose cellular boundary maps are all trivial. The cellular boundary maps in the product X^m of m copies of X are then trivial by induction on m using Proposition 3B.1, and therefore the cellular boundary maps in $J(X)$ are all trivial since the quotient map $X^m\to J_m(X)$ is cellular and each cell of $J_m(X)$ is the homeomorphic image of a cell of X^m. Thus $H_*(J(X);\mathbb{Z})$ is free with additive basis the products $e^{n_1}\times\cdots\times e^{n_k}$ of positive-dimensional cells of X, and the multiplicative structure is that of polynomials in noncommuting variables corresponding to the positive-dimensional cells of X.

Another way to describe $H_*(J(X);\mathbb{Z})$ in this example is as the tensor algebra $T\widetilde{H}_*(X;\mathbb{Z})$, where for a graded R-module M that is trivial in dimension zero, like the reduced homology of a path-connected space, the tensor algebra TM is the direct sum of the n-fold tensor products of M with itself for all $n\ge 1$, together with a copy of R in dimension zero, with the obvious multiplication coming from tensor product and scalar multiplication.

Generalizing the preceding example, we have:

Proposition 3C.8. *If X is a connected CW complex with $H_*(X;R)$ a free R-module, then $H_*(J(X);R)$ is isomorphic to the tensor algebra $T\widetilde{H}_*(X;R)$.*

This can be paraphrased as saying that the homology of the free H–space generated by a space with free homology is the free algebra generated by the homology of the space.

Proof: With coefficients in R, let $\varphi:T\widetilde{H}_*(X)\to H_*(J(X))$ be the homomorphism whose restriction to the n-fold tensor product $\widetilde{H}_*(X)^{\otimes n}$ is the composition

$$\widetilde{H}_*(X)^{\otimes n}\hookrightarrow H_*(X)^{\otimes n}\xrightarrow{\times}H_*(X^n)\longrightarrow H_*(J_n(X))\longrightarrow H_*(J(X))$$

where the next-to-last map is induced by the quotient map $X^n \to J_n(X)$. It is clear that φ is a ring homomorphism since the product in $J(X)$ is induced from the natural map $X^m \times X^n \to X^{m+n}$. To show that φ is an isomorphism, consider the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & T_{n-1}\widetilde{H}_*(X) \arrow[r] \arrow[d, "\varphi"'] & T_n\widetilde{H}_*(X) \arrow[r] \arrow[d, "\varphi"'] & \widetilde{H}_*(X)^{\otimes n} \arrow[r] \arrow[d, "\times"', "\approx"] & 0 \\
0 \arrow[r] & H_*(J_{n-1}(X)) \arrow[r] & H_*(J_n(X)) \arrow[r] & \widetilde{H}_*(X^{\wedge n}) \arrow[r] & 0
\end{tikzcd}$$

In the upper row, $T_m\widetilde{H}_*(X)$ denotes the direct sum of the products $\widetilde{H}_*(X)^{\otimes k}$ for $k \le m$, so this row is exact. The second row is the homology exact sequence for the pair $(J_n(X), J_{n-1}(X))$, with quotient $J_n(X)/J_{n-1}(X)$ the n-fold smash product $X^{\wedge n}$. This long exact sequence breaks up into short exact sequences as indicated, by commutativity of the right-hand square and the fact that the right-hand vertical map is an isomorphism by the Künneth formula, using the hypothesis that $H_*(X)$ is free over the given coefficient ring. By induction on n and the five-lemma we deduce from the diagram that $\varphi : T_n\widetilde{H}_*(X) \to H_*(J_n(X))$ is an isomorphism for all n. Letting n go to ∞, this implies that $\varphi : T\widetilde{H}_*(X) \to H_*(J(X))$ is an isomorphism since in any given dimension $T_n\widetilde{H}_*(X)$ is independent of n when n is sufficiently large, and the same is true of $H_*(J_n(X))$ by the second row of the diagram. $\square$

Dual Hopf Algebras

There is a close connection between the Pontryagin product in homology and the Hopf algebra structure on cohomology. Suppose that X is an H–space such that, with coefficients in a field R, the vector spaces $H_n(X;R)$ are finite-dimensional for all n. Alternatively, we could take $R = \mathbb{Z}$ and assume $H_n(X;\mathbb{Z})$ is finitely generated and free for all n. In either case we have $H^n(X;R) = \mathrm{Hom}_R(H_n(X;R),R)$, and as a consequence the Pontryagin product $H_*(X;R) \otimes H_*(X;R) \to H_*(X;R)$ and the coproduct $\Delta : H^*(X;R) \to H^*(X;R) \otimes H^*(X;R)$ are dual to each other, both being induced by the H–space product $\mu : X \times X \to X$. Therefore the coproduct in cohomology determines the Pontryagin product in homology, and vice versa. Specifically, the component $\Delta_{ij} : H^{i+j}(X;R) \to H^i(X;R) \otimes H^j(X;R)$ of Δ is dual to the product $H_i(X;R) \otimes H_j(X;R) \to H_{i+j}(X;R)$.

Example 3C.9. Consider $J(S^n)$ with n even, so $H^*(J(S^n);\mathbb{Z})$ is the divided polynomial algebra $\Gamma_{\mathbb{Z}}[\alpha]$. In Example 3C.5 we derived the coproduct formula $\Delta(\alpha_k) = \sum_i \alpha_i \otimes \alpha_{k-i}$. Thus Δ_{ij} takes α_{i+j} to $\alpha_i \otimes \alpha_j$, so if x_i is the generator of $H_{in}(J(S^n);\mathbb{Z})$ dual to α_i, then $x_i x_j = x_{i+j}$. This says that $H_*(J(S^n);\mathbb{Z})$ is the polynomial ring $\mathbb{Z}[x]$. We showed this in Example 3C.7 using the cell structure of $J(S^n)$, but the present proof deduces it purely algebraically from the cup product structure.

Now we wish to show that the relation between $H^*(X;R)$ and $H_*(X;R)$ is perfectly symmetric: They are *dual Hopf algebras*. This is a purely algebraic fact:

Proposition 3C.10. *Let A be a Hopf algebra over R that is a finitely generated free R-module in each dimension. Then the product $\pi : A\otimes A \to A$ and coproduct $\Delta : A \to A\otimes A$ have duals $\pi^* : A^* \to A^*\otimes A^*$ and $\Delta^* : A^*\otimes A^* \to A^*$ that give A^* the structure of a Hopf algebra.*

Proof: This will be apparent if we reinterpret the Hopf algebra structure on A formally as a pair of graded R-module homomorphisms $\pi : A\otimes A \to A$ and $\Delta : A \to A\otimes A$ together with an element $1 \in A^0$ satisfying:

(1) The two compositions $A \xrightarrow{i_\ell} A\otimes A \xrightarrow{\pi} A$ and $A \xrightarrow{i_r} A\otimes A \xrightarrow{\pi} A$ are the identity, where $i_\ell(a) = a\otimes 1$ and $i_r(a) = 1\otimes a$. This says that 1 is a two-sided identity for the multiplication in A.

(2) The two compositions $A \xrightarrow{\Delta} A\otimes A \xrightarrow{p_\ell} A$ and $A \xrightarrow{\Delta} A\otimes A \xrightarrow{p_r} A$ are the identity, where $p_\ell(a\otimes 1) = a$, $p_\ell(a\otimes b) = 0$ if $b \in A^j$ with $j > 0$, $p_r(1\otimes a) = a$, and $p_r(a\otimes b) = 0$ if $a \in A^j$ with $j > 0$. This is just the coproduct formula $\Delta(a) = a\otimes 1 + 1\otimes a + \sum_{0<i<n} a'_i \otimes a''_{n-i}$.

(3) The diagram at the right commutes, where $\tau(a\otimes b\otimes c\otimes d) = (-1)^{ij} a\otimes c\otimes b\otimes d$ for $b \in A^i$, $c \in A^j$. This is the condition that Δ is an algebra homomorphism since if we follow an element $a\otimes b \in A^m \otimes A^n$ across the top of the diagram we get $\Delta(ab)$, while the lower route gives first $\Delta(a)\otimes\Delta(b) = (\sum_i a'_i\otimes a''_{m-i})\otimes(\sum_j b'_j\otimes b''_{n-j})$, then after applying τ and $\pi\otimes\pi$ this becomes $\sum_{i,j}(-1)^{(m-i)j} a'_i b'_j \otimes a''_{m-i} b''_{n-j} = (\sum_i a'_i\otimes a''_{m-i})(\sum_j b'_j\otimes b''_{n-j})$, which is $\Delta(a)\Delta(b)$.

$$\begin{tikzcd} A\otimes A \arrow[r, "\pi"] \arrow[d, "\Delta\otimes\Delta"'] & A \arrow[r, "\Delta"] & A\otimes A \\ A\otimes A\otimes A\otimes A \arrow[rr, "\tau"] & & A\otimes A\otimes A\otimes A \arrow[u, "\pi\otimes\pi"'] \end{tikzcd}$$

Condition (1) for A dualizes to (2) for A^*, and similarly (2) for A dualizes to (1) for A^*. Condition (3) for A dualizes to (3) for A^*. $\square$

Example 3C.11. Let us compute the dual of a polynomial algebra $R[x]$. Suppose first that x has even dimension. Then $\Delta(x^n) = (x\otimes 1 + 1\otimes x)^n = \sum_i \binom{n}{i} x^i \otimes x^{n-i}$, so if α_i is dual to x^i, the term $\binom{n}{i} x^i\otimes x^{n-i}$ in $\Delta(x^n)$ gives the product relation $\alpha_i\alpha_{n-i} = \binom{n}{i}\alpha_n$. This is the rule for multiplication in a divided polynomial algebra, so the dual of $R[x]$ is $\Gamma_R[\alpha]$ if the dimension of x is even. This also holds if $2 = 0$ in R, since the even-dimensionality of x was used only to deduce that $R[x]\otimes R[x]$ is strictly commutative.

In case x is odd-dimensional, then as we saw in Example 3C.1, if we set $y = x^2$, we have $\Delta(y^n) = (y\otimes 1 + 1\otimes y)^n = \sum_i \binom{n}{i} y^i\otimes y^{n-i}$ and $\Delta(xy^n) = \Delta(x)\Delta(y^n) = \sum_i \binom{n}{i} xy^i\otimes y^{n-i} + \sum_i \binom{n}{i} y^i \otimes xy^{n-i}$. These formulas for Δ say that the dual of $R[x]$ is $\Lambda_R[\alpha]\otimes\Gamma_R[\beta]$ where α is dual to x and β is dual to y.

This algebra allows us to deduce the cup product structure on $H^*(J(S^n);R)$ from the geometric calculation $H_*(J(S^n);R) \approx R[x]$ in Example 3C.7. As another application, recall from earlier in this section that $\mathbb{R}\mathrm{P}^\infty$ and $\mathbb{C}\mathrm{P}^\infty$ are H–spaces, so from their cup product structures we can conclude that the Pontryagin rings $H_*(\mathbb{R}P^\infty;\mathbb{Z}_2)$ and $H_*(\mathbb{C}P^\infty;\mathbb{Z})$ are divided polynomial algebras.

In these examples the Hopf algebra is generated as an algebra by primitive elements, so the product determines the coproduct and hence the dual algebra. This is not true in general, however. For example, we have seen that the Hopf algebra $\Gamma_{\mathbb{Z}_p}[\alpha]$ is isomorphic as an algebra to $\bigotimes_{i\ge 0}\mathbb{Z}_p[\alpha_{p^i}]/(\alpha_{p^i}^p)$, but if we regard the latter tensor product as the tensor product of the Hopf algebras $\mathbb{Z}_p[\alpha_{p^i}]/(\alpha_{p^i}^p)$ then the elements α_{p^i} are primitive, though they are not primitive in $\Gamma_{\mathbb{Z}_p}[\alpha]$ for $i>0$. In fact, the Hopf algebra $\bigotimes_{i\ge 0}\mathbb{Z}_p[\alpha_{p^i}]/(\alpha_{p^i}^p)$ is its own dual, according to one of the exercises below, but the dual of $\Gamma_{\mathbb{Z}_p}[\alpha]$ is $\mathbb{Z}_p[\alpha]$.

Exercises

1. Suppose that X is a CW complex with basepoint $e\in X$ a 0-cell. Show that X is an H–space if there is a map $\mu: X\times X\to X$ such that the maps $X\to X$, $x\mapsto \mu(x,e)$ and $x\mapsto\mu(e,x)$, are homotopic to the identity. [Sometimes this is taken as the definition of an H–space, rather than the more restrictive condition in the definition we have given.] With the same hypotheses, show also that μ can be homotoped so that e is a strict two-sided identity.

2. Show that a retract of an H–space is an H–space if it contains the identity element.

3. Show that if X is an H–space such that the set of path-components of X is a group with respect to the multiplication induced by the H–space structure, then all the path-components are homotopy equivalent.

4. Show that an H–space or topological group structure on a path-connected, locally path-connected space can be lifted to such a structure on its universal cover. [For the group $SO(n)$ considered in the next section, the universal cover for $n>2$ is a 2-sheeted cover, a group called $Spin(n)$.]

5. Show that if (X,e) is an H–space then $\pi_1(X,e)$ is abelian. [Compare the usual composition $f\cdot g$ of loops with the product $\mu(f(t),g(t))$ coming from the H–space multiplication μ.]

6. Show that S^n is an H–space iff the attaching map of the $2n$-cell of $J_2(S^n)$ is homotopically trivial.

7. What are the primitive elements of the Hopf algebra $\mathbb{Z}_p[x]$ for p prime?

8. Show that the tensor product of two Hopf algebras is a Hopf algebra.

9. Apply the theorems of Hopf and Borel to show that for a finite CW complex H–space X with $\widetilde{H}_*(X;\mathbb{Z})\ne 0$, the Euler characteristic $\chi(X)$ is 0.

10. Let X be a path-connected H–space with $H^*(X;R)$ free and finitely generated in each dimension. For maps $f,g:X\to X$, the product $fg:X\to X$ is defined by $(fg)(x)=f(x)g(x)$, using the H–space product.

(a) Show that $(fg)^*(\alpha) = f^*(\alpha) + g^*(\alpha)$ for primitive elements $\alpha \in H^*(X;R)$.

(b) Deduce that the k^{th}-power map $x \mapsto x^k$ induces the map $\alpha \mapsto k\alpha$ on primitive elements α. In particular the quaternionic k^{th}-power map $S^3 \to S^3$ has degree k.

(c) Show that every polynomial $a_n x^n b_n + \cdots + a_1 x b_1 + a_0$ with coefficients in $\mathbb{H}$ has a root in $\mathbb{H}$ if $n > 0$. [See Theorem 1.8.]

11. If T^n is the n-dimensional torus, the product of n circles, show that the Pontryagin ring $H_*(T^n;\mathbb{Z})$ is the exterior algebra $\Lambda_{\mathbb{Z}}[x_1, \cdots, x_n]$ with $|x_i| = 1$.

12. Compute the Pontryagin product structure in $H_*(L;\mathbb{Z}_p)$ where L is an infinite-dimensional lens space $S^\infty/\mathbb{Z}_p$, for p an odd prime, using the coproduct in $H^*(L;\mathbb{Z}_p)$.

13. Verify that the Hopf algebras $\Lambda_R[\alpha]$ and $\mathbb{Z}_p[\alpha]/(\alpha^p)$ are self-dual.

14. Show that the coproduct in the Hopf algebra $H_*(X;R)$ dual to $H^*(X;R)$ is induced by the diagonal map $X \to X \times X$, $x \mapsto (x, x)$.

15. Suppose that X is a path-connected H-space such that $H^*(X;\mathbb{Z})$ is free and finitely generated in each dimension, and $H^*(X;\mathbb{Q})$ is a polynomial ring $\mathbb{Q}[\alpha]$. Show that the Pontryagin ring $H_*(X;\mathbb{Z})$ is commutative and associative, with a structure uniquely determined by the ring $H^*(X;\mathbb{Z})$.

16. Classify algebraically the Hopf algebras A over $\mathbb{Z}$ such that A^n is free for each n and $A \otimes \mathbb{Q} \approx \mathbb{Q}[\alpha]$. In particular, determine which Hopf algebras $A \otimes \mathbb{Z}_p$ arise from such A's.

3.D The Cohomology of SO(n)

After the general discussion of homological and cohomological properties of H-spaces in the preceding section, we turn now to a family of quite interesting and subtle examples, the orthogonal groups $O(n)$. We will compute their homology and cohomology by constructing very nice CW structures on them, and the results illustrate the general structure theorems of the last section quite well. After dealing with the orthogonal groups we then describe the straightforward generalization to Stiefel manifolds, which are also fairly basic objects in algebraic and geometric topology.

The orthogonal group $O(n)$ can be defined as the group of isometries of $\mathbb{R}^n$ fixing the origin. Equivalently, this is the group of $n \times n$ matrices A with entries in $\mathbb{R}$ such that $AA^t = I$, where A^t is the transpose of A. From this viewpoint, $O(n)$ is topologized as a subspace of $\mathbb{R}^{n^2}$, with coordinates the n^2 entries of an $n \times n$ matrix. Since the columns of a matrix in $O(n)$ are unit vectors, $O(n)$ can also be regarded as a subspace of the product of n copies of S^{n-1}. It is a closed subspace since the conditions that columns be orthogonal are defined by polynomial equations. Hence

$O(n)$ is compact. The map $O(n)\times O(n)\to O(n)$ given by matrix multiplication is continuous since it is defined by polynomials. The inversion map $A\mapsto A^{-1}=A^t$ is clearly continuous, so $O(n)$ is a topological group, and in particular an H–space.

The determinant map $O(n)\to\{\pm 1\}$ is a surjective homomorphism, so its kernel $SO(n)$, the 'special orthogonal group,' is a subgroup of index two. The two cosets $SO(n)$ and $O(n)-SO(n)$ are homeomorphic to each other since for fixed $B\in O(n)$ of determinant -1, the maps $A\mapsto AB$ and $A\mapsto AB^{-1}$ are inverse homeomorphisms between these two cosets. The subgroup $SO(n)$ is a union of components of $O(n)$ since the image of the map $O(n)\to\{\pm 1\}$ is discrete. In fact, $SO(n)$ is path-connected since by linear algebra, each $A\in SO(n)$ is a rotation, a composition of rotations in a family of orthogonal 2-dimensional subspaces of $\mathbb{R}^n$, with the identity map on the subspace orthogonal to all these planes, and such a rotation can obviously be joined to the identity by a path of rotations of the same planes through decreasing angles. Another reason why $SO(n)$ is connected is that it has a CW structure with a single 0-cell, as we show in Proposition 3D.1. An exercise at the end of the section is to show that a topological group with a CW structure is an orientable manifold, so $SO(n)$ is a closed orientable manifold. From the CW structure it follows that its dimension is $n(n-1)/2$. These facts can also be proved using fiber bundles.

The group $O(n)$ is a subgroup of $GL_n(\mathbb{R})$, the 'general linear group' of all invertible $n\times n$ matrices with entries in $\mathbb{R}$, discussed near the beginning of §3.C. The Gram–Schmidt orthogonalization process applied to the columns of matrices in $GL_n(\mathbb{R})$ provides a retraction $r:GL_n(\mathbb{R})\to O(n)$, continuity of r being evident from the explicit formulas for the Gram–Schmidt process. By inserting appropriate scalar factors into these formulas it is easy to see that $O(n)$ is in fact a deformation retract of $GL_n(\mathbb{R})$. Using a bit more linear algebra, namely the polar decomposition, it is possible to show that $GL_n(\mathbb{R})$ is actually homeomorphic to $O(n)\times\mathbb{R}^k$ for $k=n(n+1)/2$.

The topological structure of $SO(n)$ for small values of n can be described in terms of more familiar spaces:

- $SO(1)$ is a point.
- $SO(2)$, the rotations of $\mathbb{R}^2$, is both homeomorphic and isomorphic as a group to S^1, thought of as the unit complex numbers.
- $SO(3)$ is homeomorphic to $\mathbb{R}\mathrm{P}^3$. To see this, let $\varphi:D^3\to SO(3)$ send a nonzero vector x to the rotation through angle $|x|\pi$ about the axis formed by the line through the origin in the direction of x. An orientation convention such as the 'right-hand rule' is needed to make this unambiguous. By continuity, φ then sends 0 to the identity. Antipodal points of $S^2=\partial D^3$ are sent to the same rotation through angle π, so φ induces a map $\overline{\varphi}:\mathbb{R}\mathrm{P}^3\to SO(3)$, regarding $\mathbb{R}\mathrm{P}^3$ as D^3 with antipodal boundary points identified. The map $\overline{\varphi}$ is clearly injective since the axis of a nontrivial rotation is uniquely determined as its fixed point set, and $\overline{\varphi}$ is surjective since by easy linear algebra each nonidentity element

of $SO(3)$ is a rotation about some axis. It follows that $\overline{\varphi}$ is a homeomorphism $\mathbb{R}P^3 \approx SO(3)$.

- $SO(4)$ is homeomorphic to $S^3 \times SO(3)$. Identifying $\mathbb{R}^4$ with the quaternions $\mathbb{H}$ and S^3 with the group of unit quaternions, the quaternion multiplication $v \mapsto vw$ for fixed $w \in S^3$ defines an isometry $\rho_w \in O(4)$ since $|vw| = |v||w| = |v|$ if $|w| = 1$. Points of $O(4)$ are 4-tuples $(v_1, \cdots, v_4)$ of orthonormal vectors $v_i \in \mathbb{H} = \mathbb{R}^4$, and we view $O(3)$ as the subspace with $v_1 = 1$. A homeomorphism $S^3 \times O(3) \to O(4)$ is defined by sending $(v, (1, v_2, v_3, v_4))$ to $(v, v_2 v, v_3 v, v_4 v) = \rho_v(1, v_2, v_3, v_4)$, with inverse $(v, v_2, v_3, v_4) \mapsto (v, (1, v_2 v^{-1}, v_3 v^{-1}, v_4 v^{-1})) = (v, \rho_{v^{-1}}(v, v_2, v_3, v_4))$. Restricting to identity components, we obtain a homeomorphism $S^3 \times SO(3) \approx SO(4)$. This is not a group isomorphism, however. It can be shown, though we will not digress to do so here, that the homomorphism $\psi : S^3 \times S^3 \to SO(4)$ sending a pair (u, v) of unit quaternions to the isometry $w \mapsto uwv^{-1}$ of $\mathbb{H}$ is surjective with kernel $\mathbb{Z}_2 = \{\pm(1,1)\}$, and that ψ is a covering space projection, representing $S^3 \times S^3$ as a 2-sheeted cover of $SO(4)$, the universal cover. Restricting ψ to the diagonal $S^3 = \{(u,u)\} \subset S^3 \times S^3$ gives the universal cover $S^3 \to SO(3)$, so $SO(3)$ is isomorphic to the quotient group of S^3 by the normal subgroup $\{\pm 1\}$.

Using octonions one can construct in the same way a homeomorphism $SO(8) \approx S^7 \times SO(7)$. But in all other cases $SO(n)$ is only a 'twisted product' of $SO(n-1)$ and S^{n-1}; see Example 4.55 and the discussion following Corollary 4D.3.

Cell Structure

Our first task is to construct a CW structure on $SO(n)$. This will come with a very nice cellular map $\rho : \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-2} \times \cdots \times \mathbb{R}P^1 \to SO(n)$. To simplify notation we will write P^i for $\mathbb{R}P^i$.

To each nonzero vector $v \in \mathbb{R}^n$ we can associate the reflection $r(v) \in O(n)$ across the hyperplane consisting of all vectors orthogonal to v. Since $r(v)$ is a reflection, it has determinant -1, so to get an element of $SO(n)$ we consider the composition $\rho(v) = r(v)r(e_1)$ where e_1 is the first standard basis vector $(1, 0, \cdots, 0)$. Since $\rho(v)$ depends only on the line spanned by v, ρ defines a map $P^{n-1} \to SO(n)$. This map is injective since it is the composition of $v \mapsto r(v)$, which is obviously an injection of P^{n-1} into $O(n) - SO(n)$, with the homeomorphism $O(n) - SO(n) \to SO(n)$ given by right-multiplication by $r(e_1)$. Since ρ is injective and P^{n-1} is compact Hausdorff, we may think of ρ as embedding P^{n-1} as a subspace of $SO(n)$.

More generally, for a sequence $I = (i_1, \cdots, i_m)$ with each $i_j < n$, we define a map $\rho : P^I = P^{i_1} \times \cdots \times P^{i_m} \to SO(n)$ by letting $\rho(v_1, \cdots, v_m)$ be the composition $\rho(v_1) \cdots \rho(v_m)$. If $\varphi^i : D^i \to P^i$ is the standard characteristic map for the i-cell of P^i, restricting to the 2-sheeted covering projection $\partial D^i \to P^{i-1}$, then the product $\varphi^I : D^I \to P^I$ of the appropriate φ^{i_j}'s is a characteristic map for the top-dimensional cell of P^I. We will be especially interested in the sequences $I = (i_1, \cdots, i_m)$ satisfying $n > i_1 > \cdots > i_m > 0$. These sequences will be called *admissible*, as will the sequence consisting of a single 0.

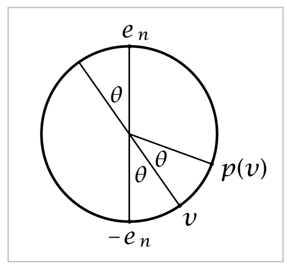
Proposition 3D.1. *The maps $\rho\varphi^I : D^I \to SO(n)$, for I ranging over all admissible sequences, are the characteristic maps of a CW structure on $SO(n)$ for which the map $\rho : P^{n-1} \times P^{n-2} \times \cdots \times P^1 \to SO(n)$ is cellular.*

In particular, there is a single 0-cell $e^0 = \{\mathbb{1}\}$, so $SO(n)$ is path-connected. The other cells $e^I = e^{i_1} \cdots e^{i_m}$ are products, via the group operation in $SO(n)$, of the cells $e^i \subset P^{n-1} \subset SO(n)$.

Proof: According to Proposition A.2 in the Appendix, there are three things to show in order to obtain the CW structure:

(1) For each decreasing sequence I, $\rho\varphi^I$ is a homeomorphism from the interior of D^I onto its image.
(2) The resulting image cells e^I are all disjoint and cover $SO(n)$.
(3) For each e^I, $\rho\varphi^I(\partial D^I)$ is contained in a union of cells of lower dimension than e^I.

To begin the verification of these properties, define $p : SO(n) \to S^{n-1}$ by evaluation at the vector $e_n = (0, \cdots, 0, 1)$, $p(\alpha) = \alpha(e_n)$. Isometries in $P^{n-2} \subset P^{n-1} \subset SO(n)$ fix e_n, so $p(P^{n-2}) = \{e_n\}$. We claim that p is a homeomorphism from $P^{n-1} - P^{n-2}$ onto $S^{n-1} - \{e_n\}$. This can be seen as follows. Thinking of a point in P^n as a vector v, the map p takes this to $\rho(v)(e_n) = r(v)r(e_1)(e_n)$, which equals $r(v)(e_n)$ since e_n is in the hyperplane orthogonal to e_1. From the picture at the right it is then clear that p simply stretches the lower half of each meridian circle in S^{n-1} onto the whole meridian circle, doubling the angle up from the south pole, so $P^{n-1} - P^{n-2}$, represented by vectors whose last coordinate is negative, is taken homeomorphically onto $S^{n-1} - \{e_n\}$.



The next statement is that the map

$$h : (P^{n-1} \times SO(n-1), P^{n-2} \times SO(n-1)) \to (SO(n), SO(n-1)), \quad h(v, \alpha) = \rho(v)\alpha$$

is a homeomorphism from $(P^{n-1} - P^{n-2}) \times SO(n-1)$ onto $SO(n) - SO(n-1)$. Here we view $SO(n-1)$ as the subgroup of $SO(n)$ fixing the vector e_n. To construct an inverse to this homeomorphism, let $\beta \in SO(n) - SO(n-1)$ be given. Then $\beta(e_n) \neq e_n$ so by the preceding paragraph there is a unique $v_\beta \in P^{n-1} - P^{n-2}$ with $\rho(v_\beta)(e_n) = \beta(e_n)$, and v_β depends continuously on β since $\beta(e_n)$ does. The composition $\alpha_\beta = \rho(v_\beta)^{-1}\beta$ then fixes e_n, hence lies in $SO(n-1)$. Since $\rho(v_\beta)\alpha_\beta = \beta$, the map $\beta \mapsto (v_\beta, \alpha_\beta)$ is an inverse to h on $SO(n) - SO(n-1)$.

Statements (1) and (2) can now be proved by induction on n. The map ρ takes P^{n-2} to $SO(n-1)$, so we may assume inductively that the maps $\rho\varphi^I$ for I ranging over admissible sequences with first term $i_1 < n-1$ are the characteristic maps for a CW structure on $SO(n-1)$, with cells the corresponding products e^I. The admissible sequences I with $i_1 = n-1$ then give disjoint cells e^I covering $SO(n) - SO(n-1)$ by what was shown in the previous paragraph. So (1) and (2) hold for $SO(n)$.

To prove (3) it suffices to show there is an inclusion $P^iP^i \subset P^iP^{i-1}$ in $SO(n)$ since for an admissible sequence I, the map $\rho : P^I \to SO(n)$ takes the boundary of the top-dimensional cell of P^I to the image of products P^J with J obtained from I by decreasing one term i_j by 1, yielding a sequence which is admissible except perhaps for having two successive terms equal. As a preliminary to showing that $P^iP^i \subset P^iP^{i-1}$, observe that for $\alpha \in O(n)$ we have $r(\alpha(v)) = \alpha r(v)\alpha^{-1}$. Hence $\rho(v)\rho(w) = r(v)r(e_1)r(w)r(e_1) = r(v)r(w')$ where $w' = r(e_1)w$. Thus to show $P^iP^i \subset P^iP^{i-1}$ it suffices to find for each pair $v, w \in \mathbb{R}^{i+1}$ a pair $x \in \mathbb{R}^{i+1}$, $y \in \mathbb{R}^i$ with $r(v)r(w) = r(x)r(y)$.

Let $V \subset \mathbb{R}^{i+1}$ be a 2-dimensional subspace containing v and w. Since $V \cap \mathbb{R}^i$ is at least 1-dimensional, we can choose a unit vector $y \in V \cap \mathbb{R}^i$. Let $\alpha \in O(i+1)$ take V to $\mathbb{R}^2$ and y to e_1. Then the conjugate $\alpha r(v)r(w)\alpha^{-1} = r(\alpha(v))r(\alpha(w))$ lies in $SO(2)$, hence has the form $\rho(z) = r(z)r(e_1)$ for some $z \in \mathbb{R}^2$ by statement (2) for $n = 2$. Therefore

$$r(v)r(w) = \alpha^{-1}r(z)r(e_1)\alpha = r(\alpha^{-1}(z))r(\alpha^{-1}(e_1)) = r(x)r(y)$$

for $x = \alpha^{-1}(z) \in \mathbb{R}^{i+1}$ and $y \in \mathbb{R}^i$.

It remains to show that the map $\rho : P^{n-1} \times P^{n-2} \times \cdots \times P^1 \to SO(n)$ is cellular. This follows from the inclusions $P^iP^i \subset P^iP^{i-1}$ derived above, together with another family of inclusions $P^iP^j \subset P^jP^i$ for $i < j$. To prove the latter we have the formulas

$$\begin{aligned}
\rho(v)\rho(w) &= r(v)r(w') && \text{where } w' = r(e_1)w, \text{ as earlier} \\
&= r(v)r(w')r(v)r(v) \\
&= r(r(v)w')r(v) && \text{from } r(\alpha(v)) = \alpha r(v)\alpha^{-1} \\
&= r(r(v)r(e_1)w)r(v) = r(\rho(v)w)r(v) \\
&= \rho(\rho(v)w)\rho(v') && \text{where } v' = r(e_1)v, \text{ hence } v = r(e_1)v'
\end{aligned}$$

In particular, taking $v \in \mathbb{R}^{i+1}$ and $w \in \mathbb{R}^{j+1}$ with $i < j$, we have $\rho(v)w \in \mathbb{R}^{j+1}$, and the product $\rho(v)\rho(w) \in P^iP^j$ equals the product $\rho(\rho(v)w)\rho(v') \in P^jP^i$. $\square$

Mod 2 Homology and Cohomology

Each cell of $SO(n)$ is the homeomorphic image of a cell in $P^{n-1} \times P^{n-2} \times \cdots \times P^1$, so the cellular chain map induced by $\rho : P^{n-1} \times P^{n-2} \times \cdots \times P^1 \to SO(n)$ is surjective. It follows that with $\mathbb{Z}_2$ coefficients the cellular boundary maps for $SO(n)$ are all trivial since this is true in P^i and hence in $P^{n-1} \times P^{n-2} \times \cdots \times P^1$ by Proposition 3B.1. Thus $H_*(SO(n);\mathbb{Z}_2)$ has a $\mathbb{Z}_2$ summand for each cell of $SO(n)$. One can rephrase this as saying that there are isomorphisms $H_i(SO(n);\mathbb{Z}_2) \approx H_i(S^{n-1}\times S^{n-2}\times\cdots\times S^1;\mathbb{Z}_2)$ for all i since this product of spheres also has cells in one-to-one correspondence with admissible sequences. The full structure of the $\mathbb{Z}_2$ homology and cohomology rings is given by:

Theorem 3D.2. **(a)** *$H^*(SO(n);\mathbb{Z}_2) \approx \bigotimes_{i \text{ odd}} \mathbb{Z}_2[\beta_i]/(\beta_i^{p_i})$ where $|\beta_i| = i$ and p_i is the smallest power of 2 such that $|\beta_i^{p_i}| \ge n$.*
(b) *The Pontryagin ring $H_*(SO(n);\mathbb{Z}_2)$ is the exterior algebra $\Lambda_{\mathbb{Z}_2}[e^1,\cdots,e^{n-1}]$.*

Here e^i denotes the cellular homology class of the cell $e^i \subset P^{n-1} \subset SO(n)$, and β_i is the dual class to e^i, represented by the cellular cochain assigning the value 1 to the cell e^i and 0 to all other i-cells.

Proof: As we noted above, ρ induces a surjection on cellular chains. Since the cellular boundary maps with $\mathbb{Z}_2$ coefficients are trivial for both $P^{n-1}\times\cdots\times P^1$ and $SO(n)$, it follows that ρ_* is surjective on $H_*(-;\mathbb{Z}_2)$ and ρ^* is injective on $H^*(-;\mathbb{Z}_2)$. We know that $H^*(P^{n-1}\times\cdots\times P^1;\mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[\alpha_1,\cdots,\alpha_{n-1}]$ truncated by the relations $\alpha_i^{i+1} = 0$. For $\beta_i \in H^i(SO(n);\mathbb{Z}_2)$ the dual class to e^i, we have $\rho^*(\beta_i) = \sum_j \alpha_j^i$, the class assigning 1 to each i-cell in a factor P^j of $P^{n-1}\times\cdots\times P^1$ and 0 to all other i-cells, which are products of lower-dimensional cells and hence map to cells in $SO(n)$ disjoint from e^i.

First we will show that the monomials $\beta_I = \beta_{i_1}\cdots\beta_{i_m}$ corresponding to admissible sequences I are linearly independent in $H^*(SO(n);\mathbb{Z}_2)$, hence are a vector space basis. Since ρ^* is injective, we may identify each β_i with its image $\sum_j \alpha_j^i$ in the truncated polynomial ring $\mathbb{Z}_2[\alpha_1,\cdots,\alpha_{n-1}]/(\alpha_1^2,\cdots,\alpha_{n-1}^n)$. Suppose we have a linear relation $\sum_I b_I\beta_I = 0$ with $b_I \in \mathbb{Z}_2$ and I ranging over the admissible sequences. Since each β_I is a product of distinct β_i's, we can write the relation in the form $x\beta_1 + y = 0$ where neither x nor y has β_1 as a factor. Since α_1 occurs only in the term β_1 of $x\beta_1 + y$, where it has exponent 1, we have $x\beta_1 + y = x\alpha_1 + z$ where neither x nor z involves α_1. The relation $x\alpha_1 + z = 0$ in $\mathbb{Z}_2[\alpha_1,\cdots,\alpha_{n-1}]/(\alpha_1^2,\cdots,\alpha_{n-1}^n)$ then implies $x = 0$. Thus we may assume the original relation does not involve β_1. Now we repeat the argument for β_2. Write the relation in the form $x\beta_2 + y = 0$ where neither x nor y involves β_2 or β_1. The variable α_2 now occurs only in the term β_2 of $x\beta_2 + y$, where it has exponent 2, so we have $x\beta_2 + y = x\alpha_2^2 + z$ where x and z do not involve α_1 or α_2. Then $x\alpha_2^2 + z = 0$ implies $x = 0$ and we have a relation involving neither β_1 nor β_2. Continuing inductively, we eventually deduce that all coefficients b_I in the original relation $\sum_I b_I\beta_I = 0$ must be zero.

Observe now that $\beta_i^2 = \beta_{2i}$ if $2i < n$ and $\beta_i^2 = 0$ if $2i \ge n$, since $\big(\sum_j \alpha_j^i\big)^2 = \sum_j \alpha_j^{2i}$. The quotient Q of the algebra $\mathbb{Z}_2[\beta_1,\beta_2,\cdots]$ by the relations $\beta_i^2 = \beta_{2i}$ and $\beta_j = 0$ for $j \ge n$ then maps onto $H^*(SO(n);\mathbb{Z}_2)$. This map $Q \to H^*(SO(n);\mathbb{Z}_2)$ is also injective since the relations defining Q allow every element of Q to be represented as a linear combination of admissible monomials β^I, and the admissible monomials are linearly independent in $H^*(SO(n);\mathbb{Z}_2)$. The algebra Q can also be described as the tensor product in statement (a) of the theorem since the relations $\beta_i^2 = \beta_{2i}$ allow admissible monomials to be written uniquely as monomials in powers of the β_i's with i odd, and the relation $\beta_j = 0$ for $j \ge n$ becomes $\beta_{ip_i} = \beta_i^{p_i} = 0$ where $j = ip_i$ with i odd and p_i a power of 2. For a given i, this relation holds iff $ip_i \ge n$, or in other words, iff $|\beta_i^{p_i}| \ge n$. This finishes the proof of (a).

For part (b), note first that the group multiplication $SO(n)\times SO(n)\to SO(n)$ is cellular in view of the inclusions $P^iP^i \subset P^iP^{i-1}$ and $P^iP^j \subset P^jP^i$ for $i < j$. So we can compute Pontryagin products at the cellular level. We know that there is at least an additive isomorphism $H_*(SO(n);\mathbb{Z}_2) \approx \Lambda_{\mathbb{Z}_2}[e^1, \cdots, e^{n-1}]$ since the products $e^I = e^{i_1}\cdots e^{i_m}$ with I admissible form a basis for $H_*(SO(n);\mathbb{Z}_2)$. The inclusion $P^iP^i \subset P^iP^{i-1}$ then implies that the Pontryagin product $(e^i)^2$ is 0. It remains only to see the commutativity relation $e^ie^j = e^je^i$. The inclusion $P^iP^j \subset P^jP^i$ for $i<j$ was obtained from the formula $\rho(v)\rho(w) = \rho(\rho(v)w)\rho(v')$ for $v \in \mathbb{R}^{i+1}$, $w \in \mathbb{R}^{j+1}$, and $v' = r(e_1)v$. The map $f: P^i\times P^j \to P^j\times P^i$, $f(v,w) = (\rho(v)w, v')$, is a homeomorphism since it is the composition of homeomorphisms $(v,w)\mapsto (v,\rho(v)w)\mapsto (v',\rho(v)w)\mapsto(\rho(v)w,v')$. The first of these maps takes $e^i\times e^j$ homeomorphically onto itself since $\rho(v)(e^j) = e^j$ if $i<j$. Obviously the second map also takes $e^i\times e^j$ homeomorphically onto itself, while the third map simply transposes the two factors. Thus f restricts to a homeomorphism from $e^i\times e^j$ onto $e^j\times e^i$, and therefore $e^ie^j = e^je^i$ in $H_*(SO(n);\mathbb{Z}_2)$. $\square$

The cup product and Pontryagin product structures in this theorem may seem at first glance to be unrelated, but in fact the relationship is fairly direct. As we saw in the previous section, the dual of a polynomial algebra $\mathbb{Z}_2[x]$ is a divided polynomial algebra $\Gamma_{\mathbb{Z}_2}[\alpha]$, and with $\mathbb{Z}_2$ coefficients the latter is an exterior algebra $\Lambda_{\mathbb{Z}_2}[\alpha_0, \alpha_1, \cdots]$ where $|\alpha_i| = 2^i|x|$. If we truncate the polynomial algebra by a relation $x^{2^n} = 0$, then this just eliminates the generators α_i for $i \ge n$. In view of this, if it were the case that the generators β_i for the algebra $H^*(SO(n);\mathbb{Z}_2)$ happened to be primitive, then $H^*(SO(n);\mathbb{Z}_2)$ would be isomorphic as a Hopf algebra to the tensor product of the single-generator Hopf algebras $\mathbb{Z}_2[\beta_i]/(\beta_i^{p_i})$, $i = 1, 3, \cdots$, hence the dual algebra $H_*(SO(n);\mathbb{Z}_2)$ would be the tensor product of the corresponding truncated divided polynomial algebras, in other words an exterior algebra as just explained. This is in fact the structure of $H_*(SO(n);\mathbb{Z}_2)$, so since the Pontryagin product in $H_*(SO(n);\mathbb{Z}_2)$ determines the coproduct in $H^*(SO(n);\mathbb{Z}_2)$ uniquely, it follows that the β_i's must indeed be primitive.

It is not difficult to give a direct argument that each β_i is primitive. The coproduct $\Delta: H^*(SO(n);\mathbb{Z}_2)\to H^*(SO(n);\mathbb{Z}_2)\otimes H^*(SO(n);\mathbb{Z}_2)$ is induced by the group multiplication $\mu: SO(n)\times SO(n)\to SO(n)$. We need to show that the value of $\Delta(\beta_i)$ on $e^I\otimes e^J$, which we denote $\langle\Delta(\beta_i), e^I\otimes e^J\rangle$, is the same as the value $\langle \beta_i\otimes 1 + 1\otimes\beta_i, e^I\otimes e^J\rangle$

for all cells e^I and e^J whose dimensions add up to i. Since $\Delta = \mu^*$, we have $\langle \Delta(\beta_i), e^I \otimes e^J \rangle = \langle \beta_i, \mu_*(e^I \otimes e^J) \rangle$. Because μ is the multiplication map, $\mu(e^I \times e^J)$ is contained in $P^I P^J$, and if we use the relations $P^j P^j \subset P^j P^{j-1}$ and $P^j P^k \subset P^k P^j$ for $j < k$ to rearrange the factors P^j of $P^I P^J$ so that their dimensions are in decreasing order, then the only way we will end up with a term P^i is if we start with $P^I P^J$ equal to $P^i P^0$ or $P^0 P^i$. Thus $\langle \beta_i, \mu_*(e^I \otimes e^J) \rangle = 0$ unless $e^I \otimes e^J$ equals $e^i \otimes e^0$ or $e^0 \otimes e^i$. Hence $\Delta(\beta_i)$ contains no other terms besides $\beta_i \otimes 1 + 1 \otimes \beta_i$, and β_i is primitive.

Integer Homology and Cohomology

With $\mathbb{Z}$ coefficients the homology and cohomology of $SO(n)$ turns out to be a good bit more complicated than with $\mathbb{Z}_2$ coefficients. One can see a little of this complexity already for small values of n, where the homeomorphisms $SO(3) \approx \mathbb{R}\mathrm{P}^3$ and $SO(4) \approx S^3 \times \mathbb{R}\mathrm{P}^3$ would allow one to compute the additive structure as a direct sum of a certain number of $\mathbb{Z}$'s and $\mathbb{Z}_2$'s. For larger values of n the additive structure is qualitatively the same:

Proposition 3D.3. *$H_*(SO(n);\mathbb{Z})$ is a direct sum of $\mathbb{Z}$'s and $\mathbb{Z}_2$'s.*

Proof: We compute the cellular chain complex of $SO(n)$, showing that it splits as a tensor product of simpler complexes. For a cell $e^i \subset P^{n-1} \subset SO(n)$ the cellular boundary de^i is $2e^{i-1}$ for even $i > 0$ and 0 for odd i. To compute the cellular boundary of a cell $e^{i_1} \cdots e^{i_m}$ we can pull it back to a cell $e^{i_1} \times \cdots \times e^{i_m}$ of $P^{n-1} \times \cdots \times P^1$ whose cellular boundary, by Proposition 3B.1, is $\sum_j (-1)^{\sigma_j} e^{i_1} \times \cdots \times de^{i_j} \times \cdots \times e^{i_m}$ where $\sigma_j = i_1 + \cdots + i_{j-1}$. Hence $d(e^{i_1} \cdots e^{i_m}) = \sum_j (-1)^{\sigma_j} e^{i_1} \cdots de^{i_j} \cdots e^{i_m}$, where it is understood that $e^{i_1} \cdots de^{i_j} \cdots e^{i_m}$ is zero if $i_j = i_{j+1} + 1$ since $P^{i_j - 1} P^{i_j - 1} \subset P^{i_j - 1} P^{i_j - 2}$, in a lower-dimensional skeleton.

To split the cellular chain complex $C_*(SO(n))$ as a tensor product of smaller chain complexes, let C^{2i} be the subcomplex of $C_*(SO(n))$ with basis the cells e^0, e^{2i}, e^{2i-1}, and $e^{2i}e^{2i-1}$. This is a subcomplex since $de^{2i-1} = 0$, $de^{2i} = 2e^{2i-1}$, and, in $P^{2i} \times P^{2i-1}$, $d(e^{2i} \times e^{2i-1}) = de^{2i} \times e^{2i-1} + e^{2i} \times de^{2i-1} = 2e^{2i-1} \times e^{2i-1}$, hence $d(e^{2i}e^{2i-1}) = 0$ since $P^{2i-1}P^{2i-1} \subset P^{2i-1}P^{2i-2}$. The claim is that there are chain complex isomorphisms

$$C_*(SO(2k+1)) \approx C^2 \otimes C^4 \otimes \cdots \otimes C^{2k}$$
$$C_*(SO(2k+2)) \approx C^2 \otimes C^4 \otimes \cdots \otimes C^{2k} \otimes C^{2k+1}$$

where C^{2k+1} has basis e^0 and e^{2k+1}. Certainly these isomorphisms hold for the chain groups themselves, so it is only a matter of checking that the boundary maps agree. For the case of $C_*(SO(2k+1))$ this can be seen by induction on k, as the reader can easily verify. Then the case of $C_*(SO(2k+2))$ reduces to the first case by a similar argument.

Since $H_*(C^{2i})$ consists of $\mathbb{Z}$'s in dimensions 0 and $4i-1$ and a $\mathbb{Z}_2$ in dimension $2i-1$, while $H_*(C^{2k+1})$ consists of $\mathbb{Z}$'s in dimensions 0 and $2k+1$, we conclude from the algebraic Künneth formula that $H_*(SO(n);\mathbb{Z})$ is a direct sum of $\mathbb{Z}$'s and $\mathbb{Z}_2$'s. $\square$

Note that the calculation shows that $SO(2k)$ and $SO(2k-1)\times S^{2k-1}$ have isomorphic homology groups in all dimensions.

In view of the preceding proposition, one can get rather complete information about $H_*(SO(n);\mathbb{Z})$ by considering the natural maps to $H_*(SO(n);\mathbb{Z}_2)$ and to the quotient of $H_*(SO(n);\mathbb{Z})$ by its torsion subgroup. Let us denote this quotient by $H_*^{free}(SO(n);\mathbb{Z})$. The same strategy applies equally well to cohomology, and the universal coefficient theorem gives an isomorphism $H^*_{free}(SO(n);\mathbb{Z}) \approx H_*^{free}(SO(n);\mathbb{Z})$.

The proof of the proposition shows that the additive structure of $H_*^{free}(SO(n);\mathbb{Z})$ is fairly simple:

$$H_*^{free}(SO(2k+1);\mathbb{Z}) \approx H_*(S^3\times S^7\times\cdots\times S^{4k-1})$$
$$H_*^{free}(SO(2k+2);\mathbb{Z}) \approx H_*(S^3\times S^7\times\cdots\times S^{4k-1}\times S^{2k+1})$$

The multiplicative structure is also as simple as it could be:

Proposition 3D.4. *The Pontryagin ring $H_*^{free}(SO(n);\mathbb{Z})$ is an exterior algebra,*

$$H_*^{free}(SO(2k+1);\mathbb{Z}) \approx \Lambda_{\mathbb{Z}}[a_3,a_7,\cdots,a_{4k-1}] \quad \textit{where } |a_i| = i$$
$$H_*^{free}(SO(2k+2);\mathbb{Z}) \approx \Lambda_{\mathbb{Z}}[a_3,a_7,\cdots,a_{4k-1},a_{2k+1}]$$

The generators a_i are primitive, so the dual Hopf algebra $H^*_{free}(SO(n);\mathbb{Z})$ is an exterior algebra on the dual generators α_i.

Proof: As in the case of $\mathbb{Z}_2$ coefficients we can work at the level of cellular chains since the multiplication in $SO(n)$ is cellular. Consider first the case $n = 2k+1$. Let E^i be the cycle $e^{2i}e^{2i-1}$ generating a $\mathbb{Z}$ summand of $H_*(SO(n);\mathbb{Z})$. By what we have shown above, the products $E^{i_1}\cdots E^{i_m}$ with $i_1 > \cdots > i_m$ form an additive basis for $H_*^{free}(SO(n);\mathbb{Z})$, so we need only verify that the multiplication is as in an exterior algebra on the classes E^i. The map f in the proof of Proposition 3D.2 gives a homeomorphism $e^i\times e^j \approx e^j\times e^i$ if $i<j$, and this homeomorphism has local degree $(-1)^{ij+1}$ since it is the composition $(v,w)\mapsto(v,\rho(v)w)\mapsto(v',\rho(v)w)\mapsto(\rho(v)w,v')$ of homeomorphisms with local degrees $+1,-1$, and $(-1)^{ij}$. Applying this four times to commute $E^iE^j = e^{2i}e^{2i-1}e^{2j}e^{2j-1}$ to $E^jE^i = e^{2j}e^{2j-1}e^{2i}e^{2i-1}$, three of the four applications give a sign of -1 and the fourth gives a $+1$, so we conclude that $E^iE^j = -E^jE^i$ if $i<j$. When $i=j$ we have $(E^i)^2 = 0$ since $e^{2i}e^{2i-1}e^{2i}e^{2i-1} = e^{2i}e^{2i}e^{2i-1}e^{2i-1}$, which lies in a lower-dimensional skeleton because of the relation $P^{2i}P^{2i}\subset P^{2i}P^{2i-1}$.

Thus we have shown that $H_*(SO(2k+1);\mathbb{Z})$ contains $\Lambda_{\mathbb{Z}}[E^1,\cdots,E^k]$ as a subalgebra. The same reasoning shows that $H_*(SO(2k+2);\mathbb{Z})$ contains the subalgebra $\Lambda_{\mathbb{Z}}[E^1,\cdots,E^k,e^{2k+1}]$. These exterior subalgebras account for all the nontorsion in $H_*(SO(n);\mathbb{Z})$, so the product structure in $H_*^{free}(SO(n);\mathbb{Z})$ is as stated.

Now we show that the generators E^i and e^{2k+1} are primitive in $H_*^{free}(SO(n);\mathbb{Z})$. Looking at the formula for the boundary maps in the cellular chain complex of $SO(n)$, we see that this chain complex is the direct sum of the subcomplexes $C(m)$ with basis the m-fold products $e^{i_1}\cdots e^{i_m}$ with $i_1>\cdots>i_m>0$. We allow $m=0$ here, with $C(0)$ having basis the 0-cell of $SO(n)$. The direct sum $C(0)\oplus\cdots\oplus C(m)$ is the cellular chain complex of the subcomplex of $SO(n)$ consisting of cells that are products of m or fewer cells e^i. In particular, taking $m=2$ we have a subcomplex $X\subset SO(n)$ whose homology, mod torsion, consists of the $\mathbb{Z}$ in dimension zero and the $\mathbb{Z}$'s generated by the cells E^i, together with the cell e^{2k+1} when $n=2k+2$. The inclusion $X\hookrightarrow SO(n)$ induces a commutative diagram

$$\begin{tikzcd}
H_*^{free}(X;\mathbb{Z}) \arrow[r, "\Delta"] \arrow[d] & H_*^{free}(X;\mathbb{Z})\otimes H_*^{free}(X;\mathbb{Z}) \arrow[d] \\
H_*^{free}(SO(n);\mathbb{Z}) \arrow[r, "\Delta"] & H_*^{free}(SO(n);\mathbb{Z})\otimes H_*^{free}(SO(n);\mathbb{Z})
\end{tikzcd}$$

where the lower Δ is the coproduct in $H_*^{free}(SO(n);\mathbb{Z})$ and the upper Δ is its analog for X, coming from the diagonal map $X\to X\times X$ and the Künneth formula. The classes E^i in the lower left group pull back to elements we label $\widetilde{E}^i$ in the upper left group. Since these have odd dimension and $H_*^{free}(X;\mathbb{Z})$ vanishes in even positive dimensions, the images $\Delta(\widetilde{E}^i)$ can have no components $a\otimes b$ with both a and b positive-dimensional. The same is therefore true for $\Delta(E^i)$ by commutativity of the diagram, so the classes E^i are primitive. This argument also works for e^{2k+1} when $n=2k+2$.

Since the exterior algebra generators of $H_*^{free}(SO(n);\mathbb{Z})$ are primitive, this algebra splits as a Hopf algebra into a tensor product of single-generator exterior algebras $\Lambda_{\mathbb{Z}}[a_i]$. The dual Hopf algebra $H^*_{free}(SO(n);\mathbb{Z})$ therefore splits as the tensor product of the dual exterior algebras $\Lambda_{\mathbb{Z}}[\alpha_i]$, hence $H^*_{free}(SO(n);\mathbb{Z})$ is also an exterior algebra. $\square$

The exact ring structure of $H^*(SO(n);\mathbb{Z})$ can be deduced from these results via Bockstein homomorphisms, as we show in Example 3E.7, though the process is somewhat laborious and the answer not very neat.

Stiefel Manifolds

Consider the **Stiefel manifold** $V_{n,k}$, whose points are the *orthonormal k-frames* in $\mathbb{R}^n$, that is, orthonormal k-tuples of vectors. Thus $V_{n,k}$ is a subset of the product of k copies of S^{n-1}, and it is given the subspace topology. As special cases, $V_{n,n}=O(n)$ and $V_{n,1}=S^{n-1}$. Also, $V_{n,2}$ can be identified with the space of unit tangent vectors to S^{n-1} since a vector v at the point $x\in S^{n-1}$ is tangent to S^{n-1} iff it is orthogonal to x. We can also identify $V_{n,n-1}$ with $SO(n)$ since there is a unique way of extending an orthonormal $(n-1)$-frame to a positively oriented orthonormal n-frame.

There is a natural projection $p:O(n)\to V_{n,k}$ sending $\alpha\in O(n)$ to the k-frame consisting of the last k columns of α, which are the images under α of the last k standard basis vectors in $\mathbb{R}^n$. This projection is onto, and the preimages of points are precisely the cosets $\alpha O(n-k)$, where we embed $O(n-k)$ in $O(n)$ as the orthogonal transformations of the first $n-k$ coordinates of $\mathbb{R}^n$. Thus $V_{n,k}$ can be viewed as the space $O(n)/O(n-k)$ of such cosets, with the quotient topology from $O(n)$. This is the same as the previously defined topology on $V_{n,k}$ since the projection $O(n)\to V_{n,k}$ is a surjection of compact Hausdorff spaces.

When $k<n$ the projection $p:SO(n)\to V_{n,k}$ is surjective, and $V_{n,k}$ can also be viewed as the coset space $SO(n)/SO(n-k)$. We can use this to induce a CW structure on $V_{n,k}$ from the CW structure on $SO(n)$. The cells are the sets of cosets of the form $e^I SO(n-k)=e^{i_1}\cdots e^{i_m}SO(n-k)$ for $n>i_1>\cdots>i_m\ge n-k$, together with the coset $SO(n-k)$ itself as a 0-cell of $V_{n,k}$. These sets of cosets are unions of cells of $SO(n)$ since $SO(n-k)$ consists of the cells $e^J=e^{j_1}\cdots e^{j_\ell}$ with $n-k>j_1>\cdots>j_\ell$. This implies that $V_{n,k}$ is the disjoint union of its cells, and the boundary of each cell is contained in cells of lower dimension, so we do have a CW structure.

Since the projection $SO(n)\to V_{n,k}$ is a cellular map, the structure of the cellular chain complex of $V_{n,k}$ can easily be deduced from that of $SO(n)$. For example, the cellular chain complex of $V_{2k+1,2}$ is just the complex C^{2k} defined earlier, while for $V_{2k,2}$ the cellular boundary maps are all trivial. Hence the nonzero homology groups of $V_{n,2}$ are

$$H_i(V_{2k+1,2};\mathbb{Z})=\begin{cases}\mathbb{Z} & \text{for } i=0,\ 4k-1\\ \mathbb{Z}_2 & \text{for } i=2k-1\end{cases}$$

$$H_i(V_{2k,2};\mathbb{Z})=\mathbb{Z}\quad \text{for } i=0,\ 2k-2,\ 2k-1,\ 4k-3$$

Thus $SO(n)$ has the same homology and cohomology groups as the product space $V_{3,2}\times V_{5,2}\times\cdots\times V_{2k+1,2}$ when $n=2k+1$, or as $V_{3,2}\times V_{5,2}\times\cdots\times V_{2k+1,2}\times S^{2k+1}$ when $n=2k+2$. However, our calculations show that $SO(n)$ is distinguished from these products by its cup product structure with $\mathbb{Z}_2$ coefficients, at least when $n\ge5$, since β_1^4 is nonzero in $H^4(SO(n);\mathbb{Z}_2)$ if $n\ge5$, while for the product spaces the nontrivial element of $H^1(-;\mathbb{Z}_2)$ must lie in the factor $V_{3,2}$, and $H^4(V_{3,2};\mathbb{Z}_2)=0$. When $n=4$ we have $SO(4)$ homeomorphic to $SO(3)\times S^3=V_{3,2}\times S^3$ as we noted at the beginning of this section. Also $SO(3)=V_{3,2}$ and $SO(2)=S^1$.

Exercises

1. Show that a topological group that has a CW structure is an orientable manifold. [Consider the homeomorphisms $x\mapsto xg$ for a fixed group element g.]

2. Using the CW structure on $SO(n)$, show that $\pi_1 SO(n)\approx\mathbb{Z}_2$ for $n\ge3$. Find a loop representing a generator, and describe how twice this loop is nullhomotopic.

3. Compute the Pontryagin ring structure in $H_*(SO(5);\mathbb{Z})$.

3.E Bockstein Homomorphisms

Homology and cohomology with coefficients in a field, particularly $\mathbb{Z}_p$ with p prime, often have more structure and are easier to compute than with $\mathbb{Z}$ coefficients. Of course, passing from $\mathbb{Z}$ to $\mathbb{Z}_p$ coefficients can involve a certain loss of information, a blurring of finer distinctions. For example, a $\mathbb{Z}_{p^n}$ in integer homology becomes a pair of $\mathbb{Z}_p$'s in $\mathbb{Z}_p$ homology or cohomology, so the exponent n is lost with $\mathbb{Z}_p$ coefficients. In this section we introduce Bockstein homomorphisms, which in many interesting cases allow one to recover $\mathbb{Z}$ coefficient information from $\mathbb{Z}_p$ coefficients. Bockstein homomorphisms also provide a small piece of extra internal structure to $\mathbb{Z}_p$ homology or cohomology itself, which can be quite useful.

We will concentrate on cohomology in order to have cup products available, but the basic constructions work equally well for homology. If we take a short exact sequence $0 \to G \to H \to K \to 0$ of abelian groups and apply the covariant functor $\mathrm{Hom}(C_n(X), -)$, we obtain

$$0 \longrightarrow C^n(X;G) \longrightarrow C^n(X;H) \longrightarrow C^n(X;K) \longrightarrow 0$$

which is exact since $C_n(X)$ is free. Letting n vary, we have a short exact sequence of chain complexes, so there is an associated long exact sequence

$$\cdots \longrightarrow H^n(X;G) \longrightarrow H^n(X;H) \longrightarrow H^n(X;K) \longrightarrow H^{n+1}(X;G) \longrightarrow \cdots$$

whose 'boundary' map $H^n(X;K) \to H^{n+1}(X;G)$ is called a **Bockstein homomorphism**.

We shall be interested primarily in the Bockstein $\beta : H^n(X;\mathbb{Z}_m) \to H^{n+1}(X;\mathbb{Z}_m)$ associated to the coefficient sequence $0 \to \mathbb{Z}_m \xrightarrow{m} \mathbb{Z}_{m^2} \longrightarrow \mathbb{Z}_m \to 0$, especially when m is prime, but for the moment we do not need this assumption. Closely related to β is the Bockstein $\widetilde{\beta} : H^n(X;\mathbb{Z}_m) \to H^{n+1}(X;\mathbb{Z})$ associated to $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}_m \to 0$. From the natural map of the latter short exact sequence onto the former one, we obtain the relationship $\beta = \rho\widetilde{\beta}$ where $\rho : H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Z}_m)$ is the homomorphism induced by the map $\mathbb{Z} \to \mathbb{Z}_m$ reducing coefficients mod m. Thus we have a commutative triangle in the following diagram, whose upper row is the exact sequence containing $\widetilde{\beta}$.

$$\begin{tikzcd}
H^n(X;\mathbb{Z}) \arrow[r, "\rho"] & H^n(X;\mathbb{Z}_m) \arrow[r, "\widetilde{\beta}"] \arrow[dr, "\beta"'] & H^{n+1}(X;\mathbb{Z}) \arrow[r, "m"] \arrow[d, "\rho"] & H^{n+1}(X;\mathbb{Z}) \\
& & H^{n+1}(X;\mathbb{Z}_m) &
\end{tikzcd}$$

Example 3E.1. Let X be a $K(\mathbb{Z}_m, 1)$, for example $\mathbb{R}\mathrm{P}^\infty$ when $m = 2$ or an infinite-dimensional lens space with fundamental group $\mathbb{Z}_m$ for arbitrary m. From the homology calculations in Examples 2.42 and 2.43 together with the universal coefficient theorem or cellular cohomology we have $H^n(X;\mathbb{Z}_m) \approx \mathbb{Z}_m$ for all n. Let us show that $\beta : H^n(X;\mathbb{Z}_m) \to H^{n+1}(X;\mathbb{Z}_m)$ is an isomorphism for n odd and zero for n even. If n is odd the vertical map ρ in the diagram above is surjective for $X = K(\mathbb{Z}_m, 1)$, as is $\widetilde{\beta}$ since the map m is trivial, so β is surjective, hence an isomorphism. On the other hand, when n is even the first map ρ in the diagram is surjective, so $\widetilde{\beta} = 0$ by exactness, hence $\beta = 0$.

A useful fact about β is that it satisfies the derivation property

$$(*) \qquad \beta(a \smile b) = \beta(a) \smile b + (-1)^{|a|} a \smile \beta(b)$$

which comes from the corresponding formula for ordinary coboundary. Namely, let φ and ψ be $\mathbb{Z}_m$ cocycles representing a and b, and let $\widetilde{\varphi}$ and $\widetilde{\psi}$ be lifts of these to $\mathbb{Z}_{m^2}$ cochains. Concretely, one can view φ and ψ as functions on singular simplices with values in $\{0, 1, \cdots, m-1\}$, and then $\widetilde{\varphi}$ and $\widetilde{\psi}$ can be taken to be the same functions, but with $\{0, 1, \cdots, m-1\}$ regarded as a subset of $\mathbb{Z}_{m^2}$. Then $\delta\widetilde{\varphi} = m\eta$ and $\delta\widetilde{\psi} = m\mu$ for $\mathbb{Z}_p$ cocycles η and μ representing $\beta(a)$ and $\beta(b)$. Taking cup products, $\widetilde{\varphi} \smile \widetilde{\psi}$ is a $\mathbb{Z}_{m^2}$ cochain lifting the $\mathbb{Z}_m$ cocycle $\varphi \smile \psi$, and

$$\delta(\widetilde{\varphi} \smile \widetilde{\psi}) = \delta\widetilde{\varphi} \smile \widetilde{\psi} \pm \widetilde{\varphi} \smile \delta\widetilde{\psi} = m\eta \smile \widetilde{\psi} \pm \widetilde{\varphi} \smile m\mu = m(\eta \smile \psi \pm \varphi \smile \mu)$$

where the sign $\pm$ is $(-1)^{|a|}$. Hence $\eta \smile \psi + (-1)^{|a|}\varphi \smile \mu$ represents $\beta(a \smile b)$, giving the formula $(*)$.

Example 3E.2: Cup Products in Lens Spaces. The cup product structure for lens spaces was computed in Example 3.41 via Poincaré duality, but using Bocksteins we can deduce it from the cup product structure in $\mathbb{C}\mathrm{P}^\infty$, which was computed in Theorem 3.12 without Poincaré duality. Consider first the infinite-dimensional lens space $L = S^\infty/\mathbb{Z}_m$ where $\mathbb{Z}_m$ acts on the unit sphere S^∞ in $\mathbb{C}^\infty$ by scalar multiplication, so the action is generated by the rotation $v \mapsto e^{2\pi i/m}v$. The quotient map $S^\infty \to \mathbb{C}\mathrm{P}^\infty$ factors through L, so we have a projection $L \to \mathbb{C}\mathrm{P}^\infty$. Looking at the cell structure on L described in Example 2.43, we see that each even-dimensional cell of L projects homeomorphically onto the corresponding cell of $\mathbb{C}\mathrm{P}^\infty$. Namely, the $2n$-cell of L is the homeomorphic image of the $2n$-cell in $S^{2n+1} \subset \mathbb{C}^{n+1}$ formed by the points $\cos\theta(z_1, \cdots, z_n, 0) + \sin\theta(0, \cdots, 0, 1)$ with $\sum_i z_i^2 = 1$ and $0 < \theta \le \pi$, and the same is true for the $2n$-cell of $\mathbb{C}\mathrm{P}^\infty$. From cellular cohomology it then follows that the map $L \to \mathbb{C}\mathrm{P}^\infty$ induces isomorphisms on even-dimensional cohomology with $\mathbb{Z}_m$ coefficients. Since $H^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}_m)$ is a polynomial ring, we deduce that if $y \in H^2(L; \mathbb{Z}_m)$ is a generator, then y^k generates $H^{2k}(L; \mathbb{Z}_m)$ for all k.

By Example 3E.1 there is a generator $x \in H^1(L; \mathbb{Z}_m)$ with $\beta(x) = y$. The product formula $(*)$ gives $\beta(xy^k) = \beta(x)y^k - x\beta(y^k) = y^{k+1}$. Thus β takes xy^k to a generator, hence xy^k must be a generator of $H^{2k+1}(L; \mathbb{Z}_m)$. This completely determines the cup product structure in $H^*(L; \mathbb{Z}_m)$ if m is odd since the commutativity property of cup product implies that $x^2 = 0$ in this case. The result is that $H^*(L; \mathbb{Z}_m) \approx \Lambda_{\mathbb{Z}_m}[x] \otimes \mathbb{Z}_m[y]$ for odd m. When m is even this statement needs to be modified slightly by inserting the relation that x^2 is the unique element of order

2 in $H^2(L;\mathbb{Z}_m) \approx \mathbb{Z}_m$, as we showed in Example 3.9 by an explicit calculation in the 2-skeleton of L.

The cup product structure in finite-dimensional lens spaces follows from this since a finite-dimensional lens space embeds as a skeleton in an infinite-dimensional lens space, and the homotopy type of an infinite-dimensional lens space is determined by its fundamental group since it is a $K(\pi,1)$. It follows that the cup product structure on a lens space $S^{2n+1}/\mathbb{Z}_m$ with $\mathbb{Z}_m$ coefficients is obtained from the preceding calculation by truncating via the relation $y^{n+1}=0$.

The relation $\beta = \rho\widetilde{\beta}$ implies that $\beta^2 = \rho\widetilde{\beta}\rho\widetilde{\beta} = 0$ since $\widetilde{\beta}\rho = 0$ in the long exact sequence containing $\widetilde{\beta}$. Because $\beta^2 = 0$, the groups $H^n(X;\mathbb{Z}_m)$ form a chain complex with the Bockstein homomorphisms β as the 'boundary' maps. We can then form the associated *Bockstein cohomology groups* $\operatorname{Ker}\beta/\operatorname{Im}\beta$, which we denote $BH^n(X;\mathbb{Z}_m)$ in dimension n. The most interesting case is when m is a prime p, so we shall assume this from now on.

Proposition 3E.3. *If $H_n(X;\mathbb{Z})$ is finitely generated for all n, then the Bockstein cohomology groups $BH^n(X;\mathbb{Z}_p)$ are determined by the following rules*:

(a) *Each $\mathbb{Z}$ summand of $H^n(X;\mathbb{Z})$ contributes a $\mathbb{Z}_p$ summand to $BH^n(X;\mathbb{Z}_p)$.*

(b) *Each $\mathbb{Z}_{p^k}$ summand of $H^n(X;\mathbb{Z})$ with $k>1$ contributes $\mathbb{Z}_p$ summands to both $BH^{n-1}(X;\mathbb{Z}_p)$ and $BH^n(X;\mathbb{Z}_p)$.*

(c) *A $\mathbb{Z}_p$ summand of $H^n(X;\mathbb{Z})$ gives $\mathbb{Z}_p$ summands of $H^{n-1}(X;\mathbb{Z}_p)$ and $H^n(X;\mathbb{Z}_p)$ with β an isomorphism between these two summands, hence there is no contribution to $BH^*(X;\mathbb{Z}_p)$.*

Proof: We will use the algebraic notion of *minimal chain complexes.* Suppose that C is a chain complex of free abelian groups for which the homology groups $H_n(C)$ are finitely generated for each n. Choose a splitting of each $H_n(C)$ as a direct sum of cyclic groups. There are countably many of these cyclic groups, so we can list them as $G_1, G_2, \cdots$. For each G_i choose a generator g_i and define a corresponding chain complex $M(g_i)$ by the following prescription. If g_i has infinite order in $G_i \subset H_{n_i}(C)$, let $M(g_i)$ consist of just a $\mathbb{Z}$ in dimension n_i, with generator z_i. On the other hand, if g_i has finite order k in $H_{n_i}(C)$, let $M(g_i)$ consist of $\mathbb{Z}$'s in dimensions n_i and n_i+1, generated by x_i and y_i respectively, with $\partial y_i = kx_i$. Let M be the direct sum of the chain complexes $M(g_i)$. Define a chain map $\sigma: M \to C$ by sending z_i and x_i to cycles ζ_i and ξ_i representing the corresponding homology classes g_i, and y_i to a chain η_i with $\partial\eta_i = k\xi_i$. The chain map σ induces an isomorphism on homology, hence also on cohomology with any coefficients, by Corollary 3.4. The dual cochain complex M^* obtained by applying $\operatorname{Hom}(-,\mathbb{Z})$ splits as the direct sum of the dual complexes $M^*(g_i)$. So in cohomology with $\mathbb{Z}$ coefficients the dual basis element z_i^* generates a $\mathbb{Z}$ summand in dimension n_i, while y_i^* generates a $\mathbb{Z}_k$ summand in dimension n_i+1 since $\delta x_i^* = ky_i^*$. With $\mathbb{Z}_p$ coefficients, p prime, z_i^* gives a $\mathbb{Z}_p$ summand of

$H^{n_i}(M;\mathbb{Z}_p)$, while x_i^* and y_i^* give $\mathbb{Z}_p$ summands of $H^{n_i}(M;\mathbb{Z}_p)$ and $H^{n_i+1}(M;\mathbb{Z}_p)$ if p divides k and otherwise they give nothing.

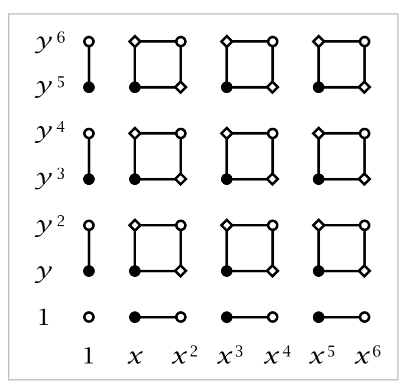
The map σ induces an isomorphism between the associated Bockstein long exact sequences of cohomology groups, with commuting squares, so we can use M^* to compute β and $\widetilde{\beta}$, and we can do the calculation separately on each summand $M^*(g_i)$. Obviously β and $\widetilde{\beta}$ are zero on y_i^* and z_i^*. When p divides k we have the class $x_i^* \in H^{n_i}(M;\mathbb{Z}_p)$, and from the definition of Bockstein homomorphisms it follows that $\widetilde{\beta}(x_i^*) = (k/p)y_i^* \in H^{n_i+1}(M;\mathbb{Z})$ and $\beta(x_i^*) = (k/p)y_i^* \in H^{n_i+1}(M;\mathbb{Z}_p)$. The latter element is nonzero iff k is not divisible by p^2. □

Corollary 3E.4. *In the situation of the preceding proposition, $H^*(X;\mathbb{Z})$ contains no elements of order p^2 iff the dimension of $BH^n(X;\mathbb{Z}_p)$ as a vector space over $\mathbb{Z}_p$ equals the rank of $H^n(X;\mathbb{Z})$ for all n. In this case $\rho : H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Z}_p)$ is injective on the p-torsion, and the image of this p-torsion under ρ is equal to* $\operatorname{Im}\beta$.

Proof: The first statement is evident from the proposition. The injectivity of ρ on p-torsion is in fact equivalent to there being no elements of order p^2. The equality $\operatorname{Im}\rho = \operatorname{Im}\beta$ follows from the fact that $\operatorname{Im}\beta = \rho(\operatorname{Im}\widetilde{\beta}) = \rho(\operatorname{Ker} m)$ in the commutative diagram near the beginning of this section, and the fact that for $m = p$ the kernel of m is exactly the p-torsion when there are no elements of order p^2. □

Example 3E.5. Let us use Bocksteins to compute $H^*(\mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty;\mathbb{Z})$. This could instead be done by first computing the homology via the general Künneth formula, then applying the universal coefficient theorem, but with Bocksteins we will only need the simpler Künneth formula for field coefficients in Theorem 3.16. The cup product structure in $H^*(\mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty;\mathbb{Z})$ will also be easy to determine via Bocksteins.

For p an odd prime we have $\widetilde{H}^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_p) = 0$, hence $\widetilde{H}^*(\mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_p) = 0$ by Theorem 3.16. The universal coefficient theorem then implies that $\widetilde{H}^*(\mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty;\mathbb{Z})$ consists entirely of elements of order a power of 2. From Example 3E.1 we know that Bockstein homomorphisms in $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) \approx \mathbb{Z}_2[x]$ are given by $\beta(x^{2k-1}) = x^{2k}$ and $\beta(x^{2k}) = 0$. In $H^*(\mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) \approx \mathbb{Z}_2[x,y]$ we can then compute β via the product formula $\beta(x^m y^n) = (\beta x^m)y^n + x^m(\beta y^n)$. The answer can be represented graphically by the figure to the right. Here the dot, diamond, or circle in the (m,n) position represents the monomial $x^m y^n$ and line segments indicate nontrivial Bocksteins. For example, the lower left square records the formulas $\beta(xy) = x^2y + xy^2$, $\beta(x^2y) = x^2y^2 = \beta(xy^2)$, and $\beta(x^2y^2) = 0$. Thus in this square we see that $\operatorname{Ker}\beta = \operatorname{Im}\beta$, with generators the 'diagonal' sum $x^2y + xy^2$ and x^2y^2. The



same thing happens in all the other squares, so it is apparent that $\operatorname{Ker}\beta = \operatorname{Im}\beta$ except for the zero-dimensional class '1.' By the preceding corollary this says that all nontrivial elements of $\widetilde{H}^*(\mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty;\mathbb{Z})$ have order 2. Furthermore, $\operatorname{Im}\beta$ consists of the subring $\mathbb{Z}_2[x^2,y^2]$, indicated by the circles in the figure, together with the multiples of $x^2y + xy^2$ by elements of $\mathbb{Z}_2[x^2,y^2]$. It follows that there is a ring isomorphism

$$H^*(\mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty;\mathbb{Z}) \approx \mathbb{Z}[\lambda,\mu,\nu]/(2\lambda,2\mu,2\nu,\nu^2+\lambda^2\mu+\lambda\mu^2)$$

where $\rho(\lambda) = x^2$, $\rho(\mu) = y^2$, $\rho(\nu) = x^2y + xy^2$, and the relation $\nu^2+\lambda^2\mu+\lambda\mu^2 = 0$ holds since $(x^2y+xy^2)^2 = x^4y^2 + x^2y^4$.

This calculation illustrates the general principle that cup product structures with $\mathbb{Z}$ coefficients tend to be considerably more complicated than with field coefficients. One can see even more striking evidence of this by computing $H^*(\mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty;\mathbb{Z})$ by the same technique.

Example 3E.6. Let us construct finite CW complexes X_1, X_2, and Y such that the rings $H^*(X_1;\mathbb{Z})$ and $H^*(X_2;\mathbb{Z})$ are isomorphic but $H^*(X_1 \times Y;\mathbb{Z})$ and $H^*(X_2 \times Y;\mathbb{Z})$ are isomorphic only as groups, not as rings. According to Theorem 3.16 this can happen only if all three of X_1, X_2, and Y have torsion in their $\mathbb{Z}$-cohomology. The space X_1 is obtained from $S^2 \times S^2$ by attaching a 3-cell e^3 to the second S^2 factor by a map of degree 2. Thus X_1 has a CW structure with cells e^0, e^2_1, e^2_2, e^3, e^4 with e^3 attached to the 2-sphere $e_0 \cup e^2_2$. The space X_2 is obtained from $S^2 \vee S^2 \vee S^4$ by attaching a 3-cell to the second S^2 summand by a map of degree 2, so it has a CW structure with the same collection of five cells, the only difference being that in X_2 the 4-cell is attached trivially. For the space Y we choose a Moore space $M(\mathbb{Z}_2,2)$, with cells labeled f^0, f^2, f^3, the 3-cell being attached by a map of degree 2.

From cellular cohomology we see that both $H^*(X_1;\mathbb{Z})$ and $H^*(X_2;\mathbb{Z})$ consist of $\mathbb{Z}$'s in dimensions 0, 2, and 4, and a $\mathbb{Z}_2$ in dimension 3. In both cases all cup products of positive-dimensional classes are zero since for dimension reasons the only possible nontrivial product is the square of the 2-dimensional class, but this is zero as one sees by restricting to the subcomplex $S^2 \times S^2$ or $S^2 \vee S^2 \vee S^4$. For the space Y we have $H^*(Y;\mathbb{Z})$ consisting of a $\mathbb{Z}$ in dimension 0 and a $\mathbb{Z}_2$ in dimension 3, so the cup product structure here is trivial as well.

With $\mathbb{Z}_2$ coefficients the cellular cochain complexes for X_i, Y, and $X_i \times Y$ are all trivial, so we can identify the cells with a basis for $\mathbb{Z}_2$ cohomology. In X_i and Y the only nontrivial $\mathbb{Z}_2$ Bocksteins are $\beta(e^2_2) = e^3$ and $\beta(f^2) = f^3$. The Bocksteins in $X_i \times Y$ can then be computed using the product formula for β, which applies to cross product as well as cup product since cross product is defined in terms of cup product. The results are shown in the following table, where an arrow denotes a nontrivial Bockstein.

$$
\begin{tikzcd}[row sep=small, column sep=small]
e^0\times f^0 & e_1^2\times f^0 & e^3\times f^0 & e^4\times f^0 & e_1^2\times f^3 & e^4\times f^2 \arrow[r] & e^4\times f^3 \\
 & e_2^2\times f^0 \arrow[ur] & e^0\times f^3 & e_1^2\times f^2 \arrow[ur] & e^3\times f^2 \arrow[r] & e^3\times f^3 & \\
 & e^0\times f^2 \arrow[ur] & & e_2^2\times f^2 \arrow[ur] \arrow[r] & e_2^2\times f^3 \arrow[ur] & &
\end{tikzcd}
$$

The two arrows from $e_2^2\times f^2$ mean that $\beta(e_2^2\times f^2) = e^3\times f^2 + e_2^2\times f^3$. It is evident that $BH^*(X_i\times Y;\mathbb{Z}_2)$ consists of $\mathbb{Z}_2$'s in dimensions 0, 2, and 4, so Proposition 3E.3 implies that the nontorsion in $H^*(X_i\times Y;\mathbb{Z})$ consists of $\mathbb{Z}$'s in these dimensions. Furthermore, by Corollary 3E.4 the 2-torsion in $H^*(X_i\times Y;\mathbb{Z})$ corresponds to the image of β and consists of $\mathbb{Z}_2\times\mathbb{Z}_2$'s in dimensions 3 and 5 together with $\mathbb{Z}_2$'s in dimensions 6 and 7. In particular, there is a $\mathbb{Z}_2$ corresponding to $e^3\times f^2+e_2^2\times f^3$ in dimension 5. There is no p-torsion for odd primes p since $H^*(X_i\times Y;\mathbb{Z}_p) \approx H^*(X_i;\mathbb{Z}_p)\otimes H^*(Y;\mathbb{Z}_p)$ is nonzero only in even dimensions.

We can see now that with $\mathbb{Z}$ coefficients, the cup product $H^2\times H^5 \to H^7$ is nontrivial for $X_1\times Y$ but trivial for $X_2\times Y$. For in $H^*(X_i\times Y;\mathbb{Z}_2)$ we have, using the relation $(a\times b)\smile(c\times d) = (a\smile c)\times(b\smile d)$ which follows immediately from the definition of cross product,

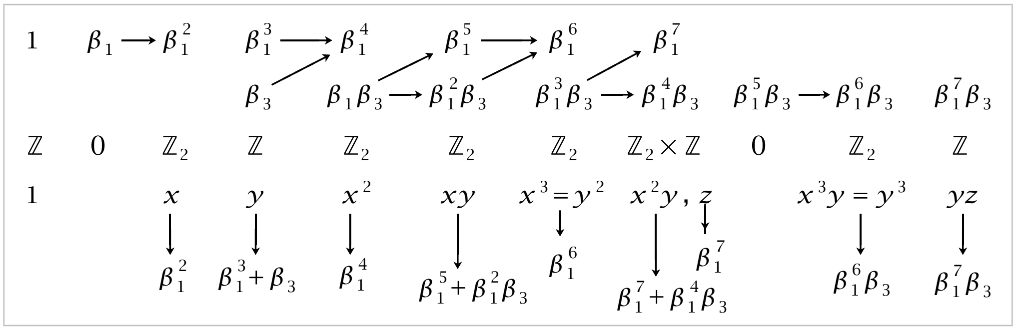
(1) $e_1^2\times f^0\smile e_1^2\times f^3 = (e_1^2\smile e_1^2)\times(f^0\smile f^3) = 0$ since $e_1^2\smile e_1^2 = 0$

(2) $e_1^2\times f^0 \smile (e^3\times f^2 + e_2^2\times f^3) = (e_1^2\smile e^3)\times(f^0\smile f^2) + (e_1^2\smile e_2^2)\times(f^0\smile f^3) = (e_1^2\smile e_2^2)\times f^3$ since $e_1^2\smile e^3 = 0$

and in $H^7(X_i\times Y;\mathbb{Z}_2) \approx H^7(X_i\times Y;\mathbb{Z})$ we have $(e_1^2\smile e_2^2)\times f^3 = e^4\times f^3 \neq 0$ for $i = 1$ but $(e_1^2\smile e_2^2)\times f^3 = 0\times f^3 = 0$ for $i = 2$.

Thus the cohomology ring of a product space is not always determined by the cohomology rings of the factors.

Example 3E.7. Bockstein homomorphisms can be used to get a more complete picture of the structure of $H^*(SO(n);\mathbb{Z})$ than we obtained in the preceding section. Continuing the notation employed there, we know from the calculation for $\mathbb{R}P^\infty$ in Example 3E.1 that $\beta(\sum_j \alpha_j^{2i-1}) = \sum_j \alpha_j^{2i}$ and $\beta(\sum_j \alpha_j^{2i}) = 0$, hence $\beta(\beta_{2i-1}) = \beta_{2i}$ and $\beta(\beta_{2i}) = 0$. Taking the case $n = 5$ as an example, we have $H^*(SO(5);\mathbb{Z}_2) \approx \mathbb{Z}_2[\beta_1,\beta_3]/(\beta_1^8,\beta_3^2)$. The upper part of the table at the top of the next page shows the nontrivial Bocksteins. Once again two arrows from an element mean 'sum,' for example $\beta(\beta_1\beta_3) = \beta(\beta_1)\beta_3 + \beta_1\beta(\beta_3) = \beta_2\beta_3 + \beta_1\beta_4 = \beta_1^2\beta_3 + \beta_1^5$. This Bockstein data allows us to calculate $H^i(SO(5);\mathbb{Z})$ modulo odd torsion, with the results indicated in the remainder of the table, where the vertical arrows denote the map ρ. As we showed in Proposition 3D.3, there is no odd torsion, so this in fact gives the full calculation of $H^i(SO(5);\mathbb{Z})$.

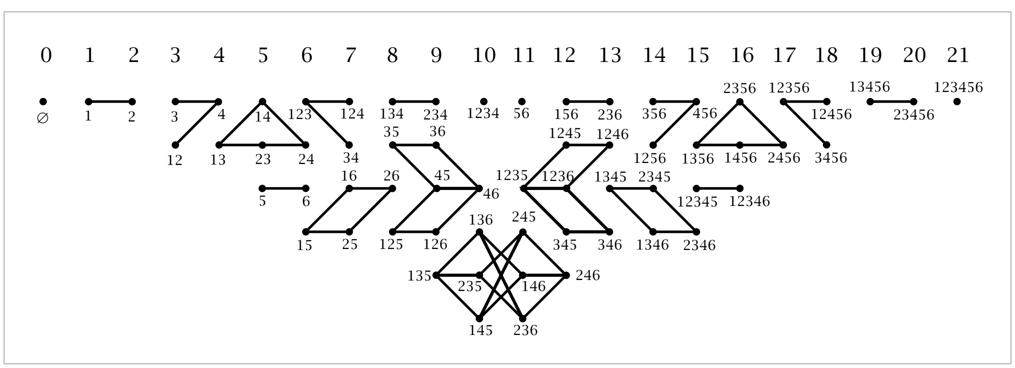


It is interesting that the generator $y \in H^3(SO(5);\mathbb{Z}) \approx \mathbb{Z}$ has y^2 nontrivial, since this implies that the ring structures of $H^*(SO(5);\mathbb{Z})$ and $H^*(\mathbb{R}\mathrm{P}^7 \times S^3;\mathbb{Z})$ are not isomorphic, even though the cohomology groups and the $\mathbb{Z}_2$ cohomology rings of these two spaces are the same. An exercise at the end of the section is to show that in fact $SO(5)$ is not homotopy equivalent to the product of any two CW complexes with nontrivial cohomology.

A natural way to describe $H^*(SO(5);\mathbb{Z})$ would be as a quotient of a free graded commutative associative algebra $F[x,y,z]$ over $\mathbb{Z}$ with $|x| = 2$, $|y| = 3$, and $|z| = 7$. Elements of $F[x,y,z]$ are representable as polynomials $p(x,y,z)$, subject only to the relations imposed by commutativity. In particular, since y and z are odd-dimensional we have $yz = -zy$, and y^2 and z^2 are nonzero elements of order 2 in $F[x,y,z]$. Any monomial containing y^2 or z^2 as a factor also has order 2. In these terms, the calculation of $H^*(SO(5);\mathbb{Z})$ can be written

$$H^*(SO(5);\mathbb{Z}) \approx F[x,y,z]/(2x, x^4, y^4, z^2, xz, x^3 - y^2)$$

The next figure shows the nontrivial Bocksteins for $H^*(SO(7);\mathbb{Z}_2)$. Here the numbers across the top indicate dimension, stopping with 21, the dimension of $SO(7)$. The labels on the dots refer to the basis of products of distinct β_i's. For example, the dot labeled 135 is $\beta_1\beta_3\beta_5$.



The left-right symmetry of the figure displays Poincaré duality quite graphically. Note that the corresponding diagram for $SO(5)$, drawn in a slightly different way from the preceding figure, occurs in the upper left corner as the subdiagram with labels 1 through 4. This subdiagram has the symmetry of Poincaré duality as well.

From the diagram one can with some effort work out the cup product structure in $H^*(SO(7);\mathbb{Z})$, but the answer is rather complicated, just as the diagram is:

$$F[x,y,z,v,w]/(2x,2v,x^4,y^4,z^2,v^2,w^2,xz,vz,vw,y^2w,x^3y^2v,\\ y^2z-x^3v,xw-y^2v-x^3v)$$

where x, y, z, v, w have dimensions 2, 3, 7, 7, 11, respectively. It is curious that the relation $x^3=y^2$ in $H^*(SO(5);\mathbb{Z})$ no longer holds in $H^*(SO(7);\mathbb{Z})$.

Exercises

1. Show that $H^*(K(\mathbb{Z}_m,1);\mathbb{Z}_k)$ is isomorphic as a ring to $H^*(K(\mathbb{Z}_m,1);\mathbb{Z}_m)\otimes\mathbb{Z}_k$ if k divides m. In particular, if m/k is even, this is $\Lambda_{\mathbb{Z}_k}[x]\otimes\mathbb{Z}_k[y]$.

2. In this problem we will derive one half of the classification of lens spaces up to homotopy equivalence, by showing that if $L_m(\ell_1,\cdots,\ell_n)\simeq L_m(\ell_1',\cdots,\ell_n')$ then $\ell_1\cdots\ell_n\equiv\pm\ell_1'\cdots\ell_n'k^n \bmod m$ for some integer k. The converse is Exercise 29 for §4.2.

(a) Let $L=L_m(\ell_1,\cdots,\ell_n)$ and let $\mathbb{Z}_m^*$ be the multiplicative group of invertible elements of $\mathbb{Z}_m$. Define $t\in\mathbb{Z}_m^*$ by the equation $xy^{n-1}=tz$ where x is a generator of $H^1(L;\mathbb{Z}_m)$, $y=\beta(x)$, and $z\in H^{2n-1}(L;\mathbb{Z}_m)$ is the image of a generator of $H^{2n-1}(L;\mathbb{Z})$. Show that the image $\tau(L)$ of t in the quotient group $\mathbb{Z}_m^*/\pm(\mathbb{Z}_m^*)^n$ depends only on the homotopy type of L.

(b) Given nonzero integers $k_1,\cdots,k_n$, define a map $\widetilde{f}:S^{2n-1}\to S^{2n-1}$ sending the unit vector $(r_1e^{i\theta_1},\cdots,r_ne^{i\theta_n})$ in $\mathbb{C}^n$ to $(r_1e^{ik_1\theta_1},\cdots,r_ne^{ik_n\theta_n})$. Show:
   (i) $\widetilde{f}$ has degree $k_1\cdots k_n$.
   (ii) $\widetilde{f}$ induces a quotient map $f:L\to L'$ for $L'=L_m(\ell_1',\cdots,\ell_n')$ provided that $k_j\ell_j\equiv\ell_j' \bmod m$ for each j.
   (iii) f induces an isomorphism on π_1, hence on $H^1(-;\mathbb{Z}_m)$.
   (iv) f has degree $k_1\cdots k_n$, i.e., f_* is multiplication by $k_1\cdots k_n$ on $H_{2n-1}(-;\mathbb{Z})$.

(c) Using the f in (b), show that $\tau(L)=k_1\cdots k_n\tau(L')$.

(d) Deduce that if $L_m(\ell_1,\cdots,\ell_n)\simeq L_m(\ell_1',\cdots,\ell_n')$, then $\ell_1\cdots\ell_n\equiv\pm\ell_1'\cdots\ell_n'k^n \bmod m$ for some integer k.

3. Let X be the smash product of k copies of a Moore space $M(\mathbb{Z}_p,n)$ with p prime. Compute the Bockstein homomorphisms in $H^*(X;\mathbb{Z}_p)$ and use this to describe $H^*(X;\mathbb{Z})$.

4. Using the cup product structure in $H^*(SO(5);\mathbb{Z})$, show that $SO(5)$ is not homotopy equivalent to the product of any two CW complexes with nontrivial cohomology.

3.F Limits and Ext

It often happens that one has a CW complex X expressed as a union of an increasing sequence of subcomplexes $X_0 \subset X_1 \subset X_2 \subset \cdots$. For example, X_i could be the i-skeleton of X, or the X_i's could be finite complexes whose union is X. In situations of this sort, Proposition 3.33 says that $H_n(X;G)$ is the direct limit $\varinjlim H_n(X_i;G)$. Our goal in this section is to show this holds more generally for any homology theory, and to derive the corresponding formula for cohomology theories, which is a bit more complicated even for ordinary cohomology with $\mathbb{Z}$ coefficients. For ordinary homology and cohomology the results apply somewhat more generally than just to CW complexes, since if a space X is the union of an increasing sequence of subspaces X_i with the property that each compact set in X is contained in some X_i, then the singular complex of X is the union of the singular complexes of the X_i's, and so this gives a reduction to the CW case.

Passing to limits can often result in nonfinitely generated homology and cohomology groups. At the end of this section we describe some of the rather subtle behavior of Ext for nonfinitely generated groups.

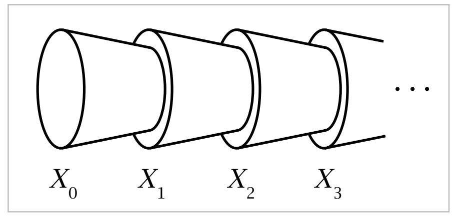
Direct and Inverse Limits

As a special case of the general definition in §3.3, the direct limit $\varinjlim G_i$ of a sequence of homomorphisms of abelian groups $G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} G_3 \longrightarrow \cdots$ is defined to be the quotient of the direct sum $\bigoplus_i G_i$ by the subgroup consisting of elements of the form $(g_1, g_2 - \alpha_1(g_1), g_3 - \alpha_2(g_2), \cdots)$. It is easy to see from this definition that every element of $\varinjlim G_i$ is represented by an element $g_i \in G_i$ for some i, and two such representatives $g_i \in G_i$ and $g_j \in G_j$ define the same element of $\varinjlim G_i$ iff they have the same image in some G_k under the appropriate composition of α_ℓ's. If all the α_i's are injective and are viewed as inclusions of subgroups, $\varinjlim G_i$ is just $\bigcup_i G_i$.

Example 3F.1. For a prime p, consider the sequence $\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \cdots$ with all maps multiplication by p. Then $\varinjlim G_i$ can be identified with the subgroup $\mathbb{Z}[1/p]$ of $\mathbb{Q}$ consisting of rational numbers with denominator a power of p. More generally, we can realize any subgroup of $\mathbb{Q}$ as the direct limit of a sequence $\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \cdots$ with an appropriate choice of maps. For example, if the n^{th} map is multiplication by n, then the direct limit is $\mathbb{Q}$ itself.

Example 3F.2. The sequence of injections $\mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_{p^2} \xrightarrow{p} \mathbb{Z}_{p^3} \longrightarrow \cdots$, with p prime, has direct limit a group we denote $\mathbb{Z}_{p^\infty}$. This is isomorphic to $\mathbb{Z}[1/p]/\mathbb{Z}$, the subgroup of $\mathbb{Q}/\mathbb{Z}$ represented by fractions with denominator a power of p. In fact $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the direct sum of the subgroups $\mathbb{Z}[1/p]/\mathbb{Z} \approx \mathbb{Z}_{p^\infty}$ for all primes p. It is not hard to determine all the subgroups of $\mathbb{Q}/\mathbb{Z}$ and see that each one can be realized as a direct limit of finite cyclic groups with injective maps between them. Conversely, every such direct limit is isomorphic to a subgroup of $\mathbb{Q}/\mathbb{Z}$.

We can realize these algebraic examples topologically by the following construction. The **mapping telescope** of a sequence of maps $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \longrightarrow \cdots$ is the union of the mapping cylinders M_{f_i} with the copies of X_i in M_{f_i} and $M_{f_{i-1}}$ identified for all i. Thus the mapping telescope is the quotient space of the disjoint union $\coprod_i (X_i \times [i, i+1])$ in which each point $(x_i, i+1) \in X_i \times [i, i+1]$ is identified with $(f_i(x_i), i+1) \in X_{i+1} \times [i+1, i+2]$. In the mapping telescope T, let T_i be the union of the first i mapping cylinders. This deformation retracts onto X_i by deformation retracting each mapping cylinder onto its right end in turn. If the maps f_i are cellular, each mapping cylinder is a CW complex and the telescope T is the increasing union of the subcomplexes $T_i \simeq X_i$. Then Proposition 3.33, or Theorem 3F.8 below, implies that $H_n(T;G) \approx \varinjlim H_n(X_i;G)$.



Example 3F.3. Suppose each f_i is a map $S^n \to S^n$ of degree p for a fixed prime p. Then $H_n(T)$ is the direct limit of the sequence $\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \cdots$ considered in Example 3F.1 above, and $\widetilde{H}_k(T) = 0$ for $k \neq n$, so T is a Moore space $M(\mathbb{Z}[1/p], n)$.

Example 3F.4. In the preceding example, if we attach a cell e^{n+1} to the first S^n in T via the identity map of S^n, we obtain a space X which is a Moore space $M(\mathbb{Z}_{p^\infty}, n)$ since X is the union of its subspaces $X_i = T_i \cup e^{n+1}$, which are $M(\mathbb{Z}_{p^i}, n)$'s, and the inclusion $X_i \subset X_{i+1}$ induces the inclusion $\mathbb{Z}_{p^i} \subset \mathbb{Z}_{p^{i+1}}$ on H_n.

Generalizing these two examples, we can obtain Moore spaces $M(G, n)$ for arbitrary subgroups G of $\mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$ by choosing maps $f_i : S^n \to S^n$ of suitable degrees.

The behavior of cohomology groups is more complicated. If X is the increasing union of subcomplexes X_i, then the cohomology groups $H^n(X_i;G)$, for fixed n and G, form a sequence of homomorphisms

$$\cdots \longrightarrow G_2 \xrightarrow{\alpha_2} G_1 \xrightarrow{\alpha_1} G_0$$

Given such a sequence of group homomorphisms, the **inverse limit** $\varprojlim G_i$ is defined to be the subgroup of $\prod_i G_i$ consisting of sequences (g_i) with $\alpha_i(g_i) = g_{i-1}$ for all i. There is a natural map $\lambda : H^n(X;G) \to \varprojlim H^n(X_i;G)$ sending an element of $H^n(X;G)$ to its sequence of images in $H^n(X_i;G)$ under the maps $H^n(X;G) \to H^n(X_i;G)$ induced by inclusion. One might hope that λ is an isomorphism, but this is not true in general, as we shall see. However, for some choices of G it is:

Proposition 3F.5. *If the CW complex X is the union of an increasing sequence of subcomplexes X_i and if G is one of the fields $\mathbb{Q}$ or $\mathbb{Z}_p$, then $\lambda : H^n(X;G) \to \varprojlim H^n(X_i;G)$ is an isomorphism for all n.*

Proof: First we have an easy algebraic fact: Given a sequence of homomorphisms of abelian groups $G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} G_3 \longrightarrow \cdots$, then $\mathrm{Hom}(\varinjlim G_i, G) = \varprojlim \mathrm{Hom}(G_i, G)$

for any G. Namely, it follows from the definition of $\varinjlim G_i$ that a homomorphism $\varphi : \varinjlim G_i \to G$ is the same thing as a sequence of homomorphisms $\varphi_i : G_i \to G$ with $\varphi_i = \varphi_{i+1}\alpha_i$ for all i. Such a sequence (φ_i) is exactly an element of $\varprojlim \mathrm{Hom}(G_i, G)$.

Now if G is a field $\mathbb{Q}$ or $\mathbb{Z}_p$ we have

$$\begin{aligned} H^n(X;G) &= \mathrm{Hom}(H_n(X;G), G) \\ &= \mathrm{Hom}(\varinjlim H_n(X_i;G), G) \\ &= \varprojlim \mathrm{Hom}(H_n(X_i;G), G) \\ &= \varprojlim H^n(X_i;G) \end{aligned}$$ $\square$

Let us analyze what happens for cohomology with an arbitrary coefficient group, or more generally for any cohomology theory. Given a sequence of homomorphisms of abelian groups

$$\cdots \longrightarrow G_2 \xrightarrow{\alpha_2} G_1 \xrightarrow{\alpha_1} G_0$$

define a map $\delta : \prod_i G_i \to \prod_i G_i$ by $\delta(\cdots, g_i, \cdots) = (\cdots, g_i - \alpha_{i+1}(g_{i+1}), \cdots)$, so that $\varprojlim G_i$ is the kernel of δ. Denoting the cokernel of δ by $\varprojlim^1 G_i$, we have then an exact sequence

$$0 \to \varprojlim G_i \longrightarrow \prod_i G_i \xrightarrow{\delta} \prod_i G_i \longrightarrow \varprojlim{}^1 G_i \longrightarrow 0$$

This may be compared with the corresponding situation for the direct limit of a sequence $G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} G_3 \longrightarrow \cdots$. In this case one has a short exact sequence

$$0 \longrightarrow \bigoplus_i G_i \xrightarrow{\delta} \bigoplus_i G_i \longrightarrow \varinjlim G_i \longrightarrow 0$$

where $\delta(\cdots, g_i, \cdots) = (\cdots, g_i - \alpha_{i-1}(g_{i-1}), \cdots)$, so δ is injective and there is no term $\varinjlim^1 G_i$ analogous to $\varprojlim^1 G_i$.

Here are a few simple observations about $\varprojlim$ and $\varprojlim^1$:

- If all the α_i's are isomorphisms then $\varprojlim G_i \approx G_0$ and $\varprojlim^1 G_i = 0$. In fact, $\varprojlim^1 G_i = 0$ if each α_i is surjective, for to realize a given element $(h_i) \in \prod_i G_i$ as $\delta(g_i)$ we can take $g_0 = 0$ and then solve $\alpha_1(g_1) = -h_0$, $\alpha_2(g_2) = g_1 - h_1$, $\cdots$.
- If all the α_i's are zero then $\varprojlim G_i = \varprojlim^1 G_i = 0$.
- Deleting a finite number of terms from the end of the sequence $\cdots \to G_1 \to G_0$ does not affect $\varprojlim G_i$ or $\varprojlim^1 G_i$. More generally, $\varprojlim G_i$ and $\varprojlim^1 G_i$ are unchanged if we replace the sequence $\cdots \to G_1 \to G_0$ by a subsequence, with the appropriate compositions of α_j's as the maps.

Example 3F.6. Consider the sequence of natural surjections $\cdots \to \mathbb{Z}_{p^3} \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p$ with p a prime. The inverse limit of this sequence is a famous object in number theory, called the ***p*-adic integers**. Our notation for it will be $\widehat{\mathbb{Z}}_p$. It is actually a commutative ring, not just a group, since the projections $\mathbb{Z}_{p^{i+1}} \to \mathbb{Z}_{p^i}$ are ring homomorphisms, but we will be interested only in the additive group structure. Elements of $\widehat{\mathbb{Z}}_p$ are infinite sequences $(\cdots, a_2, a_1)$ with $a_i \in \mathbb{Z}_{p^i}$ such that a_i is the mod p^i reduction of a_{i+1}.

For each choice of a_i there are exactly p choices for a_{i+1}, so $\widehat{\mathbb{Z}}_p$ is uncountable. There is a natural inclusion $\mathbb{Z} \subset \widehat{\mathbb{Z}}_p$ as the constant sequences $a_i = n \in \mathbb{Z}$.

There is another way of looking at $\widehat{\mathbb{Z}}_p$. An element of $\widehat{\mathbb{Z}}_p$ has a unique representation as a sequence $(\cdots, a_2, a_1)$ of integers a_i with $0 \le a_i < p^i$ for each i. We can write each a_i uniquely in the form $b_{i-1}p^{i-1} + \cdots + b_1 p + b_0$ with $0 \le b_j < p$. The fact that a_{i+1} reduces mod p^i to a_i means that the numbers b_j depend only on the element $(\cdots, a_2, a_1) \in \widehat{\mathbb{Z}}_p$, so we can view the elements of $\widehat{\mathbb{Z}}_p$ as the 'base p infinite numbers' $\cdots b_1 b_0$ with $0 \le b_i < p$ for all i, with the familiar rule for addition in base p notation. From this viewpoint, the subgroup $\mathbb{Z} \subset \widehat{\mathbb{Z}}_p$ consists of the finite expressions $b_n \cdots b_1 b_0$. It is also clear from this representation of $\widehat{\mathbb{Z}}_p$ that $\widehat{\mathbb{Z}}_p$ is torsionfree.

Since the maps $\mathbb{Z}_{p^{i+1}} \to \mathbb{Z}_{p^i}$ are surjective, $\varprojlim^1 \mathbb{Z}_{p^i} = 0$. The next example shows how p-adic integers can also give rise to a nonvanishing $\varprojlim^1$ term.

Example 3F.7. Consider the sequence $\cdots \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z}$ for p prime. In this case the inverse limit is zero since a nonzero integer can only be divided by p finitely often. The $\varprojlim^1$ term is the cokernel of the map $\delta : \prod_\infty \mathbb{Z} \to \prod_\infty \mathbb{Z}$ given by $\delta(y_1, y_2, \cdots) = (y_1 - p y_2, y_2 - p y_3, \cdots)$. We claim that the map $\widehat{\mathbb{Z}}_p/\mathbb{Z} \to \operatorname{Coker}\delta$ sending a p-adic number $\cdots b_1 b_0$ as in the preceding example to $(b_0, b_1, \cdots)$ is an isomorphism. To see this, note that the image of δ consists of the sums $y_1(1, 0, \cdots) + y_2(-p, 1, 0, \cdots) + y_3(0, -p, 1, 0, \cdots) + \cdots$. The terms after $y_1(1, 0, \cdots)$ give exactly the relations that hold among the p-adic numbers $\cdots b_1 b_0$, and in particular allow one to reduce an arbitrary sequence $(b_0, b_1, \cdots)$ to a unique sequence with $0 \le b_i < p$ for all i. The term $y_1(1, 0, \cdots)$ corresponds to the subgroup $\mathbb{Z} \subset \widehat{\mathbb{Z}}_p$.

We come now to the main result of this section:

Theorem 3F.8. *For a CW complex X which is the union of an increasing sequence of subcomplexes $X_0 \subset X_1 \subset \cdots$ there is an exact sequence*

$$0 \longrightarrow \varprojlim{}^1 h^{n-1}(X_i) \longrightarrow h^n(X) \xrightarrow{\lambda} \varprojlim h^n(X_i) \longrightarrow 0$$

where h^* is any reduced or unreduced cohomology theory. For any homology theory h_*, reduced or unreduced, the natural maps $\varinjlim h_n(X_i) \to h_n(X)$ are isomorphisms.

Proof: Let T be the mapping telescope of the inclusion sequence $X_0 \hookrightarrow X_1 \hookrightarrow \cdots$. This is a subcomplex of $X \times [0, \infty)$ when $[0, \infty)$ is given the CW structure with the integer points as 0-cells. We have $T \simeq X$ since T is a deformation retract of $X \times [0, \infty)$, as we showed in the proof of Lemma 2.34 in the special case that X_i is the i-skeleton of X, but the argument works just as well for arbitrary subcomplexes X_i.

Let $T_1 \subset T$ be the union of the products $X_i \times [i, i+1]$ for i odd, and let T_2 be the corresponding union for i even. Thus $T_1 \cap T_2 = \coprod_i X_i$ and $T_1 \cup T_2 = T$. For an unreduced cohomology theory h^* we have then a Mayer–Vietoris sequence

$$\begin{array}{ccccccccc} h^{n-1}(T_1)\oplus h^{n-1}(T_2) & \longrightarrow & h^{n-1}(T_1\cap T_2) & \longrightarrow & h^n(T) & \longrightarrow & h^n(T_1)\oplus h^n(T_2) & \longrightarrow & h^n(T_1\cap T_2) \\ \wr\wr & & \wr\wr & & \wr\wr & & \wr\wr & & \wr\wr \\ \prod_i h^{n-1}(X_i) & \xrightarrow{\ \varphi\ } & \prod_i h^{n-1}(X_i) & \longrightarrow & h^n(X) & \longrightarrow & \prod_i h^n(X_i) & \xrightarrow{\ \varphi\ } & \prod_i h^n(X_i) \end{array}$$

The maps φ making the diagram commute are given by the formula $\varphi(\cdots,g_i,\cdots)=(\cdots,(-1)^{i-1}(g_i-\rho(g_{i+1})),\cdots)$, the ρ's being the appropriate restriction maps. This differs from δ only in the sign of its even coordinates, so if we change the isomorphism $h^k(T_1\cap T_2)\approx\prod_i h^k(X_i)$ by inserting a minus sign in the even coordinates, we can replace φ by δ in the second row of the diagram. This row then yields a short exact sequence $0\to\operatorname{Coker}\delta\to H^n(X;G)\to\operatorname{Ker}\delta\to 0$, finishing the proof for unreduced cohomology.

The same argument works for reduced cohomology if we use the reduced telescope obtained from T by collapsing $\{x_0\}\times[0,\infty)$ to a point, for x_0 a basepoint 0-cell of X_0. Then $T_1\cap T_2=\bigvee_i X_i$ rather than $\coprod_i X_i$, and the rest of the argument goes through unchanged. The proof also applies for homology theories, with direct products replaced by direct sums in the second row of the diagram. As we noted earlier, $\operatorname{Ker}\delta=0$ in the direct limit case, and $\operatorname{Coker}\delta=\varinjlim$. ◻

Example 3F.9. As in Example 3F.3, consider the mapping telescope T for the sequence of degree p maps $S^n\to S^n\to\cdots$. Letting T_i be the union of the first i mapping cylinders in the telescope, the inclusions $T_1\hookrightarrow T_2\hookrightarrow\cdots$ induce on $H^n(-;\mathbb{Z})$ the sequence $\cdots\to\mathbb{Z}\xrightarrow{p}\mathbb{Z}$ in Example 3F.7. From the theorem we deduce that $H^{n+1}(T;\mathbb{Z})\approx\widehat{\mathbb{Z}}_p/\mathbb{Z}$ and $\widetilde{H}^k(T;\mathbb{Z})=0$ for $k\neq n+1$. Thus we have the rather strange situation that the CW complex T is the union of subcomplexes T_i each having cohomology consisting only of a $\mathbb{Z}$ in dimension n, but T itself has no cohomology in dimension n and instead has a huge uncountable group $\widehat{\mathbb{Z}}_p/\mathbb{Z}$ in dimension $n+1$. This contrasts sharply with what happens for homology, where the groups $H_n(T_i)\approx\mathbb{Z}$ fit together nicely to give $H_n(T)\approx\mathbb{Z}[1/p]$.

Example 3F.10. A more reasonable behavior is exhibited if we consider the space $X=M(\mathbb{Z}_{p^\infty},n)$ in Example 3F.4 expressed as the union of its subspaces X_i. By the universal coefficient theorem, the reduced cohomology of X_i with $\mathbb{Z}$ coefficients consists of a $\mathbb{Z}_{p^i}=\operatorname{Ext}(\mathbb{Z}_{p^i},\mathbb{Z})$ in dimension $n+1$. The inclusion $X_i\hookrightarrow X_{i+1}$ induces the inclusion $\mathbb{Z}_{p^i}\hookrightarrow\mathbb{Z}_{p^{i+1}}$ on H_n, and on Ext this induced map is a surjection $\mathbb{Z}_{p^{i+1}}\to\mathbb{Z}_{p^i}$ as one can see by looking at the diagram of free resolutions on the left:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{p^i} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_{p^i} & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \mathbb{1}} & & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{p^{i+1}} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_{p^{i+1}} & \longrightarrow & 0 \end{array}
\qquad
\begin{array}{ccccccc} 0 & \longleftarrow & \operatorname{Ext}(\mathbb{Z}_{p^i},\mathbb{Z}) & \longleftarrow & \operatorname{Hom}(\mathbb{Z},\mathbb{Z}) & \longleftarrow & \cdots \\ & & \big\uparrow & & \big\uparrow{\scriptstyle \mathbb{1}} & & \\ 0 & \longleftarrow & \operatorname{Ext}(\mathbb{Z}_{p^{i+1}},\mathbb{Z}) & \longleftarrow & \operatorname{Hom}(\mathbb{Z},\mathbb{Z}) & \longleftarrow & \cdots \end{array}$$

Applying $\operatorname{Hom}(-,\mathbb{Z})$ to this diagram, we get the diagram on the right, with exact rows, and the left-hand vertical map is a surjection since the vertical map to the right of it is surjective. Thus the sequence $\cdots\to H^{n+1}(X_2;\mathbb{Z})\to H^{n+1}(X_1;\mathbb{Z})$ is the sequence in Example 3F.6, and we deduce that $H^{n+1}(X;\mathbb{Z}) \approx \widehat{\mathbb{Z}}_p$, the p-adic integers, and $\widetilde{H}^k(X;\mathbb{Z}) = 0$ for $k \neq n+1$.

This example can be related to the preceding one. If we view X as the mapping cone of the inclusion $S^n \hookrightarrow T$ of one end of the telescope, then the long exact sequences of homology and cohomology groups for the pair (T, S^n) reduce to the short exact sequences at the right.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_n(S^n) & \longrightarrow & H_n(T) & \longrightarrow & H_n(X) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & \mathbb{Z} & & \mathbb{Z}[1/p] & & \mathbb{Z}_{p^\infty} & & \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^n(S^n) & \longrightarrow & H^{n+1}(X) & \longrightarrow & H^{n+1}(T) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & \mathbb{Z} & & \widehat{\mathbb{Z}}_p & & \widehat{\mathbb{Z}}_p/\mathbb{Z} & & \end{array}$$

From these examples and the universal coefficient theorem we obtain isomorphisms $\operatorname{Ext}(\mathbb{Z}_{p^\infty}, \mathbb{Z}) \approx \widehat{\mathbb{Z}}_p$ and $\operatorname{Ext}(\mathbb{Z}[1/p], \mathbb{Z}) \approx \widehat{\mathbb{Z}}_p/\mathbb{Z}$. These can also be derived directly from the definition of Ext. A free resolution of $\mathbb{Z}_{p^\infty}$ is

$$0 \longrightarrow \mathbb{Z}^\infty \xrightarrow{\varphi} \mathbb{Z}^\infty \longrightarrow \mathbb{Z}_{p^\infty} \longrightarrow 0$$

where $\mathbb{Z}^\infty$ is the direct sum of an infinite number of $\mathbb{Z}$'s, the sequences $(x_1, x_2, \cdots)$ of integers all but finitely many of which are zero, and φ sends $(x_1, x_2, \cdots)$ to $(px_1 - x_2, px_2 - x_3, \cdots)$. We can view φ as the linear map corresponding to the infinite matrix with p's on the diagonal, -1's just above the diagonal, and 0's everywhere else. Clearly $\operatorname{Ker}\varphi = 0$ since integers cannot be divided by p infinitely often. The image of φ is generated by the vectors $(p, 0, \cdots), (-1, p, 0, \cdots), (0, -1, p, 0, \cdots), \cdots$ so $\operatorname{Coker}\varphi \approx \mathbb{Z}_{p^\infty}$. Dualizing by taking $\operatorname{Hom}(-, \mathbb{Z})$, we have $\operatorname{Hom}(\mathbb{Z}^\infty, \mathbb{Z})$ the infinite direct product of $\mathbb{Z}$'s, and $\varphi^*(y_1, y_2, \cdots) = (py_1, py_2 - y_1, py_3 - y_2, \cdots)$, corresponding to the transpose of the matrix of φ. By definition, $\operatorname{Ext}(\mathbb{Z}_{p^\infty}, \mathbb{Z}) = \operatorname{Coker}\varphi^*$. The image of φ^* consists of the infinite sums $y_1(p, -1, 0 \cdots) + y_2(0, p, -1, 0, \cdots) + \cdots$, so $\operatorname{Coker}\varphi^*$ can be identified with $\widehat{\mathbb{Z}}_p$ by rewriting a sequence $(z_1, z_2, \cdots)$ as the p-adic number $\cdots z_2 z_1$.

The calculation $\operatorname{Ext}(\mathbb{Z}[1/p], \mathbb{Z}) \approx \widehat{\mathbb{Z}}_p/\mathbb{Z}$ is quite similar. A free resolution of $\mathbb{Z}[1/p]$ can be obtained from the free resolution of $\mathbb{Z}_{p^\infty}$ by omitting the first column of the matrix of φ and, for convenience, changing sign. This gives the formula $\varphi(x_1, x_2, \cdots) = (x_1, x_2 - px_1, x_3 - px_2, \cdots)$, with the image of φ generated by the elements $(1, -p, 0, \cdots)$, $(0, 1, -p, 0, \cdots), \cdots$. The dual map φ^* is given by $\varphi^*(y_1, y_2, \cdots) = (y_1 - py_2, y_2 - py_3, \cdots)$, and this has image consisting of the sums $y_1(1, 0 \cdots) + y_2(-p, 1, 0, \cdots) + y_3(0, -p, 1, 0, \cdots) + \cdots$, so we get $\operatorname{Ext}(\mathbb{Z}[1/p], \mathbb{Z}) = \operatorname{Coker}\varphi^* \approx \widehat{\mathbb{Z}}_p/\mathbb{Z}$. Note that φ^* is exactly the map δ in Example 3F.7.

It is interesting to note also that the map $\varphi : \mathbb{Z}^\infty \to \mathbb{Z}^\infty$ in the two cases $\mathbb{Z}_{p^\infty}$ and $\mathbb{Z}[1/p]$ is precisely the cellular boundary map $H_{n+1}(X^{n+1}, X^n) \to H_n(X^n, X^{n-1})$ for the Moore space $M(\mathbb{Z}_{p^\infty}, n)$ or $M(\mathbb{Z}[1/p], n)$ constructed as the mapping telescope of the sequence of degree p maps $S^n \to S^n \to \cdots$, with a cell e^{n+1} attached to the first S^n in the case of $\mathbb{Z}_{p^\infty}$.

More About Ext

The functors Hom and Ext behave fairly simply for finitely generated groups, when cohomology and homology are essentially the same except for a dimension shift in the torsion. But matters are more complicated in the nonfinitely generated case. A useful tool for getting a handle on this complication is the following:

Proposition 3F.11. *Given an abelian group G and a short exact sequence of abelian groups $0\to A\to B\to C\to 0$, there are exact sequences*

$$0\to\mathrm{Hom}(G,A)\to\mathrm{Hom}(G,B)\to\mathrm{Hom}(G,C)\to\mathrm{Ext}(G,A)\to\mathrm{Ext}(G,B)\to\mathrm{Ext}(G,C)\to 0$$

$$0\to\mathrm{Hom}(C,G)\to\mathrm{Hom}(B,G)\to\mathrm{Hom}(A,G)\to\mathrm{Ext}(C,G)\to\mathrm{Ext}(B,G)\to\mathrm{Ext}(A,G)\to 0$$

Proof: A free resolution $0\to F_1\to F_0\to G\to 0$ gives rise to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(F_0,A) \arrow[r] \arrow[d] & \mathrm{Hom}(F_0,B) \arrow[r] \arrow[d] & \mathrm{Hom}(F_0,C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(F_1,A) \arrow[r] & \mathrm{Hom}(F_1,B) \arrow[r] & \mathrm{Hom}(F_1,C) \arrow[r] & 0
\end{tikzcd}$$

Since F_0 and F_1 are free, the two rows are exact, as they are simply direct products of copies of the exact sequence $0\to A\to B\to C\to 0$, in view of the general fact that $\mathrm{Hom}(\bigoplus_i G_i,H)=\prod_i\mathrm{Hom}(G_i,H)$. Enlarging the diagram by zeros above and below, it becomes a short exact sequence of chain complexes, and the associated long exact sequence of homology groups is the first of the two six-term exact sequences in the proposition.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F_1 \arrow[r] \arrow[d] & F_1' \arrow[r] \arrow[d] & F_1'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_0 \arrow[r] \arrow[d] & F_0' \arrow[r] \arrow[d] & F_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

To obtain the other exact sequence we will construct the commutative diagram at the right, where the columns are free resolutions and the rows are exact. To start, let $F_0\to A$ and $F_0''\to C$ be surjections from free abelian groups onto A and C. Then let $F_0'=F_0\oplus F_0''$, with the obvious maps in the second row, inclusion and projection. The map $F_0'\to B$ is defined on the summand F_0 to make the lower left square commute, and on the summand F_0'' it is defined by sending basis elements of F_0'' to elements of B mapping to the images of these basis elements in C, so the lower right square also commutes. Now we have the bottom two rows of the diagram, and we can regard these two rows as a short exact sequence of two-term chain complexes. The associated long exact sequence of homology groups has six terms, the first three being the kernels of the three vertical maps to A, B, and C, and the last three being the cokernels of these maps. Since the vertical maps to A and C are surjective, the fourth and sixth of the six homology groups vanish, hence also the fifth, which says the vertical map to B is surjective. The first three of the original six homology groups form a short exact sequence, and we let this be the top row of the diagram, formed by the kernels of the vertical maps to A, B, and C. These kernels are subgroups of free abelian groups, hence are also free.

Thus the three columns are free resolutions. The upper two squares automatically commute, so the construction of the diagram is complete.

The first two rows of the diagram split by freeness, so applying $\mathrm{Hom}(-,G)$ yields a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(F_0'',G) & \longrightarrow & \mathrm{Hom}(F_0',G) & \longrightarrow & \mathrm{Hom}(F_0,G) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}(F_1'',G) & \longrightarrow & \mathrm{Hom}(F_1',G) & \longrightarrow & \mathrm{Hom}(F_1,G) & \longrightarrow & 0 \end{array}$$

with exact rows. Again viewing this as a short exact sequence of chain complexes, the associated long exact sequence of homology groups is the second six-term exact sequence in the statement of the proposition. $\square$

The second sequence in the proposition says in particular that an injection $A \to B$ induces a surjection $\mathrm{Ext}(B,C) \to \mathrm{Ext}(A,C)$ for any C. For example, if A has torsion, this says $\mathrm{Ext}(A,\mathbb{Z})$ is nonzero since it maps onto $\mathrm{Ext}(\mathbb{Z}_n,\mathbb{Z}) \approx \mathbb{Z}_n$ for some $n > 1$. The calculation $\mathrm{Ext}(\mathbb{Z}_{p^\infty},\mathbb{Z}) \approx \widehat{\mathbb{Z}}_p$ earlier in this section shows that torsion in A does not necessarily yield torsion in $\mathrm{Ext}(A,\mathbb{Z})$, however.

Also useful are the formulas

$$\mathrm{Ext}(\textstyle\bigoplus_i A_i, B) \approx \prod_i \mathrm{Ext}(A_i,B) \qquad\qquad \mathrm{Ext}(A, \bigoplus_i B_i) \approx \bigoplus_i \mathrm{Ext}(A,B_i)$$

whose proofs we leave as exercises. For example, since $\mathbb{Q}/\mathbb{Z} = \bigoplus_p \mathbb{Z}_{p^\infty}$ we obtain $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z},\mathbb{Z}) \approx \prod_p \widehat{\mathbb{Z}}_p$ from the calculation $\mathrm{Ext}(\mathbb{Z}_{p^\infty},\mathbb{Z}) \approx \widehat{\mathbb{Z}}_p$. Then from the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ we get $\mathrm{Ext}(\mathbb{Q},\mathbb{Z}) \approx (\prod_p \widehat{\mathbb{Z}}_p)/\mathbb{Z}$ using the second exact sequence in the proposition.

In these examples the groups $\mathrm{Ext}(A,\mathbb{Z})$ are rather large, and the next result says this is part of a general pattern:

Proposition 3F.12. *If A is not finitely generated then either* $\mathrm{Hom}(A,\mathbb{Z})$ *or* $\mathrm{Ext}(A,\mathbb{Z})$ *is uncountable. Hence if $H_n(X;\mathbb{Z})$ is not finitely generated then either $H^n(X;\mathbb{Z})$ or $H^{n+1}(X;\mathbb{Z})$ is uncountable.*

Both possibilities can occur, as we see from the examples $\mathrm{Hom}(\bigoplus_\infty \mathbb{Z},\mathbb{Z}) \approx \prod_\infty \mathbb{Z}$ and $\mathrm{Ext}(\mathbb{Z}_{p^\infty},\mathbb{Z}) \approx \widehat{\mathbb{Z}}_p$.

This proposition has some interesting topological consequences. First, it implies that if a space X has $\widetilde{H}^*(X;\mathbb{Z}) = 0$, then $\widetilde{H}_*(X;\mathbb{Z}) = 0$, since the case of finitely generated homology groups follows from our earlier results. And second, it says that one cannot always construct a space X with prescribed cohomology groups $H^n(X;\mathbb{Z})$, as one can for homology. For example there is no space whose only nonvanishing $\widetilde{H}^n(X;\mathbb{Z})$ is a countable nonfinitely generated group such as $\mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$. Even in the finitely generated case the dimension $n = 1$ is somewhat special since the group $H^1(X;\mathbb{Z}) \approx \mathrm{Hom}(H_1(X),\mathbb{Z})$ is always torsionfree.

Proof: Consider the map $A \xrightarrow{p} A$, $a \mapsto pa$, multiplication by the positive integer p. Denote the kernel, image, and cokernel of this map by ${}_pA$, pA, and A_p, respectively. The short exact sequences $0 \to {}_pA \to A \to pA \to 0$ and $0 \to pA \to A \to A_p \to 0$ give two six-term exact sequences involving $\mathrm{Hom}(-,\mathbb{Z})$ and $\mathrm{Ext}(-,\mathbb{Z})$. The parts of these exact sequences we need are

$$0 \longrightarrow \mathrm{Hom}(pA,\mathbb{Z}) \xrightarrow{\approx} \mathrm{Hom}(A,\mathbb{Z}) \longrightarrow \mathrm{Hom}({}_pA,\mathbb{Z}) = 0$$
$$\mathrm{Hom}(pA,\mathbb{Z}) \longrightarrow \mathrm{Ext}(A_p,\mathbb{Z}) \longrightarrow \mathrm{Ext}(A,\mathbb{Z})$$

where the term $\mathrm{Hom}({}_pA,\mathbb{Z})$ in the first sequence is zero since ${}_pA$ is a torsion group.

Now let p be a prime, so A_p is a vector space over $\mathbb{Z}_p$. If this vector space is infinite-dimensional, it is an infinite direct sum of $\mathbb{Z}_p$'s and $\mathrm{Ext}(A_p,\mathbb{Z})$ is the direct product of an infinite numbers of $\mathbb{Z}_p$'s, hence uncountable. Exactness of the second sequence above then implies that one of the two adjacent terms $\mathrm{Ext}(A,\mathbb{Z})$ or $\mathrm{Hom}(pA,\mathbb{Z}) \approx \mathrm{Hom}(A,\mathbb{Z})$ must be uncountable, so we are done when A_p is infinite.

At the other extreme is the possibility that $A_p = 0$. This means that $A = pA$, so every element of A is divisible by p. Hence if A is nontrivial, it then contains a subgroup isomorphic to either $\mathbb{Z}[1/p]$ or $\mathbb{Z}_{p^\infty}$. We have seen that $\mathrm{Ext}(\mathbb{Z}[1/p],\mathbb{Z}) \approx \widehat{\mathbb{Z}}_p/\mathbb{Z}$ and $\mathrm{Ext}(\mathbb{Z}_{p^\infty},\mathbb{Z}) \approx \widehat{\mathbb{Z}}_p$, an uncountable group in either case. As noted earlier, an inclusion $B \hookrightarrow A$ induces a surjection $\mathrm{Ext}(A,\mathbb{Z}) \to \mathrm{Ext}(B,\mathbb{Z})$, so it follows that $\mathrm{Ext}(A,\mathbb{Z})$ is uncountable when $A_p = 0$ and $A \neq 0$.

The remaining case that A_p is a finite direct sum of $\mathbb{Z}_p$'s will be reduced to the case $A_p = 0$. Choose finitely many elements of A whose images in A_p are a set of generators, and let $B \subset A$ be the subgroup generated by these elements. Thus the map $B_p \to A_p$ induced by the inclusion $B \hookrightarrow A$ is surjective. The functor $A \mapsto A_p$ is the same as $A \mapsto A \otimes \mathbb{Z}_p$, so exactness of $B \to A \to A/B \to 0$ implies exactness of $B_p \to A_p \to (A/B)_p \to 0$, and hence $(A/B)_p = 0$. If A is not finitely generated, A/B is nonzero, so the preceding case implies that $\mathrm{Ext}(A/B,\mathbb{Z})$ is uncountable. This implies that $\mathrm{Ext}(A,\mathbb{Z})$ is also uncountable via the exact sequence $\mathrm{Hom}(B,\mathbb{Z}) \to \mathrm{Ext}(A/B,\mathbb{Z}) \to \mathrm{Ext}(A,\mathbb{Z})$, since $\mathrm{Hom}(B,\mathbb{Z})$ is finitely generated and therefore countable. $\square$

From this proposition one might conjecture that cohomology groups with $\mathbb{Z}$ coefficients are either finitely generated or uncountable.

As was explained in §3.1, the functor Ext generalizes to a sequence of functors Ext_R^n for modules over a ring R. In this generality the six-term sequences of Proposition 3F.11 become long exact sequences of Ext_R^n groups associated to short exact sequences of R-modules. These are derived in a similar fashion, by constructing short exact sequences of free resolutions. There are also analogous long exact sequences for the functors Tor_n^R, specializing to six-term sequences when $R = \mathbb{Z}$. These six-term sequences are perhaps less useful than their Ext analogs, however, since Tor is less mysterious than Ext for nonfinitely generated groups, as it commutes with direct limits, according to an exercise for §3.A.

Exercises

1. Given maps $f_i : X_i \to X_{i+1}$ for integers $i < 0$, show that the 'reverse mapping telescope' obtained by glueing together the mapping cylinders of the f_i's in the obvious way deformation retracts onto X_0. Similarly, if maps $f_i : X_i \to X_{i+1}$ are given for all $i \in \mathbb{Z}$, show that the resulting 'double mapping telescope' deformation retracts onto any of the ordinary mapping telescopes contained in it, the union of the mapping cylinders of the f_i's for i greater than a given number n.

2. Show that $\varprojlim^1 G_i = 0$ if the sequence $\cdots \longrightarrow G_2 \xrightarrow{\alpha_2} G_1 \xrightarrow{\alpha_1} G_0$ satisfies the *Mittag-Leffler condition* that for each i the images of the maps $G_{i+n} \to G_i$ are independent of n for sufficiently large n.

3. Show that $\mathrm{Ext}(A, \mathbb{Q}) = 0$ for all A. [Consider the homology with $\mathbb{Q}$ coefficients of a Moore space $M(A, n)$.]

4. An abelian group G is defined to be *divisible* if the map $G \xrightarrow{n} G$, $g \mapsto ng$, is surjective for all $n > 1$. Show that a group is divisible iff it is a quotient of a direct sum of $\mathbb{Q}$'s. Deduce from the previous problem that if G is divisible then $\mathrm{Ext}(A, G) = 0$ for all A.

5. Show that $\mathrm{Ext}(A, \mathbb{Z})$ is isomorphic to the cokernel of $\mathrm{Hom}(A, \mathbb{Q}) \to \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$, the map induced by the quotient map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. Use this to get another proof that $\mathrm{Ext}(\mathbb{Z}_{p^\infty}, \mathbb{Z}) \approx \widehat{\mathbb{Z}}_p$ for p prime.

6. Show that $\mathrm{Ext}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_p) \approx \mathbb{Z}_p$.

7. Show that for a short exact sequence of abelian groups $0 \to A \to B \to C \to 0$, a Moore space $M(C, n)$ can be realized as a quotient $M(B, n)/M(A, n)$. Applying the long exact sequence of cohomology for the pair $\big(M(B, n), M(A, n)\big)$ with any coefficient group G, deduce an exact sequence

$$0 \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G) \to \mathrm{Ext}(C, G) \to \mathrm{Ext}(B, G) \to \mathrm{Ext}(A, G) \to 0$$

8. Show that for a Moore space $M(G, n)$ the Bockstein long exact sequence in cohomology associated to the short exact sequence of coefficient groups $0 \to A \to B \to C \to 0$ reduces to an exact sequence

$$0 \to \mathrm{Hom}(G, A) \to \mathrm{Hom}(G, B) \to \mathrm{Hom}(G, C) \to \mathrm{Ext}(G, A) \to \mathrm{Ext}(G, B) \to \mathrm{Ext}(G, C) \to 0$$

9. For an abelian group A let $p : A \to A$ be multiplication by p, and let ${}_pA = \mathrm{Ker}\, p$, $pA = \mathrm{Im}\, p$, and $A_p = \mathrm{Coker}\, p$ as in the proof of Proposition 3F.12. Show that the six-term exact sequences involving $\mathrm{Hom}(-, \mathbb{Z})$ and $\mathrm{Ext}(-, \mathbb{Z})$ associated to the short exact sequences $0 \to {}_pA \to A \to pA \to 0$ and $0 \to pA \to A \to A_p \to 0$ can be spliced together to yield the exact sequence across the top of the following diagram

$$
\begin{tikzcd}[column sep=small]
\operatorname{Hom}(pA,\mathbb{Z}) \arrow[r] & \operatorname{Ext}(A_p,\mathbb{Z}) \arrow[r] & \operatorname{Ext}(A,\mathbb{Z}) \arrow[rr, "p"] \arrow[dr] & & \operatorname{Ext}(A,\mathbb{Z}) \arrow[r] & \operatorname{Ext}({}_pA,\mathbb{Z}) \arrow[r] & 0 \\
& & & \operatorname{Ext}(pA,\mathbb{Z}) \arrow[ur] \arrow[dr] & & & \\
& 0 \longrightarrow \operatorname{Hom}(pA,\mathbb{Z}) \xrightarrow{\ \approx\ } \operatorname{Hom}(A,\mathbb{Z}) \arrow[r] & 0 \arrow[ur] & & 0 & &
\end{tikzcd}
$$

where the map labeled 'p' is multiplication by p. Use this to show:

(a) $\operatorname{Ext}(A,\mathbb{Z})$ is divisible iff A is torsionfree.

(b) $\operatorname{Ext}(A,\mathbb{Z})$ is torsionfree if A is divisible, and the converse holds if $\operatorname{Hom}(A,\mathbb{Z}) = 0$.

3.G Transfer Homomorphisms

There is a simple construction called 'transfer' that provides very useful information about homology and cohomology of finite-sheeted covering spaces. After giving the definition and proving a few elementary properties, we will use the transfer in the construction of a number of spaces whose $\mathbb{Z}_p$ cohomology is a polynomial ring.

Let $\pi:\widetilde{X}\to X$ be an n-sheeted covering space, for some finite n. In addition to the induced map on singular chains $\pi_\sharp:C_k(\widetilde{X})\to C_k(X)$ there is also a homomorphism in the opposite direction $\tau:C_k(X)\to C_k(\widetilde{X})$ which assigns to a singular simplex $\sigma:\Delta^k\to X$ the sum of the n distinct lifts $\widetilde{\sigma}:\Delta^k\to\widetilde{X}$. This is obviously a chain map, commuting with boundary homomorphisms, so it induces **transfer homomorphisms** $\tau_*:H_k(X;G)\to H_k(\widetilde{X};G)$ and $\tau^*:H^k(\widetilde{X};G)\to H^k(X;G)$ for any coefficient group G. We focus on cohomology in what follows, but similar statements hold for homology as well.

The composition $\pi_\sharp\tau$ is clearly multiplication by n, hence $\tau^*\pi^* = n$. This has the consequence that the kernel of $\pi^*:H^k(X;G)\to H^k(\widetilde{X};G)$ consists of torsion elements of order dividing n, since $\pi^*(\alpha) = 0$ implies $\tau^*\pi^*(\alpha) = n\alpha = 0$. Thus the cohomology of $\widetilde{X}$ must be 'larger' than that of X except possibly for torsion of order dividing n. This can be a genuine exception as one sees from the examples of S^m covering $\mathbb{R}\mathrm{P}^m$ and lens spaces. More generally, if $S^m\to X$ is any n-sheeted covering space, then the relation $\tau^*\pi^* = n$ implies that $\widetilde{H}^*(X;\mathbb{Z})$ consists entirely of torsion elements of order dividing n, apart from a possible $\mathbb{Z}$ in dimension m. (Since X is a closed manifold, its homology groups are finitely generated by Corollaries A.8 and A.9 in the Appendix.)

By studying the other composition $\pi^*\tau^*$ we will prove:

Proposition 3G.1. *Let $\pi:\widetilde{X}\to X$ be an n-sheeted covering space defined by an action of a group Γ on $\widetilde{X}$. Then with coefficients in a field F whose characteristic is 0 or a prime not dividing n, the map $\pi^*:H^k(X;F)\to H^k(\widetilde{X};F)$ is injective with image the subgroup $H^*(\widetilde{X};F)^\Gamma$ consisting of classes α such that $\gamma^*(\alpha) = \alpha$ for all $\gamma\in\Gamma$.*

Proof: We have already seen that elements of the kernel of π^* have finite order dividing n, so π^* is injective for the coefficient fields we are considering here. It remains to describe the image of π^*. Note first that $\tau\pi_\sharp$ sends a singular simplex $\Delta^k \to \widetilde{X}$ to the sum of all its images under the Γ-action. Hence $\pi^*\tau^*(\alpha) = \sum_{\gamma\in\Gamma}\gamma^*(\alpha)$ for $\alpha \in H^k(X;F)$. If α is fixed under the action of Γ on $H^k(\widetilde{X};F)$, the sum $\sum_{\gamma\in\Gamma}\gamma^*(\alpha)$ equals $n\alpha$, so if the coefficient field F has characteristic 0 or a prime not dividing n, we can write $\alpha = \pi^*\tau^*(\alpha/n)$ and thus α lies in the image of π^*. Conversely, since $\pi\gamma = \pi$ for all $\gamma \in \Gamma$, we have $\gamma^*\pi^*(\alpha) = \pi^*(\alpha)$ for all α, and so the image of π^* is contained in $H^*(\widetilde{X};F)^\Gamma$. $\square$

Example 3G.2. Let $X = S^1 \vee S^k$, $k > 1$, with $\widetilde{X}$ the n-sheeted cover corresponding to the index n subgroup of $\pi_1(X)$, so $\widetilde{X}$ is a circle with n S^k's attached at equally spaced points around the circle. The deck transformation group $\mathbb{Z}_n$ acts by rotating the circle, permuting the S^k's cyclically. Hence for any coefficient group G, the invariant cohomology $H^*(\widetilde{X};G)^{\mathbb{Z}_n}$ is all of H^0 and H^1, plus a copy of G in dimension k, the cellular cohomology classes assigning the same element of G to each S^k. Thus $H^i(\widetilde{X};G)^{\mathbb{Z}_n}$ is exactly the image of π^* for $i = 0$ and k, while the image of π^* in dimension 1 is the subgroup $nH^1(\widetilde{X};G)$. Whether this equals $H^1(\widetilde{X};G)^{\mathbb{Z}_n}$ or not depends on G. For $G = \mathbb{Q}$ or $\mathbb{Z}_p$ with p not dividing n, we have equality, but not for $G = \mathbb{Z}$ or $\mathbb{Z}_p$ with p dividing n. In this last case the map π^* is not injective on H^1.

Spaces with Polynomial mod p Cohomology

An interesting special case of the general problem of realizing graded commutative rings as cup product rings of spaces is the case of polynomial rings $\mathbb{Z}_p[x_1, \cdots, x_n]$ over the coefficient field $\mathbb{Z}_p$, p prime. The basic question here is, which sets of numbers $d_1, \cdots, d_n$ are realizable as the dimensions $|x_i|$ of the generators x_i? From §3.2 we have the examples of products of $\mathbb{C}\mathrm{P}^\infty$'s and $\mathbb{H}\mathrm{P}^\infty$'s with d_i's equal to 2 or 4, for arbitrary p, and when $p = 2$ we can also take $\mathbb{R}\mathrm{P}^\infty$'s with d_i's equal to 1.

As an application of transfer homomorphisms we will construct some examples with larger d_i's. In the case of polynomials in one variable, it turns out that these examples realize everything that can be realized. But for two or more variables, more sophisticated techniques are necessary to realize all the realizable cases; see the end of this section for further remarks on this.

The construction can be outlined as follows. Start with a space Y already known to have polynomial cohomology $H^*(Y;\mathbb{Z}_p) = \mathbb{Z}_p[y_1, \cdots, y_n]$, and suppose there is an action of a finite group Γ on Y. A simple trick called the Borel construction shows that without loss of generality we may assume the action is free, defining a covering space $Y \to Y/\Gamma$. Then by Proposition 3G.1 above, if p does not divide the order of Γ, $H^*(Y/\Gamma;\mathbb{Z}_p)$ is isomorphic to the subring of $\mathbb{Z}_p[y_1, \cdots, y_n]$ consisting of polynomials that are invariant under the induced action of Γ on $H^*(Y;\mathbb{Z}_p)$. And in some cases this subring is itself a polynomial ring.

For example, if Y is the product of n copies of $\mathbb{C}P^\infty$ then the symmetric group Σ_n acts on Y by permuting the factors, with the induced action on $H^*(Y;Z_p) \approx \mathbb{Z}_p[y_1,\cdots,y_n]$ permuting the y_i's. A standard theorem in algebra says that the invariant polynomials form a polynomial ring $\mathbb{Z}_p[\sigma_1,\cdots,\sigma_n]$ where σ_i is the i^{th} elementary symmetric polynomial, the sum of all products of i distinct y_j's. Thus σ_i is a homogeneous polynomial of degree i. The order of Σ_n is $n!$ so the condition that p not divide the order of Γ amounts to $p > n$. Thus we realize the polynomial ring $\mathbb{Z}_p[x_1,\cdots,x_n]$ with $|x_i| = 2i$, provided that $p > n$.

This example is less than optimal since there happens to be another space, the Grassmann manifold of n-dimensional linear subspaces of $\mathbb{C}^\infty$, whose cohomology with any coefficient ring R is $R[x_1,\cdots,x_n]$ with $|x_i| = 2i$, as we show in §4.D, so the restriction $p > n$ is not really necessary.

To get further examples the idea is to replace $\mathbb{C}P^\infty$ by a space with the same $\mathbb{Z}_p$ cohomology but with 'more symmetry,' allowing for larger groups Γ to act. The constructions will be made using $K(\pi,1)$ spaces, which were introduced in §1.B. For a group π we constructed there a Δ-complex $B\pi$ with contractible universal cover $E\pi$. The construction is functorial: A homomorphism $\varphi:\pi\to\pi'$ induces a map $B\varphi:B\pi\to B\pi'$, $B\varphi([g_1|\cdots|g_n]) = [\varphi(g_1)|\cdots|\varphi(g_n)]$, satisfying the functor properties $B(\varphi\psi) = B\varphi B\psi$ and $B\mathbb{1} = \mathbb{1}$. In particular, if Γ is a group of automorphisms of π, then Γ acts on $B\pi$.

The other ingredient we shall need is the **Borel construction**, which converts an action of a group Γ on a space Y into a free action of Γ on a homotopy equivalent space Y'. Namely, take $Y' = Y\times E\Gamma$ with the diagonal action of Γ, $\gamma(y,z) = (\gamma y,\gamma z)$ where Γ acts on $E\Gamma$ as deck transformations. The diagonal action is free, in fact a covering space action, since this is true for the action in the second coordinate. The orbit space of this diagonal action is denoted $Y\times_\Gamma E\Gamma$.

Example 3G.3. Let $\pi = \mathbb{Z}_p$ and let Γ be the full automorphism group $\mathrm{Aut}(\mathbb{Z}_p)$. Automorphisms of $\mathbb{Z}_p$ have the form $x\mapsto mx$ for $(m,p) = 1$, so Γ is the multiplicative group of invertible elements in the field $\mathbb{Z}_p$. By elementary field theory this is a cyclic group, of order $p-1$. The preceding constructions then give a covering space $K(\mathbb{Z}_p,1)\to K(\mathbb{Z}_p,1)/\Gamma$ with $H^*(K(\mathbb{Z}_p,1)/\Gamma;\mathbb{Z}_p) \approx H^*(K(\mathbb{Z}_p,1);\mathbb{Z}_p)^\Gamma$. We may assume we are in the nontrivial case $p > 2$. From the calculation of the cup product structure of lens spaces in Example 3.41 or Example 3E.2 we have $H^*(K(\mathbb{Z}_p,1);\mathbb{Z}_p) \approx \Lambda_{\mathbb{Z}_p}[\alpha]\otimes\mathbb{Z}_p[\beta]$ with $|\alpha| = 1$ and $|\beta| = 2$, and we need to figure out how Γ acts on this cohomology ring.

Let $\gamma\in\Gamma$ be a generator, say $\gamma(x) = mx$. The induced action of γ on $\pi_1K(\mathbb{Z}_p,1)$ is also multiplication by m since we have taken $K(\mathbb{Z}_p,1) = B\mathbb{Z}_p\times E\Gamma$ and γ takes an edge loop $[g]$ in $B\mathbb{Z}_p$ to $[\gamma(g)] = [mg]$. Hence γ acts on $H_1(K(\mathbb{Z}_p,1);\mathbb{Z})$ by multiplication by m. It follows that $\gamma(\alpha) = m\alpha$ and $\gamma(\beta) = m\beta$ since $H^1(K(\mathbb{Z}_p,1);\mathbb{Z}_p) \approx \mathrm{Hom}(H_1(K(\mathbb{Z}_p,1)),\mathbb{Z}_p)$ and $H^2(K(\mathbb{Z}_p,1);\mathbb{Z}_p) \approx \mathrm{Ext}(H_1(K(\mathbb{Z}_p,1)),\mathbb{Z}_p)$, and it is a general fact, following easily from the definitions, that multiplication by an integer m in an abelian group H induces multiplication by m in $\mathrm{Hom}(H,G)$ and $\mathrm{Ext}(H,G)$.

Thus $\gamma(\beta^k) = m^k\beta^k$ and $\gamma(\alpha\beta^k) = m^{k+1}\alpha\beta^k$. Since m was chosen to be a generator of the multiplicative group of invertible elements of $\mathbb{Z}_p$, it follows that the only elements of $H^*(K(\mathbb{Z}_p,1);\mathbb{Z}_p)$ fixed by γ, hence by Γ, are the scalar multiples of $\beta^{i(p-1)}$ and $\alpha\beta^{i(p-1)-1}$. Thus $H^*(K(\mathbb{Z}_p,1);\mathbb{Z}_p)^\Gamma = \Lambda_{\mathbb{Z}_p}[\alpha\beta^{p-2}]\otimes\mathbb{Z}_p[\beta^{p-1}]$, so we have produced a space whose $\mathbb{Z}_p$ cohomology ring is $\Lambda_{\mathbb{Z}_p}[x_{2p-3}]\otimes\mathbb{Z}_p[y_{2p-2}]$, subscripts indicating dimension.

Example 3G.4. As an easy generalization of the preceding example, replace the group Γ there by a subgroup of $\mathrm{Aut}(\mathbb{Z}_p)$ of order d, where d is any divisor of $p-1$. The new Γ is generated by the automorphism $x \mapsto m^{(p-1)/d}x$, and the same analysis shows that we obtain a space with $\mathbb{Z}_p$ cohomology $\Lambda_{\mathbb{Z}_p}[x_{2d-1}]\otimes\mathbb{Z}_p[y_{2d}]$, subscripts again denoting dimension. For a given choice of d the condition that d divides $p-1$ says $p \equiv 1 \bmod d$, which is satisfied by infinitely many p's, according to a classical theorem of Dirichlet.

Example 3G.5. The two preceding examples can be modified so as to eliminate the exterior algebra factors, by replacing $\mathbb{Z}_p$ by $\mathbb{Z}_{p^\infty}$, the union of the increasing sequence $\mathbb{Z}_p \subset \mathbb{Z}_{p^2} \subset \mathbb{Z}_{p^3} \subset \cdots$. The first step is to show that $H^*(K(\mathbb{Z}_{p^\infty},1);\mathbb{Z}_p) \approx \mathbb{Z}_p[\beta]$ with $|\beta| = 2$. We know that $\widetilde{H}_*(K(\mathbb{Z}_{p^i},1);\mathbb{Z})$ consists of $\mathbb{Z}_{p^i}$'s in odd dimensions. The inclusion $\mathbb{Z}_{p^i} \hookrightarrow \mathbb{Z}_{p^{i+1}}$ induces a map $K(\mathbb{Z}_{p^i},1)\to K(\mathbb{Z}_{p^{i+1}},1)$ that is unique up to homotopy. We can take this map to be a p-sheeted covering space since the covering space of a $K(\mathbb{Z}_{p^{i+1}},1)$ corresponding to the unique index p subgroup of $\pi_1K(\mathbb{Z}_{p^{i+1}},1)$ is a $K(\mathbb{Z}_{p^i},1)$. The homology transfer formula $\pi_*\tau_* = p$ shows that the image of the induced map $H_n(K(\mathbb{Z}_{p^i},1);\mathbb{Z})\to H_n(K(\mathbb{Z}_{p^{i+1}},1);\mathbb{Z})$ for n odd contains the multiples of p, hence this map is the inclusion $\mathbb{Z}_{p^i}\hookrightarrow\mathbb{Z}_{p^{i+1}}$. We can use the universal coefficient theorem to compute the induced map $H^*(K(\mathbb{Z}_{p^{i+1}},1);\mathbb{Z}_p)\to H^*(K(\mathbb{Z}_{p^i},1);\mathbb{Z}_p)$. Namely, the inclusion $\mathbb{Z}_{p^i} \hookrightarrow \mathbb{Z}_{p^{i+1}}$ induces the trivial map $\mathrm{Hom}(\mathbb{Z}_{p^{i+1}},\mathbb{Z}_p)\to\mathrm{Hom}(\mathbb{Z}_{p^i},\mathbb{Z}_p)$, so on odd-dimensional cohomology the induced map is trivial. On the other hand, the induced map on even-dimensional cohomology is an isomorphism since the map of free resolutions

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "p^i"] \arrow[d, "1\!\!1"] & \mathbb{Z} \arrow[r] \arrow[d, "p"] & \mathbb{Z}_{p^i} \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "p^{i+1}"] & \mathbb{Z} \arrow[r] & \mathbb{Z}_{p^{i+1}} \arrow[r] & 0
\end{tikzcd}$$

dualizes to

$$\begin{tikzcd}
0 & \mathrm{Ext}(\mathbb{Z}_{p^i},\mathbb{Z}_p) \arrow[l] & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_p) \arrow[l] & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_p) \arrow[l, "0"'] \\
0 & \mathrm{Ext}(\mathbb{Z}_{p^{i+1}},\mathbb{Z}_p) \arrow[l] \arrow[u] & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_p) \arrow[l] \arrow[u, "1\!\!1"] & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_p) \arrow[l, "0"'] \arrow[u]
\end{tikzcd}$$

Since $\mathbb{Z}_{p^\infty}$ is the union of the increasing sequence of subgroups $\mathbb{Z}_{p^i}$, the space $B\mathbb{Z}_{p^\infty}$ is the union of the increasing sequence of subcomplexes $B\mathbb{Z}_{p^i}$. We can therefore apply

Proposition 3F.5 to conclude that $H^*(K(\mathbb{Z}_{p^\infty},1);\mathbb{Z}_p)$ is zero in odd dimensions, while in even dimensions the map $H^*(K(\mathbb{Z}_{p^\infty},1);\mathbb{Z}_p)\to H^*(K(\mathbb{Z}_p,1);\mathbb{Z}_p)$ induced by the inclusion $\mathbb{Z}_p \hookrightarrow \mathbb{Z}_{p^\infty}$ is an isomorphism. Thus $H^*(K(\mathbb{Z}_{p^\infty},1);\mathbb{Z}_p)\approx \mathbb{Z}_p[\beta]$ as claimed.

Next we show that the map $\mathrm{Aut}(\mathbb{Z}_{p^\infty})\to\mathrm{Aut}(\mathbb{Z}_p)$ obtained by restriction to the subgroup $\mathbb{Z}_p\subset\mathbb{Z}_{p^\infty}$ is a split surjection. Automorphisms of $\mathbb{Z}_{p^i}$ are the maps $x\mapsto mx$ for $(m,p)=1$, so the restriction map $\mathrm{Aut}(\mathbb{Z}_{p^{i+1}})\to\mathrm{Aut}(\mathbb{Z}_{p^i})$ is surjective. Since $\mathrm{Aut}(\mathbb{Z}_{p^\infty}) = \varprojlim \mathrm{Aut}(\mathbb{Z}_{p^i})$, the restriction map $\mathrm{Aut}(\mathbb{Z}_{p^\infty})\to\mathrm{Aut}(\mathbb{Z}_p)$ is also surjective. The order of $\mathrm{Aut}(\mathbb{Z}_{p^i})$, the multiplicative group of invertible elements of $\mathbb{Z}_{p^i}$, is $p^i-p^{i-1}=p^{i-1}(p-1)$ and $p-1$ is relatively prime to p^{i-1}, so the abelian group $\mathrm{Aut}(\mathbb{Z}_{p^i})$ contains a subgroup of order $p-1$. This subgroup maps onto the cyclic group $\mathrm{Aut}(\mathbb{Z}_p)$ of the same order, so $\mathrm{Aut}(\mathbb{Z}_{p^i})\to\mathrm{Aut}(\mathbb{Z}_p)$ is a split surjection, hence so is $\mathrm{Aut}(\mathbb{Z}_{p^\infty})\to\mathrm{Aut}(\mathbb{Z}_p)$.

Thus we have an action of $\Gamma=\mathrm{Aut}(\mathbb{Z}_p)$ on $B\mathbb{Z}_{p^\infty}$ extending its natural action on $B\mathbb{Z}_p$. The Borel construction then gives an inclusion $B\mathbb{Z}_p\times_\Gamma E\Gamma \hookrightarrow B\mathbb{Z}_{p^\infty}\times_\Gamma E\Gamma$ inducing an isomorphism of $H^*(B\mathbb{Z}_{p^\infty}\times_\Gamma E\Gamma;\mathbb{Z}_p)$ onto the even-dimensional part of $H^*(B\mathbb{Z}_p\times_\Gamma E\Gamma;\mathbb{Z}_p)$, a polynomial algebra $\mathbb{Z}_p[y_{2p-2}]$. Similarly, if d is any divisor of $p-1$, then taking Γ to be the subgroup of $\mathrm{Aut}(\mathbb{Z}_p)$ of order d yields a space with $\mathbb{Z}_p$ cohomology the polynomial ring $\mathbb{Z}_p[y_{2d}]$.

Example 3G.6. Now we enlarge the preceding example by taking products and bringing in the permutation group to produce a space with $\mathbb{Z}_p$ cohomology the polynomial ring $\mathbb{Z}_p[y_{2d},y_{4d},\cdots,y_{2nd}]$ where d is any divisor of $p-1$ and $p>n$. Let X be the product of n copies of $B\mathbb{Z}_{p^\infty}$ and let Γ be the group of homeomorphisms of X generated by permutations of the factors together with the actions of $\mathbb{Z}_d$ in each factor constructed in the preceding example. We can view Γ as a group of $n\times n$ matrices with entries in $\mathbb{Z}_p$, the matrices obtained by replacing some of the 1's in a permutation matrix by elements of $\mathbb{Z}_p$ of multiplicative order a divisor of d. Thus there is a split short exact sequence $0\to(\mathbb{Z}_d)^n\to\Gamma\to\Sigma_n\to 0$, and the order of Γ is $d^n n!$. The product space X has $H^*(X;\mathbb{Z}_p)\approx\mathbb{Z}_p[\beta_1,\cdots,\beta_n]$ with $|\beta_i|=2$, so $H^*(X\times_\Gamma E\Gamma;Z_p)\approx\mathbb{Z}_p[\beta_1,\cdots,\beta_n]^\Gamma$ provided that p does not divide the order of Γ, which means $p>n$. For a polynomial to be invariant under the $\mathbb{Z}_d$ action in each factor it must be a polynomial in the powers β_i^d, and to be invariant under permutations of the variables it must be a symmetric polynomial in these powers. Since symmetric polynomials are exactly the polynomials in the elementary symmetric functions, the polynomials in the β_i's invariant under Γ form a polynomial ring $\mathbb{Z}_p[y_{2d},y_{4d},\cdots,y_{2nd}]$ with y_{2k} the sum of all products of k distinct powers β_i^d.

Example 3G.7. As a further variant on the preceding example, choose a divisor q of d and replace Γ by its subgroup consisting of matrices for which the product of the q^{th} powers of the nonzero entries is 1. This has the effect of enlarging the ring of polynomials invariant under the action, and it can be shown that the invariant polynomials form a polynomial ring $\mathbb{Z}_p[y_{2d}, y_{4d}, \cdots, y_{2(n-1)d}, y_{2nq}]$, with the last generator y_{2nd} replaced by $y_{2nq} = \prod_i \beta_i^q$. For example, if $n = 2$ and $q = 1$ we obtain $\mathbb{Z}_p[y_4, y_{2d}]$ with $y_4 = \beta_1\beta_2$ and $y_{2d} = \beta_1^d + \beta_2^d$. The group Γ in this case happens to be isomorphic to the dihedral group of order $2d$.

General Remarks

The problem of realizing graded polynomial rings $\mathbb{Z}_p[y]$ in one variable as cup product rings of spaces was discussed in §3.2, and Example 3G.5 provides the remaining examples, showing that $|y|$ can be any even divisor of $2(p-1)$. In more variables the problem of realizing $\mathbb{Z}_p[y_1, \cdots, y_n]$ with specified dimensions $|y_i|$ is more difficult, but has been solved for odd primes p. Here is a sketch of the answer.

Assuming that p is odd, the dimensions $|y_i|$ are even. Call the number $d_i = |y_i|/2$ the *degree* of y_i. In the examples above this was in fact the degree of y_i as a polynomial in the 2-dimensional classes β_j invariant under the action of Γ. It was proved in [Dwyer, Miller, & Wilkerson 1992] that every realizable polynomial algebra $\mathbb{Z}_p[y_1, \cdots, y_n]$ is the ring of invariant polynomials $\mathbb{Z}_p[\beta_1, \cdots, \beta_n]^\Gamma$ for an action of some finite group Γ on $\mathbb{Z}_p[\beta_1, \cdots, \beta_n]$, where $|\beta_i| = 2$. The basic examples, whose products yield all realizable polynomial algebras, can be divided into two categories. First there are classifying spaces of Lie groups, each of which realizes a polynomial algebra for all but finitely many primes p. These are listed in the following table.

Lie group	degrees	primes
S^1	1	all
$SU(n)$	$2, 3, \cdots, n$	all
$Sp(n)$	$2, 4, \cdots, 2n$	all
$SO(2k)$	$2, 4, \cdots, 2k-2, k$	$p > 2$
G_2	$2, 6$	$p > 2$
F_4	$2, 6, 8, 12$	$p > 3$
E_6	$2, 5, 6, 8, 9, 12$	$p > 3$
E_7	$2, 6, 8, 10, 12, 14$	$p > 3$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$	$p > 5$

The remaining examples have to be constructed by hand. They form two infinite families plus 30 sporadic exceptions shown in the table on the next page. The first row is the examples we have constructed, though our construction needed the extra condition that p not divide the order of the group Γ. For all entries in both tables the order of Γ, the group such that $\mathbb{Z}_p[y_1, \cdots, y_n] = \mathbb{Z}_p[\beta_1, \cdots, \beta_n]^\Gamma$, turns out to equal the product of the degrees. When p does not divide this order, the method we used for the first row can also be applied to give examples for all the other rows. In some cases the congruence conditions on p, which are needed in order for Γ to be a subgroup of $\mathrm{Aut}(\mathbb{Z}_p^n) = GL_n(\mathbb{Z}_p)$, automatically imply that p does not divide the order of Γ. But when this is not the case a different construction of a space with the desired cohomology is needed. To find out more about this the reader can begin by consulting [Kane 1988] and [Notbohm 1999].

degrees	primes
$d, 2d, \cdots, (n-1)d, nq$ with $q \mid d$	$p \equiv 1 \bmod d$
$2, d$	$p \equiv -1 \bmod d$

degrees	primes
4, 6	$p \equiv 1 \bmod 3$
6, 12	$p \equiv 1 \bmod 3$
4, 12	$p \equiv 1 \bmod 12$
12, 12	$p \equiv 1 \bmod 12$
8, 12	$p \equiv 1 \bmod 4$
8, 24	$p \equiv 1 \bmod 8$
12, 24	$p \equiv 1 \bmod 12$
24, 24	$p \equiv 1 \bmod 24$
6, 8	$p \equiv 1, 3 \bmod 8$
8, 12	$p \equiv 1 \bmod 8$
6, 24	$p \equiv 1, 19 \bmod 24$
12, 24	$p \equiv 1 \bmod 24$
20, 30	$p \equiv 1 \bmod 5$
20, 60	$p \equiv 1 \bmod 20$
30, 60	$p \equiv 1 \bmod 15$

degrees	primes
60, 60	$p \equiv 1 \bmod 60$
12, 30	$p \equiv 1, 4 \bmod 15$
12, 60	$p \equiv 1, 49 \bmod 60$
12, 20	$p \equiv 1, 9 \bmod 20$
2, 6, 10	$p \equiv 1, 4 \bmod 5$
4, 6, 14	$p \equiv 1, 2, 4 \bmod 7$
6, 9, 12	$p \equiv 1 \bmod 3$
6, 12, 18	$p \equiv 1 \bmod 3$
6, 12, 30	$p \equiv 1, 4 \bmod 15$
4, 8, 12, 20	$p \equiv 1 \bmod 4$
2, 12, 20, 30	$p \equiv 1, 4 \bmod 5$
8, 12, 20, 24	$p \equiv 1 \bmod 4$
12, 18, 24, 30	$p \equiv 1 \bmod 3$
4, 6, 10, 12, 18	$p \equiv 1 \bmod 3$
6, 12, 18, 24, 30, 42	$p \equiv 1 \bmod 3$

For the prime 2 the realization problem is still not completely solved. The known examples are listed in the short table at the right, where again we give only the irreducible examples, which generate others by taking products. All but the last entry in the table arise from classifying spaces of Lie groups, as described in §4.D. The construction for the last entry is in [Dwyer & Wilkerson 1993].

Lie group	degrees
$O(1)$	1
$SO(n)$	$2, 3, \cdots, n$
$SU(n)$	$4, 6, \cdots, 2n$
$Sp(n)$	$4, 8, \cdots, 4n$
—	8, 12, 14, 15

3.H Local Coefficients

Homology and cohomology with local coefficients are fancier versions of ordinary homology and cohomology that can be defined for nonsimply-connected spaces. In various situations these more refined homology and cohomology theories arise naturally and inevitably. For example, the only way to extend Poincaré duality with $\mathbb{Z}$ coefficients to nonorientable manifolds is to use local coefficients. In the overall scheme of algebraic topology, however, the role played by local coefficients is fairly small. Local coefficients bring an extra level of complication that one tries to avoid whenever possible. With this in mind, the goal of this section will not be to give a full exposition but rather just to sketch the main ideas, leaving the technical details for the interested reader to fill in.

The plan for this section is first to give the quick algebraic definition of homology and cohomology with local coefficients, and then to reinterpret this definition more geometrically in a way that looks more like ordinary homology and cohomology. The reinterpretation also allows the familiar properties of homology and cohomology to be extended to the local coefficient case with very little effort.

Local Coefficients via Modules

Let X be a path-connected space having a universal cover $\widetilde{X}$ and fundamental group π, so that X is the quotient of $\widetilde{X}$ by the action of π by deck transformations $\widetilde{x} \mapsto \gamma \cdot \widetilde{x}$ for $\gamma \in \pi$ and $\widetilde{x} \in \widetilde{X}$. The action of π on $\widetilde{X}$ induces an action of π on the group $C_n(\widetilde{X})$ of singular n-chains in $\widetilde{X}$, by sending a singular n-simplex $\sigma : \Delta^n \to \widetilde{X}$ to the composition $\Delta^n \xrightarrow{\sigma} \widetilde{X} \xrightarrow{\gamma} \widetilde{X}$. The action of π on $C_n(\widetilde{X})$ makes $C_n(\widetilde{X})$ a module over the group ring $\mathbb{Z}[\pi]$, which consists of the finite formal sums $\sum_i m_i \gamma_i$ with $m_i \in \mathbb{Z}$ and $\gamma_i \in \pi$, with the natural addition $\sum_i m_i \gamma_i + \sum_i n_i \gamma_i = \sum_i (m_i + n_i)\gamma_i$ and multiplication $(\sum_i m_i \gamma_i)(\sum_j n_j \gamma_j) = \sum_{i,j} m_i n_j \gamma_i \gamma_j$. The boundary maps $\partial : C_n(\widetilde{X}) \to C_{n-1}(\widetilde{X})$ are $\mathbb{Z}[\pi]$-module homomorphisms since the action of π on these groups comes from an action on $\widetilde{X}$.

If M is an arbitrary module over $\mathbb{Z}[\pi]$, we would like to define $C_n(X;M)$ to be $C_n(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} M$, but for tensor products over a noncommutative ring one has to be a little careful with left and right module structures. In general, if R is a ring, possibly noncommutative, one defines the tensor product $A \otimes_R B$ of a right R-module A and a left R-module B to be the abelian group with generators $a \otimes b$ for $a \in A$ and $b \in B$, subject to distributivity and associativity relations:

(i) $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$ and $a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$.

(ii) $ar \otimes b = a \otimes rb$.

In case $R = \mathbb{Z}[\pi]$, a left $\mathbb{Z}[\pi]$-module A can be regarded as a right $\mathbb{Z}[\pi]$-module by setting $a\gamma = \gamma^{-1}a$ for $\gamma \in \pi$. So the tensor product of two left $\mathbb{Z}[\pi]$-modules A and B is defined, and the relation $a\gamma \otimes b = a \otimes \gamma b$ becomes $\gamma^{-1}a \otimes b = a \otimes \gamma b$, or equivalently $a' \otimes b = \gamma a' \otimes \gamma b$ where $a' = \gamma^{-1}a$. Thus tensoring over $\mathbb{Z}[\pi]$ has the effect of factoring out the action of π. To simplify notation we shall write $A \otimes_{\mathbb{Z}[\pi]} B$ as $A \otimes_\pi B$, emphasizing the fact that the essential part of a $\mathbb{Z}[\pi]$-module structure is the action of π.

In particular, $C_n(\widetilde{X}) \otimes_\pi M$ is defined if M is a left $\mathbb{Z}[\pi]$-module. These chain groups $C_n(X;M) = C_n(\widetilde{X}) \otimes_\pi M$ form a chain complex with the boundary maps $\partial \otimes \mathbb{1}$. The homology groups $H_n(X;M)$ of this chain complex are by definition **homology groups with local coefficients**.

For cohomology one can set $C^n(X;M) = \mathrm{Hom}_{\mathbb{Z}[\pi]}(C_n(\widetilde{X}), M)$, the $\mathbb{Z}[\pi]$-module homomorphisms $C_n(\widetilde{X}) \to M$. These groups $C^n(X;M)$ form a cochain complex whose cohomology groups $H^n(X;M)$ are **cohomology groups with local coefficients**.

Example 3H.1. Let us check that when M is a trivial $\mathbb{Z}[\pi]$-module, with $\gamma m = m$ for all $\gamma \in \pi$ and $m \in M$, then $H_n(X;M)$ is just ordinary homology with coefficients in the abelian group M. For a singular n-simplex $\sigma:\Delta^n \to X$, the various lifts $\widetilde{\sigma}:\Delta^n \to \widetilde{X}$ form an orbit of the action of π on $C_n(\widetilde{X})$. In $C_n(\widetilde{X})\otimes_\pi M$ all these lifts are identified via the relation $\widetilde{\sigma}\otimes m = \gamma\widetilde{\sigma}\otimes\gamma m = \gamma\widetilde{\sigma}\otimes m$. Thus we can identify $C_n(\widetilde{X})\otimes_\pi M$ with $C_n(X)\otimes M$, the chain group denoted $C_n(X;M)$ in ordinary homology theory, so $H_n(X;M)$ reduces to ordinary homology with coefficients in M. The analogous statement for cohomology is also true since elements of $\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_n(\widetilde{X}),M)$ are functions from singular n-simplices $\widetilde{\sigma}:\Delta^n \to \widetilde{X}$ to M taking the same value on all elements of a π-orbit since the action of π on M is trivial, so $\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_n(\widetilde{X}),M)$ is identifiable with $\mathrm{Hom}(C_n(X),M)$, ordinary cochains with coefficients in M.

Example 3H.2. Suppose we take $M = \mathbb{Z}[\pi]$, viewed as a module over itself via its ring structure. For a ring R with identity element, $A\otimes_R R$ is naturally isomorphic to A via the correspondence $a\otimes r \mapsto ar$. So we have a natural identification of $C_n(\widetilde{X})\otimes_\pi \mathbb{Z}[\pi]$ with $C_n(\widetilde{X})$, and hence an isomorphism $H_n(X;\mathbb{Z}[\pi]) \approx H_n(\widetilde{X})$. Generalizing this, let $X' \to X$ be the cover corresponding to a subgroup $\pi' \subset \pi$. Then the free abelian group $\mathbb{Z}[\pi/\pi']$ with basis the cosets $\gamma\pi'$ is a $\mathbb{Z}[\pi]$-module and $C_n(\widetilde{X})\otimes_{\mathbb{Z}[\pi]}\mathbb{Z}[\pi/\pi'] \approx C_n(X')$, so $H_n(X;\mathbb{Z}[\pi/\pi']) \approx H_n(X')$. More generally, if A is an abelian group then $A[\pi/\pi']$ is a $\mathbb{Z}[\pi]$-module and $H_n(X;A[\pi/\pi']) \approx H_n(X';A)$. So homology of covering spaces is a special case of homology with local coefficients. The corresponding assertions for cohomology are not true, however, as we shall see later in the section.

For a $\mathbb{Z}[\pi]$-module M, let π' be the kernel of the homomorphism $\rho:\pi \to \mathrm{Aut}(M)$ defining the module structure, given by $\rho(\gamma)(m) = \gamma m$, where $\mathrm{Aut}(M)$ is the group of automorphisms of the abelian group M. If $X' \to X$ is the cover corresponding to the normal subgroup π' of π, then $C_n(\widetilde{X})\otimes_\pi M \approx C_n(X')\otimes_\pi M \approx C_n(X')\otimes_{\mathbb{Z}[\pi/\pi']} M$. This gives a more efficient description of $H_n(X;M)$.

Example 3H.3. As a special case, suppose that we take $M = \mathbb{Z}$, so $\mathrm{Aut}(\mathbb{Z}) \approx \mathbb{Z}_2 = \{\pm 1\}$. For a nontrivial $\mathbb{Z}[\pi]$-module structure on M, π' is a subgroup of index 2 and $X' \to X$ is a 2-sheeted covering space. If τ is the nontrivial deck transformation of X', let $C_n^+(X') = \{\alpha \in C_n(X') \mid \tau_\sharp(\alpha) = \alpha\}$ and $C_n^-(X') = \{\alpha \in C_n(X') \mid \tau_\sharp(\alpha) = -\alpha\}$. It follows easily that $C_n^\pm(X')$ has basis the chains $\sigma \pm \tau\sigma$ for $\sigma:\Delta^n \to X'$, and we have short exact sequences

$$0 \longrightarrow C_n^-(X') \hookrightarrow C_n(X') \xrightarrow{\Sigma} C_n^+(X') \longrightarrow 0$$
$$0 \longrightarrow C_n^+(X') \hookrightarrow C_n(X') \xrightarrow{\Delta} C_n^-(X') \longrightarrow 0$$

where $\Sigma(\alpha) = \alpha + \tau_\sharp(\alpha)$ and $\Delta(\alpha) = \alpha - \tau_\sharp(\alpha)$. The homomorphism $C_n(X) \to C_n^+(X')$ sending a singular simplex in X to the sum of its two lifts to X' is an isomorphism. The quotient map $C_n(X') \to C_n(X')\otimes_\pi \mathbb{Z}$ has kernel $C_n^+(X')$, so the second short exact sequence gives an isomorphism $C_n^-(X') \approx C_n(X')\otimes_\pi \mathbb{Z}$. These isomorphisms are isomorphisms of chain complexes and the short exact sequences are short exact sequence of chain complexes, so from the first short exact sequence we get a long exact sequence of homology groups

$$\cdots \longrightarrow H_n(X;\tilde{\mathbb{Z}}) \longrightarrow H_n(X') \xrightarrow{p_*} H_n(X) \longrightarrow H_{n-1}(X;\tilde{\mathbb{Z}}) \longrightarrow \cdots$$

where the symbol $\tilde{\mathbb{Z}}$ indicates local coefficients in the module $\mathbb{Z}$ and p_* is induced by the covering projection $p:X'\to X$.

Let us apply this exact sequence when X is a nonorientable n-manifold M which is closed and connected. We shall use terminology and notation from §3.3. We can view $\mathbb{Z}$ as a $\mathbb{Z}[\pi_1 M]$-module by letting a loop γ in M act on $\mathbb{Z}$ by multiplication by $+1$ or -1 according to whether γ preserves or reverses local orientations of M. The double cover $X'\to X$ is then the 2-sheeted cover $\widetilde{M}\to M$ with $\widetilde{M}$ orientable. The nonorientability of M implies that $H_n(M) = 0$. Since $H_{n+1}(M) = 0$, the exact sequence above then gives $H_n(M;\tilde{\mathbb{Z}}) \approx H_n(\widetilde{M}) \approx \mathbb{Z}$. This can be interpreted as saying that by taking homology with local coefficients we obtain a fundamental class for a nonorientable manifold.

Local Coefficients via Bundles of Groups

Now we wish to reinterpret homology and cohomology with local coefficients in more geometric terms, making it look more like ordinary homology and cohomology.

Let us first define a special kind of covering space with extra algebraic structure. A **bundle of groups** is a map $p:E\to X$ together with a group structure on each subset $p^{-1}(x)$, such that all these groups $p^{-1}(x)$ are isomorphic to a fixed group G in the following special way: Each point of X has a neighborhood U for which there exists a homeomorphism $h_U:p^{-1}(U)\to U\times G$ taking each $p^{-1}(x)$ to $\{x\}\times G$ by a group isomorphism. Since G is given the discrete topology, the projection p is a covering space. Borrowing terminology from the theory of fiber bundles, the subsets $p^{-1}(x)$ are called the **fibers** of $p:E\to X$, and one speaks of E as a bundle of groups with fiber G. It may be worth remarking that if we modify the definition by replacing the word 'group' with 'vector space' throughout, then we obtain the much more common notion of a vector bundle; see [VBKT].

Trivial examples are provided by products $E = X\times G$. Nontrivial examples we have considered are the covering spaces $M_{\mathbb{Z}}\to M$ of nonorientable manifolds M defined in §3.3. Here the group G is the homology coefficient group $\mathbb{Z}$, though one could equally well define a bundle of groups $M_G\to M$ for any abelian coefficient group G.

Homology groups of X with coefficients in a bundle E of abelian groups may be defined as follows. Consider finite sums $\sum_i n_i\sigma_i$ where each $\sigma_i:\Delta^n\to X$ is a singular n-simplex in X and $n_i:\Delta^n\to E$ is a lifting of σ_i. The sum of two lifts n_i and m_i of the same σ_i is defined by $(n_i + m_i)(s) = n_i(s) + m_i(s)$, and is also a lift of σ_i. In this way the finite sums $\sum_i n_i\sigma_i$ form an abelian group $C_n(X;E)$, provided we allow the deletion of terms $n_i\sigma_i$ when n_i is the zero-valued lift. A boundary homomorphism $\partial : C_n(X;E) \to C_{n-1}(X;E)$ is defined by the formula $\partial(\sum_i n_i\sigma_i) = \sum_{i,j}(-1)^j n_i\sigma_i \,|\, [v_0,\cdots,\hat{v}_j,\cdots,v_n]$ where 'n_i' in the right side of the equation means the restricted lifting $n_i \,|\, [v_0,\cdots,\hat{v}_j,\cdots,v_n]$. The proof that the usual boundary homomorphism ∂ satisfies $\partial^2 = 0$ still works in the present context, so the groups $C_n(X;E)$ form a chain complex. We denote the homology groups of this chain complex by $H_n(X;E)$.

In case E is the product bundle $X\times G$, lifts n_i are simply elements of G, so $H_n(X;E) = H_n(X;G)$, ordinary homology. In the general case, lifts $n_i : \Delta^n \to E$ are uniquely determined by their value at one point $s \in \Delta^n$, and these values can be specified arbitrarily since Δ^n is simply-connected, so the n_i's can be thought of as elements of $p^{-1}(\sigma_i(s))$, a group isomorphic to G. However if E is not a product, there is no canonical isomorphism between different fibers $p^{-1}(x)$, so one cannot identify $H_n(X;E)$ with ordinary homology.

An alternative approach would be to take the coefficients n_i to be elements of the fiber group over a specific point of $\sigma_i(\Delta^n)$, say $\sigma_i(v_0)$. However, with such a definition the formula for the boundary operator ∂ becomes more complicated since there is no point of Δ^n that lies in all the faces.

Our task now is to relate the homology groups $H_n(X;E)$ to homology groups with coefficients in a module, as defined earlier. In §1.3 we described how covering spaces of X with a given fiber F can be classified in terms of actions of $\pi_1(X)$ on F, assuming X is path-connected and has the local properties guaranteeing the existence of a universal cover. It is easy to check that covering spaces that are bundles of groups with fiber a group G are equivalent to actions of $\pi_1(X)$ on G by automorphisms of G, that is, homomorphisms from $\pi_1(X)$ to $\mathrm{Aut}(G)$.

For example, for the bundle $M_{\mathbb{Z}} \to M$ the action of a loop γ on the fiber $\mathbb{Z}$ is multiplication by ± 1 according to whether γ preserves or reverses orientation in M, that is, whether γ lifts to a closed loop in the orientable double cover $\widetilde{M} \to M$ or not. As another example, the action of $\pi_1(X)$ on itself by inner automorphisms corresponds to a bundle of groups $p : E \to X$ with fibers $p^{-1}(x) = \pi_1(X,x)$. This example is rather similar in spirit to the examples $M_{\mathbb{Z}} \to M$. In both cases one has a functor associating a group to each point of a space, and all the groups at different points are isomorphic, but not canonically so. Different choices of isomorphisms are obtained by choosing different paths between two points, and loops give rise to an action of π_1 on the fibers.

In the case of bundles of groups $p : E \to X$ whose fiber G is abelian, an action of $\pi_1(X)$ on G by automorphisms is the same as a $\mathbb{Z}[\pi_1 X]$-module structure on G.

Proposition 3H.4. *If X is a path-connected space having a universal covering space, then the groups $H_n(X;E)$ are naturally isomorphic to the homology groups $H_n(X;G)$ with local coefficients in the $\mathbb{Z}[\pi]$-module G associated to E, where $\pi = \pi_1(X)$.*

Proof: As noted earlier, a bundle of groups $E \to X$ with fiber G is equivalent to an action of π on G. In more explicit terms this means that if $\widetilde{X}$ is the universal cover of X, then E is identifiable with the quotient of $\widetilde{X} \times G$ by the diagonal action of π, $\gamma(\widetilde{x}, g) = (\gamma\widetilde{x}, \gamma g)$ where the action in the first coordinate is by deck transformations of $\widetilde{X}$. For a chain $\sum_i n_i \sigma_i \in C_n(X;E)$, the coefficient n_i gives a lift of σ_i to E, and n_i in turn has various lifts to $\widetilde{X} \times G$. Thus we have natural surjections $C_n(\widetilde{X} \times G) \to C_n(E) \to C_n(X;E)$ expressing each of these groups as a quotient of the preceding one. More precisely, identifying $C_n(\widetilde{X} \times G)$ with $C_n(\widetilde{X}) \otimes \mathbb{Z}[G]$ in the obvious way, then $C_n(E)$ is the quotient of $C_n(\widetilde{X}) \otimes \mathbb{Z}[G]$ under the identifications $\widetilde{\sigma} \otimes g \sim \gamma \cdot \widetilde{\sigma} \otimes \gamma \cdot g$. This quotient is the tensor product $C_n(\widetilde{X}) \otimes_\pi \mathbb{Z}[G]$. To pass to the quotient $C_n(X;E)$ of $C_n(E) = C_n(\widetilde{X}) \otimes_\pi \mathbb{Z}[G]$ we need to take into account the sum operation in $C_n(X;E)$, addition of lifts $n_i : \Delta^n \to E$. This means that in sums $\widetilde{\sigma} \otimes g_1 + \widetilde{\sigma} \otimes g_2 = \widetilde{\sigma} \otimes (g_1 + g_2)$, the term $g_1 + g_2$ should be interpreted not in $\mathbb{Z}[G]$ but in the natural quotient G of $\mathbb{Z}[G]$. Hence $C_n(X;E)$ is identified with the quotient $C_n(\widetilde{X}) \otimes_\pi G$ of $C_n(\widetilde{X}) \otimes_\pi \mathbb{Z}[G]$. This natural identification commutes with the boundary homomorphisms, so the homology groups are also identified. $\square$

More generally, if X has a number of path-components X_α with universal covers $\widetilde{X}_\alpha$, then $C_n(X;E) = \bigoplus_\alpha \big(C_n(\widetilde{X}_\alpha) \otimes_{\mathbb{Z}[\pi_1(X_\alpha)]} G\big)$, so $H_n(X;E)$ splits accordingly as a direct sum of the local coefficient homology groups for the path-components X_α.

We turn now to the question of whether homology with local coefficients satisfies axioms similar to those for ordinary homology. The main novelty is with the behavior of induced homomorphisms. In order for a map $f : X \to X'$ to induce a map on homology with local coefficients we must have bundles of groups $E \to X$ and $E' \to X'$ that are related in some way. The natural assumption to make is that there is a commutative diagram as at the right, such that $\widetilde{f}$ restricts to a homomorphism in each fiber. With this hypothesis there is then a chain homomorphism $f_\sharp : C_n(X;E) \to C_n(X';E')$ obtained by composing singular simplices with f and their lifts with $\widetilde{f}$, hence there is an induced homomorphism $f_* : H_n(X;E) \to H_n(X';E')$. The fibers of E and E' need not be isomorphic groups, so in the case of trivial bundles this construction specializes to Bockstein homomorphisms. To avoid this extra complication we shall consider only the case that $\widetilde{f}$ restricts to an isomorphism on each fiber. With this condition, a commutative diagram as above will be called a **bundle map**.

$$\begin{tikzcd} E \arrow[r, "\widetilde{f}"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

Here is a method for constructing bundle maps. Starting with a map $f : X \to X'$ and a bundle of groups $p' : E' \to X'$, let

$$E = \{\, (x, e') \in X \times E' \mid f(x) = p'(e') \,\}.$$

This fits into a commutative diagram as above if we define $p(x, e') = x$ and $\widetilde{f}(x, e') = e'$. In particular, the fiber $p^{-1}(x)$ consists of pairs (x, e') with $p'(e') = f(x)$, so $\widetilde{f}$ is a bijection of this fiber with the fiber of $E' \to X'$ over $f(x)$. We use this bijection to give $p^{-1}(x)$ a group structure. To check that $p: E \to X$ is a bundle of groups, let $h': (p')^{-1}(U') \to U' \times G$ be an isomorphism as in the definition of a bundle of groups. Define $h: p^{-1}(U) \to U \times G$ over $U = f^{-1}(U')$ by $h(x, e') = (x, h_2'(e'))$ where h_2' is the second coordinate of h'. An inverse for h is $(x, g) \mapsto (x, (h')^{-1}(f(x), g))$, and h is clearly an isomorphism on each fiber. Thus $p: E \to X$ is a bundle of groups, called the **pullback** of $E' \to X'$ via f, or the **induced bundle**. The notation $f^*(E')$ is often used for the pullback bundle.

Given any bundle map $E \to E'$ as in the diagram above, it is routine to check that the map $E \to f^*(E')$, $e \mapsto (p(e), \widetilde{f}(e))$, is an isomorphism of bundles over X, so the pullback construction produces all bundle maps. Thus we see one reason why homology with local coefficients is somewhat complicated: $H_n(X; E)$ is really a functor of two variables, covariant in X and contravariant in E.

Viewing bundles of groups over X as $\mathbb{Z}[\pi_1 X]$-modules, the pullback construction corresponds to making a $\mathbb{Z}[\pi_1 X']$-module into a $\mathbb{Z}[\pi_1 X]$-module by defining $\gamma g = f_*(\gamma) g$ for $f_*: \pi_1(X) \to \pi_1(X')$. This follows easily from the definitions. In particular, this implies that homotopic maps $f_0, f_1: X \to X'$ induce isomorphic pullback bundles $f_0^*(E'), f_1^*(E')$. Hence the map $f_*: H_n(X; E) \to H_n(X'; E')$ induced by a bundle map depends only on the homotopy class of f.

Generalizing the definition of $H_n(X; E)$ to pairs (X, A) is straightforward, starting with the definition of $H_n(X, A; E)$ as the n^{th} homology group of the chain complex of quotients $C_n(X; E)/C_n(A; E)$ where $p: E \to X$ becomes a bundle of groups over A by restriction to $p^{-1}(A)$. Associated to the pair (X, A) there is then a long exact sequence of homology groups with local coefficients in the bundle E. The excision property is proved just as for ordinary homology, via iterated barycentric subdivision. The final axiom for homology, involving disjoint unions, extends trivially to homology with local coefficients. Simplicial and cellular homology also extend without difficulty to the case of local coefficients, as do the proofs that these forms of homology agree with singular homology for Δ-complexes and CW complexes, respectively. We leave the verifications of all these statements to the energetic reader.

Now we turn to cohomology. One might try defining $H^n(X; E)$ by simply dualizing, taking $\mathrm{Hom}(C_n(X), E)$, but this makes no sense since E is not a group. Instead, the cochain group $C^n(X; E)$ is defined to consist of all functions φ assigning to each singular simplex $\sigma: \Delta^n \to X$ a lift $\varphi(\sigma): \Delta^n \to E$. In case E is the product $X \times G$, this amounts to assigning an element of G to each σ, so this definition generalizes ordinary cohomology. Coboundary maps $\delta: C^n(X; E) \to C^{n+1}(X; E)$ are defined just as with ordinary cohomology, and satisfy $\delta^2 = 0$, so we have cohomology groups $H^n(X; E)$, and in the relative case, $H^n(X, A; E)$, defined via relative cochains $C^n(X, A; E) = \mathrm{Ker}\big(C^n(X; E) \to C^n(A; E)\big)$.

For a path-connected space X with universal cover $\widetilde{X}$ and fundamental group π, we can identify $H^n(X; E)$ with $H^n(X; G)$, cohomology with local coefficients in the

$\mathbb{Z}[\pi]$-module G corresponding to E, by identifying $C^n(X;E)$ with $\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_n(\widetilde{X}),G)$ in the following way. An element $\varphi \in C^n X;E)$ assigns to each $\sigma:\Delta^n \to X$ a lift to E. Regarding E as the quotient of $\widetilde{X}\times G$ under the diagonal action of π, a lift of σ to E is the same as an orbit of a lift to $\widetilde{X}\times G$. Such an orbit is a function f assigning to each lift $\widetilde{\sigma}:\Delta^n \to \widetilde{X}$ an element $f(\widetilde{\sigma}) \in G$ such that $f(\gamma\widetilde{\sigma}) = \gamma f(\widetilde{\sigma})$ for all $\gamma \in G$, that is, an element of $\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_n(\widetilde{X}),G)$.

The basic properties of ordinary cohomology in §3.1 extend without great difficulty to cohomology groups with local coefficients. In order to define the map $f^*:H^n(X';E') \to H^n(X;E)$ induced by a bundle map as before, it suffices to observe that a singular simplex $\sigma:\Delta^n \to X$ and a lift $\widetilde{\sigma}':\Delta^n \to E'$ of $f\sigma$ define a lift $\widetilde{\sigma} = (\sigma,\widetilde{\sigma}'):\Delta^n \to f^*(E)$ of σ. To show that $f \simeq g$ implies $f^* = g^*$ requires some modification of the proof of the corresponding result for ordinary cohomology in §3.1, which proceeded by dualizing the proof for homology. In the local coefficient case one constructs a chain homotopy P^* satisfying $g^\sharp - f^\sharp = P^*\delta + \delta P^*$ directly from the subdivision of $\Delta^n \times I$ used in the proof of the homology result. Similar remarks apply to proving excision and Mayer–Vietoris sequences for cohomology with local coefficients. To prove the equivalence of simplicial and cellular cohomology with singular cohomology in the local coefficient context, one should use the telescope argument from the proof of Lemma 2.34 to show that $H^n(X^k;E) \approx H^n(X;E)$ for $k > n$. Once again details will be left to the reader.

The difference between homology with local coefficients and cohomology with local coefficients is illuminated by comparing the following proposition with our earlier identification of $H_*(X;\mathbb{Z}[\pi_1 X])$ with the ordinary homology of the universal cover of X.

Proposition 3H.5. *If X is a finite CW complex with universal cover $\widetilde{X}$ and fundamental group π, then for all n, $H^n(X;\mathbb{Z}[\pi])$ is isomorphic to $H^n_c(\widetilde{X};\mathbb{Z})$, cohomology of $\widetilde{X}$ with compact supports and ordinary integer coefficients.*

For example, consider the the n-dimensional torus T^n, the product of n circles, with fundamental group $\pi = \mathbb{Z}^n$ and universal cover $\mathbb{R}^n$. We have $H_i(T^n;\mathbb{Z}[\pi]) \approx H_i(\mathbb{R}^n)$, which is zero except for a $\mathbb{Z}$ in dimension 0, but $H^i(T^n;\mathbb{Z}[\pi]) \approx H^i_c(\mathbb{R}^n)$ vanishes except for a $\mathbb{Z}$ in dimension n, as we saw in Example 3.34.

To prove the proposition we shall use a few general facts about cohomology with compact supports. One significant difference between ordinary cohomology and cohomology with compact supports is in induced maps. A map $f:X \to Y$ induces $f^\sharp:C^n_c(Y;G) \to C^n_c(X;G)$ and hence $f^*:H^n_c(Y;G) \to H^n_c(X;G)$ provided that f is **proper**: The preimage $f^{-1}(K)$ of each compact set K in Y is compact in X. Thus if $\varphi \in C^n(Y;G)$ vanishes on chains in $Y - K$ then $f^\sharp(\varphi) \in C^n(X;G)$ vanishes on chains in $X - f^{-1}(K)$. Further, to guarantee that $f \simeq g$ implies $f^* = g^*$ we should restrict attention to homotopies that are proper as maps $X\times I \to Y$. Relative groups

$H_c^n(X,A;G)$ are defined when A is a closed subset of X, which guarantees that the inclusion $A \hookrightarrow X$ is a proper map. With these constraints the basic theory of §3.1 translates without difficulty to cohomology with compact supports.

In particular, for a locally compact CW complex X one can compute $H_c^*(X;G)$ using *finite cellular cochains*, the cellular cochains vanishing on all but finitely many cells. Namely, to compute $H_c^n(X^n,X^{n-1};G)$ using excision one first has to identify this group with $H_c^n(X^n,N(X^{n-1});G)$ where $N(X^{n-1})$ is a closed neighborhood of X^{n-1} in X^n obtained by deleting an open n-disk from the interior of each n-cell. If X is locally compact, the obvious deformation retraction of $N(X^{n-1})$ onto X^{n-1} is a proper homotopy equivalence. Hence via long exact sequences and the five-lemma we obtain isomorphisms $H_c^n(X^n,X^{n-1};G) \approx H_c^n(X^n,N(X^{n-1});G)$, and by excision the latter group can be identified with the finite cochains.

Proof of 3H.5: As noted above, we can compute $H_c^*(\widetilde{X};\mathbb{Z})$ using the groups $C_f^n(\widetilde{X};\mathbb{Z})$ of finite cellular cochains $\varphi: C_n \to \mathbb{Z}$, where $C_n = H_n(\widetilde{X}^n,\widetilde{X}^{n-1})$. Giving $\widetilde{X}$ the CW structure lifting the CW structure on X, then since X is compact, finite cellular cochains are exactly homomorphisms $\varphi: C_n \to \mathbb{Z}$ such that for each cell e^n of $\widetilde{X}$, $\varphi(\gamma e^n)$ is nonzero for only finitely many covering transformations $\gamma \in \pi$. Such a φ determines a map $\widehat{\varphi}: C_n \to \mathbb{Z}[\pi]$ by setting $\widehat{\varphi}(e^n) = \sum_\gamma \varphi(\gamma^{-1}e^n)\gamma$. The map $\widehat{\varphi}$ is a $\mathbb{Z}[\pi]$-homomorphism since if we replace the summation index γ in the right side of $\varphi(\eta e^n) = \sum_\gamma \varphi(\gamma^{-1}\eta e^n)\gamma$ by $\eta\gamma$, we get $\sum_\gamma \varphi(\gamma^{-1}e^n)\eta\gamma$. The function $\varphi \mapsto \widehat{\varphi}$ defines a homomorphism $C_f^n(\widetilde{X};\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}[\pi]}(C_n,\mathbb{Z}[\pi])$ which is injective since φ is recoverable from $\widehat{\varphi}$ as the coefficient of $\gamma = 1$. Furthermore, this homomorphism is surjective since a $\mathbb{Z}[\pi]$-homomorphism $\psi: M \to \mathbb{Z}[\pi]$ has the form $\psi(x) = \sum_\gamma \psi_\gamma(x)\gamma$ with $\psi_\gamma \in \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$ satisfying $\psi_\gamma(x) = \psi_1(\gamma^{-1}x)$, so ψ_1 determines ψ. The isomorphisms $C_f^n(\widetilde{X};\mathbb{Z}) \approx \mathrm{Hom}_{\mathbb{Z}[\pi]}(C_n,\mathbb{Z}[\pi])$ are isomorphisms of cochain complexes, so the respective cohomology groups $H_c^n(\widetilde{X};\mathbb{Z})$ and $H^n(X;\mathbb{Z}[\pi])$ are isomorphic. $\square$

Cup and cap product work easily with local coefficients in a bundle of rings, the latter concept being defined in the obvious way. The cap product can be used to extend Poincaré duality to nonorientable manifolds M, using local coefficients in $M_{\mathbb{Z}}$ or more generally M_R for a ring R:

Theorem 3H.6. *For an arbitrary closed n-manifold M there is a fundamental class $[M] \in H_n(M;M_R)$ such that $[M]\frown: H^k(M;M_R) \to H_{n-k}(M;M_R)$ is an isomorphism for all k.*

With the definitions we have given the proof is essentially the same as in §3.3, so we shall not stop to give details here.

Exercises

1. Compute $H_*(S^1;E)$ and $H^*(S^1;E)$ for $E \to S^1$ the nontrivial bundle with fiber $\mathbb{Z}$.

2. Compute the homology groups with local coefficients $H_n(M;M_{\mathbb{Z}})$ for a closed nonorientable surface M.

3. Let $\mathcal{B}(X;G)$ be the set of isomorphism classes of bundles of groups $E \to X$ with fiber G, and let $E_0 \to B\mathrm{Aut}(G)$ be the bundle corresponding to the 'identity' action $\rho:\mathrm{Aut}(G) \to \mathrm{Aut}(G)$. Show that the map $[X, B\mathrm{Aut}(G)] \to \mathcal{B}(X,G)$, $[f] \mapsto f^*(E_0)$, is a bijection if X is a CW complex, where $[X,Y]$ denotes the set of homotopy classes of maps $X \to Y$.

4. Show that if finite connected CW complexes X and Y are homotopy equivalent, then their universal covers $\widetilde{X}$ and $\widetilde{Y}$ are proper homotopy equivalent.

5. If X is a finite nonsimply-connected graph, show that $H^n(X;\mathbb{Z}[\pi_1 X])$ is zero unless $n = 1$, when it is the direct sum of a countably infinite number of $\mathbb{Z}$'s. [Use Proposition 3H.5 and compute $H^n_c(\widetilde{X})$ as $\varinjlim H^n(\widetilde{X}, \widetilde{X} - T_i)$ for a suitable sequence of finite subtrees $T_1 \subset T_2 \subset \cdots$ of $\widetilde{X}$ with $\bigcup_i T_i = \widetilde{X}$.]

6. Show that homology groups $H_n^{\ell f}(X;G)$ can be defined using **locally finite** chains, which are formal sums $\sum_\sigma g_\sigma \sigma$ of singular simplices $\sigma:\Delta^n \to X$ with coefficients $g_\sigma \in G$, such that each $x \in X$ has a neighborhood meeting the images of only finitely many σ's with $g_\sigma \neq 0$. Develop this homology theory far enough to show that for a locally compact CW complex X, $H_n^{\ell f}(X;G)$ can be computed using infinite cellular chains $\sum_\alpha g_\alpha e^n_\alpha$.